International Journal of Mathematics and
Computer Science, **15**(2020), no. 3, 821–826

$\mathscr{B}$-closed sets in $\mathscr{B}$-spaces

Prapart Pue-on, Chawalit Boonpok

Department of Mathematics
Faculty of Science
Mahasarakham University
Mahasarakham, 44150, Thailand

email: prapart.p@msu.ac.th, chawalit.b@msu.ac.th

(Received March 1, 2020, Accepted April 8, 2020)

Abstract

The objective of this article is to introduce the notion of $\mathscr{B}$-closed sets in $\mathscr{B}$-spaces and investigate some of their properties. Moreover, some characterizations of $\mathscr{B}$-regular and $\mathscr{B}$-normal spaces are discussed.

1 Introduction

General topology has shown its fruitfulness in both the pure and applied directions. The theory of generalized topological spaces, which was founded by Császár [4], is one of the most important development of general topology in recent years. In particular, the author defined some basic operator on generalized topological spaces. A large number of papers is devoted to the study of generalized open like sets of a topological space containing the class of open sets and possessing properties more or less similar to those of open sets. Császár [1] defined the notion of weak structures and showed that, in many situations, these structures can replace generalized topologies or minimal structures. Every general topology [4] and every minimal structure [5] is a weak structure. In [1], the present author defined some structure and operators under more general conditions. Levine [6] introduced the concept

Key words and phrases: $\mathscr{B}$-space, $\mathscr{B}$-closed set.
AMS (MOS) Subject Classifications: 54A05, 54D10.
ISSN 1814-0432, 2020, http://ijmcs.future-in-tech.net

of generalized closed sets in a topological space by comparing the closure of a subset with its open supersets. In 2011, Noiri and Roy [7] introduced and studied the notion of generalized μ-closed sets in a topological space by using the concept of generalized open sets introduced by Császár. The class of all generalized μ-closed sets is strictly larger than the class of all μ-closed sets. Moreover, generalized closed sets in the sense of Levine is a special type of generalized μ-closed sets in topological space. The purpose of the present article is to introduce the notion of $\mathscr{B}$-closed sets in $\mathscr{B}$-spaces. Furthermore, several interesting properties of $\mathscr{B}$-closed sets are investigated. Finally, some characterizations of $\mathscr{B}$-regular and $\mathscr{B}$-normal spaces are discussed.

2 Preliminaries

Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. A subfamily $\mu \subseteq \mathcal{P}(X)$ is called a *generalized topology* [4], (briefly, GT) on X if $\emptyset \in \mu$ and the union of elements of μ belongs to μ. A set X, with a GT μ on X is called a *generalized topological space* (briefly, GTS) and is denoted by (X, μ). For a GTS (X, μ), the elements of μ are called *μ-open* sets and the complement of μ-open sets are called *μ-closed* sets [4]. For $A \subseteq X$, the intersection of all μ-closed sets containing A is denoted by $c_\mu(A)$ and the union of all μ-open sets contained in A is denoted by $c_\mu(A)$. It is easy to observe that i_μ and c_μ are idempotent and monotonic, where $\gamma : \mathcal{P}(X) \to \mathcal{P}(X)$ is said to be *idempotent* iff $A \subseteq B \subseteq X$ implies $\gamma(\gamma(A)) = \gamma(A)$ and *monotonic* iff $\gamma(A) \subseteq \gamma(B)$. It is also well known from [2, 3] that if μ is a GT on X and $A \subseteq X$, $x \in X$, then $x \in c_\mu(A)$ iff $x \in M \in \mu$ implies $M \cap A \neq \emptyset$ and $c_\mu(X - A) = X - i_\mu(A)$.

A subfamily w of $\mathcal{P}(X)$ is called a *weak structure* (briefly, WS) [1] if $\emptyset \in w$. The pair (X, w) is called a *weak structure* (WS) *space*. Each member of a WS w is said to *w-open* [1] and the complement of a w-open set is said to be *w-closed*. Let A be a subset of X. The union of all w-open sets contained in A is called the *w-interior* of A and is denoted by $i_w(A)$ [1]. The intersection of all w-closed sets containing A is called the *w-closure* of A and is denoted by $c_w(A)$ [1].

Lemma 2.1. *[1] If w is a WS on X and $A, B \subseteq X$, then $i_w(A) \subseteq A \subseteq c_w(A)$, $A \subseteq B$ implies $i_w(A) \subseteq i_w(B)$ and $c_w(A) \subseteq c_w(B)$, $i_w(i_w(A)) = i_w(A)$ and $c_w(c_w(A)) = c_w(A)$, $i_w(X-A) = X - c_w(A)$ and $c_w(X-A) = X - i_w(A)$.*

Lemma 2.2. *[1] If w is a WS on X, then $x \in c_w(A)$ iff $W \cap A \neq \emptyset$ whenever $x \in W \in w$.*

3 $\mathscr{B}$-closed sets

In this section, we introduce the notion of $\mathscr{B}$-closed sets in $\mathscr{B}$-spaces. Moreover, some properties of $\mathscr{B}$-closed sets are investigated.

Definition 3.1. Let X be a non-empty set. Let μ be a generalized topology on X and let w be a weak structure on X. A triple (X, μ, w) is said to be a *$\mathscr{B}$-space.*

Definition 3.2. Let (X, μ, w) be a $\mathscr{B}$-space. A subset A of X is said to be *$\mathscr{B}$-closed* if $c_w(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \mu$. The complement of a $\mathscr{B}$-closed set is said to be *$\mathscr{B}$-open.*

Proposition 3.3. *Let A be a subset of a $\mathscr{B}$-space (X, μ, w). If A is $\mathscr{B}$-closed, then $c_w(A) - A$ does not contain any non-empty μ-closed set.*

Proof. Let F be a μ-closed subset of X such that $F \subseteq c_\mu(A) - A$. Since $X - F$ is μ-open, $A \subseteq X - F$ and A is $\mathscr{B}$-closed, $c_w(A) \subseteq X - F$. This implies that $F \subseteq X - c_w(A)$. Thus $F \subseteq [X - c_w(A)] \cap c_w(A) = \emptyset$ and hence $F = \emptyset$. □

Theorem 3.4. *A subset A of a $\mathscr{B}$-space (X, μ, w) is $\mathscr{B}$-closed if and only if $c_\mu(\{x\}) \cap A \neq \emptyset$ for every $x \in c_w(A)$.*

Proof. Suppose that $c_\mu(\{x\}) \cap A = \emptyset$ for some $x \in c_w(A)$. Then $A \subseteq X - c_\mu(\{x\})$. Since A is $\mathscr{B}$-closed and $X - c_\mu(\{x\})$ is μ-open, $c_w(A) \subseteq X - c_\mu(\{x\})$. This contradicts that $x \in c_w(A)$.

Conversely, suppose that A is not $\mathscr{B}$-closed. Thus, $\emptyset \neq c_w(A) - U$ for some $U \in \mu$ containing A. There exists $x \in c_w(A) - U$. Since $x \notin U$, $U \cap c_\mu(\{x\}) = \emptyset$ and hence $A \cap c_\mu(\{x\}) = \emptyset$. This shows that $A \cap c_\mu(\{x\}) = \emptyset$ for some $x \in c_w(A)$. □

Proposition 3.5. *Let (X, μ, w) be a $\mathscr{B}$-space and $A, B \subseteq X$. If A is $\mathscr{B}$-closed and $A \subseteq B \subseteq c_w(A)$, then B is $\mathscr{B}$-closed.*

Proof. Let $B \subseteq U \in \mu$. Since A is $\mathscr{B}$-closed and $A \subseteq U$, $c_w(A) \subseteq U$. Now, $B \subseteq c_w(A)$, $c_w(B) \subseteq c_w(A)$ and hence $c_w(B) \subseteq U$. Thus B is $\mathscr{B}$-closed. □

Theorem 3.6. *A subset A of a $\mathscr{B}$-space (X, μ, w) is $\mathscr{B}$-open if and only if $F \subseteq i_w(A)$ whenever $F \subseteq A$ and F is μ-closed.*

Proof. Suppose that A is a $\mathscr{B}$-open set. Let F be a μ-closed set and $F \subseteq A$. Then $X - A \subseteq X - F$. Since $X - A$ is $\mathscr{B}$-closed and $X - F$ is μ-open, we have $X - i_w(A) = c_w(X - A) \subseteq X - F$ and hence $F \subseteq i_w(A)$.

Conversely, let $X - A \subseteq U$ and $U \in \mu$. Then $X - U \subseteq A$ and $X - U$ is μ-closed. By hypothesis, $X - U \subseteq i_w(A)$ and hence $c_w(X - A) = X - i_w(A) \subseteq U$. Therefore, $X - A$ is $\mathscr{B}$-closed and hence A is $\mathscr{B}$-open. □

Proposition 3.7. *For a $\mathscr{B}$-space (X, μ, w), the following properties are equivalent:*

(1) for every μ-open subset U of X, $c_w(U) \subseteq U$;

(2) every subset of X is $\mathscr{B}$-closed.

Proof. (1) ⇒ (2): Let A be any subset of X and $A \subseteq U \in \mu$. By (1), $c_w(U) \subseteq U$ and hence $c_w(A) \subseteq c_w(U) \subseteq U$. Thus, A is $\mathscr{B}$-closed.

(2) ⇒ (1): Let $U \in \mu$. By (2), we have U is $\mathscr{B}$-closed and hence $c_w(U) \subseteq U$. □

Proposition 3.8. *Let (X, μ, w) be a $\mathscr{B}$-space and $A \subseteq X$. If A is $\mathscr{B}$-open, then $U = X$ whenever U is μ-open and $i_w(A) \cup (X - A) \subseteq U$.*

Proof. Let A be a $\mathscr{B}$-open set and $U \in \mu$ such that $i_w(A) \cup (X - A) \subseteq U$. Then $X - U \subseteq [X - i_w(A)] \cap A$. Thus, $X - U \subseteq c_w(X - A) - (X - A)$. Since $X - A$ is $\mathscr{B}$-closed, by Proposition 3.3, $X - U = \emptyset$ and hence $X = U$. □

Proposition 3.9. *Let (X, μ, w) be a $\mathscr{B}$-space and $A \subseteq X$. If A is $\mathscr{B}$-open and $i_w(A) \subseteq B \subseteq A$, then B is $\mathscr{B}$-open.*

Proof. Since $X - A \subseteq X - B \subseteq X - i_w(A) = c_w(X - A)$ and $X - A$ is $\mathscr{B}$-closed, by Proposition 3.5, $X - B$ is $\mathscr{B}$-closed. Therefore, B is $\mathscr{B}$-open. □

4 Some separation axioms in $\mathscr{B}$-spaces

In this section, we introduce separation axioms called $\mathscr{B}$-regular and $\mathscr{B}$-normal in a $\mathscr{B}$-space (X, μ, w) and investigate their characterizations.

Definition 4.1. A weak structure w on a non-empty set X is said to have *property $\mathscr{H}$* if $X \in w$ and the union of elements of w belongs to w.

Lemma 4.2. *Let X be a non-empty set and let w be a weak structure on X satisfying property $\mathscr{H}$. For a subset A of X, the following properties hold:*

(1) $A \in w$ if and only if $i_w(A) = A$;

(2) A is w-closed if and only if $c_w(A) = A$.

Definition 4.3. A $\mathscr{B}$-space (X, μ, w) is said to be *$\mathscr{B}$-regular* if for each μ-closed subset F of X and each $x \notin F$, there exist disjoint w-open subsets U and V of X such that $x \in U$ and $F \subseteq V$.

Theorem 4.4. *Let (X, μ, w) be a $\mathscr{B}$-space and let w have property $\mathscr{H}$. The following properties are equivalent:*

(1) (X, μ, w) is $\mathscr{B}$-regular;

(2) for each μ-closed set F and $x \notin F$, there exist $U \in w$ and a $\mathscr{B}$-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$;

(3) for each $A \subseteq X$ and each μ-closed set F with $A \cap F = \emptyset$, there exist $U \in w$ and a $\mathscr{B}$-open set V such that $A \cap U \neq \emptyset$, $F \subseteq V$ and $U \cap V = \emptyset$.

Proof. (1) $\Rightarrow$ (2): The proof is obvious.

(2) $\Rightarrow$ (3): Let $A \subseteq X$ and F be a μ-closed set with $A \cap F = \emptyset$. Then, for each $x \in A$, $x \notin F$ and hence by (2), there exist $U \in w$ and a $\mathscr{B}$-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$. Thus, $A \cap U \neq \emptyset$, $F \subseteq V$ and $U \cap V = \emptyset$.

(3) $\Rightarrow$ (1): Let F be a μ-closed set and $x \notin F$. Since $F \cap \{x\} = \emptyset$, by (3) there exist $U \in w$ and a $\mathscr{B}$-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$. By Theorem 3.6, we have $F \subseteq i_w(V) = W \in w$ and hence $U \cap W = \emptyset$. This shows that (X, μ, w) is $\mathscr{B}$-regular. □

Definition 4.5. A $\mathscr{B}$-space (X, μ, w) is said to be *$\mathscr{B}$-normal* if for any two disjoint μ-closed subsets A and B of X, there exist two disjoint w-open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$.

Theorem 4.6. *Let (X, μ, w) be a $\mathscr{B}$-space and let w have property $\mathscr{H}$. The following properties are equivalent:*

(1) (X, μ, w) is $\mathscr{B}$-normal;

(2) for any pair of disjoint μ-closed subset A and B of X, there exist disjoint $\mathscr{B}$-open subset U and V of X such that $A \subseteq U$ and $B \subseteq V$;

(3) for each μ-closed set F and each μ-open set U containing F, there exists a $\mathscr{B}$-open set V such that $F \subseteq V \subseteq c_w(V) \subseteq U$.

Proof. (1) $\Rightarrow$ (2): The proof is obvious.

(2) $\Rightarrow$ (3): Let F be a μ-closed set and U be a μ-open set containing F. Then F and $X-U$ are two disjoint μ-closed sets. By (2), there exist disjoint $\mathscr{B}$-open subset V and W of X such that $F \subseteq V$ and $X-U \subseteq W$. Since W is $\mathscr{B}$-open and $X-U$ is μ-closed, by Theorem 3.6, $X-U \subseteq i_w(W)$. Thus, $c_w(X-W) = X - i_w(W) \subseteq U$ and hence $F \subseteq V \subseteq c_w(V) \subseteq c_w(X-W) \subseteq U$.

(3) $\Rightarrow$ (1): Let A and B be two disjoint μ-closed subsets of X. Then A is a μ-closed set and $X-B$ is a μ-open set containing A. Then, by (3), there exists a $\mathscr{B}$-open set U such that $A \subseteq U \subseteq c_w(U) \subseteq X-B$. By Theorem 3.6, $A \subseteq i_w(U)$ and $B \subseteq X - c_w(U)$. Thus, we obtain $i_w(U)$ and $X-c_w(U)$ are disjoint w-open subsets of X. Consequently, (X,μ,w) is $\mathscr{B}$-normal. $\square$

Acknowledgment. This research was financially supported by Mahasarakham University.

References

[1] Á. Császár, Weak structures, Acta Math. Hungar., **131,** (2011), 193–195.

[2] Á. Császár, δ- and θ-modification of generalized topologies, Acta Math. Hungar., **120,** (2008), 275–279.

[3] Á. Császár, Generalized open sets in generalized topologies, Acta Math. Hungar., **106,** (2005), 53–66.

[4] Á. Császár, Generalized topology, generalized continuity, Acta Math. Hungar., **96,** (2002), 351–357.

[5] H. Maki, J. Umehara, T. Noiri, Every topological space is pre-$T_{\frac{1}{2}}$, Mem. Fac. Sci.Kochi. Univ. Ser. A Math., **17,** (1996), 33–42.

[6] N. Levine, Generalized closed sets in topology, Rend. Circ. Mat. Palermo, (2), **19,** (1970), 89–96.

[7] T. Noiri, B. Roy, Unification of generalized open sets on topological spaces, Acta Math. Hungar., **130,** (2011), 349–357.