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Weakly generalized closed sets in ideal topological spaces

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Abstract

The purpose of the present paper is to introduce the notion of weakly generalized closed sets in ideal topological spaces. Some properties of weakly generalized closed sets with respect to an ideal are investigated.

1 Introduction

Generalized closed sets and generalized open sets, as significant and fundamental subjects in the study topology, have been researched by many mathematicians. In 1970, Levine [5] introduced the concept of generalized closed sets in a topological space by comparing the closure of a subset with its open supersets. The study of generalized closed sets has produced some new separation axioms which lie between T_0 and T_1 such as $T_{\frac{1}{2}}$, T_{gs} and $T_{\frac{3}{4}}$. Some of these properties have been found to be useful in computer science and digital topology [3]. Other new properties are defined by variations of the property of submaximality. Furthermore, the study of generalized closed sets also provides new characterizations of some known classes of spaces, for example, the

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class of extremally disconnected spaces. As the weak form generalized closed sets, the notion of weakly generalized closed sets was introduced and studied by Sundaram and Pushpalatha [8]. Sundaram and Nagaveni [9] introduced and studied the notion of strongly generalized closed sets, which is implies by that of closed sets and implies that of generalized closed sets. Park and Park [7] introduced and studied mildly generalized closed sets, which is properly placed between the classes of strongly generalized closed sets and weakly generalized closed sets. The concept of ideal topological space was studied by Kuratowski [4] and Vaidyanathaswamy [10]. Janković and Hamlett [2] investigated further properties of ideal topological spaces. Noiri and Rajesh [6] introduced and studied the concept of generalized closed sets with respect to an ideal in an ideal bitopological space. In 2011, Jafari and Rajesh [1] introduced and investigated the concept of generalized closed sets with respect to an ideal, which is extension of the concept of generalized closed sets. In this paper, we introduce the notion of weakly generalized closed sets with respect to an ideal $\mathscr{I}$, and also study some of their properties.

2 Preliminaries

Throughout the present paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. In a topological space (X, τ), the closure and the interior of any subset A of X will denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. An ideal $\mathscr{I}$ on a topological space (X, τ) is a non-empty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows: $A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I}$ for every open neighbourhood U of $x\}$. In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [4], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and $\mathrm{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be $\star$-closed [2] if $A^\star \subseteq A$. The complement of a $\star$-closed set is said to be $\star$-open.

3 On weakly generalized closed sets with respect to an ideal

Recall that a subset A of a topological space (X, τ) is called *weakly generalized closed* (briefly, *weakly g-closed*) [9] if $\mathrm{Cl}(\mathrm{Int}(A)) \subseteq U$ whenever $A \subseteq U$ and U is open in X. This motivates our first definition.

Definition 3.1. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *weakly generalized closed with respect to an ideal* (briefly, *wg$\mathscr{I}$-closed*) if $\mathrm{Cl}(\mathrm{Int}(A)) - U \in \mathscr{I}$ whenever $A \subseteq U$ and U is open in X.

Remark 3.1. *Every weakly g-closed set is wg$\mathscr{I}$-closed, but the converse need not be true, as this may be seen from the following example.*

Example 3.2. *Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{b\}, \{b, c\}, X\}$ and ideal $\mathscr{I} = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. Then $A = \{b\}$ is wg$\mathscr{I}$-closed, which is not weakly g-closed.*

Theorem 3.3. *A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is wg$\mathscr{I}$-closed if and only if $F \subseteq Cl(Int(A)) - A$ and F is closed in X implies $F \in \mathscr{I}$.*

Proof. Suppose that A is a wg$\mathscr{I}$-closed set. Let F be a closed set such that $F \subseteq \mathrm{Cl}(\mathrm{Int}(A)) - A$. Then, we have $F \subseteq X - A$ and hence $A \subseteq X - F$. Since A is wg$\mathscr{I}$-closed and $X - F$ is open, $\mathrm{Cl}(\mathrm{Int}(A)) - (X - F) \in \mathscr{I}$. Since $\mathrm{Cl}(\mathrm{Int}(A)) - (X - F) = \mathrm{Cl}(\mathrm{Int}(A)) \cap F$ and $F \subseteq \mathrm{Cl}(\mathrm{Int}(A))$, we have $F \subseteq \mathrm{Cl}(\mathrm{Int}(A)) \cap F$. Thus, $F \in \mathscr{I}$.

Conversely, suppose that $F \subseteq \mathrm{Cl}(\mathrm{Int}(A)) - A$ and F is closed in X implies $F \in \mathscr{I}$. Let U be an open set and $A \subseteq U$. Then, we have $X - U \subseteq X - A$ and hence $\mathrm{Cl}(\mathrm{Int}(A)) - U = \mathrm{Cl}(\mathrm{Int}(A)) \cap (X - U) \subseteq \mathrm{Cl}(\mathrm{Int}(A)) \cap (X - A) = \mathrm{Cl}(\mathrm{Int}(A)) - A$. Since $\mathrm{Cl}(\mathrm{Int}(A)) \cap (X - U)$ is closed and by the hypothesis, $\mathrm{Cl}(\mathrm{Int}(A)) - U \in \mathscr{I}$. Consequently, we obtain A is wg$\mathscr{I}$-closed. □

Remark 3.2. *The union of two wg$\mathscr{I}$-closed sets need not be wg$\mathscr{I}$-closed as shown by the following example.*

Example 3.4. *Let $X = \{1, 2, 3\}$ with topology $\tau = \{\emptyset, \{2, 3\}, X\}$ and ideal $\mathscr{I} = \{\emptyset, \{2\}\}$. Then $A = \{3\}$ and $B = \{2\}$ are wg$\mathscr{I}$-closed sets, but $A \cup B = \{2, 3\}$ is not wg$\mathscr{I}$-closed.*

Proposition 3.5. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A, B \subseteq X$. If A and B are both wg$\mathscr{I}$-closed and open sets, then $A \cup B$ is also wg$\mathscr{I}$-closed.*

Proof. Suppose that A and B are both wg$\mathscr{I}$-closed and open sets. Let U be an open set such that $A \cup B \subseteq U$. Then, we have $A \subseteq U$ and $B \subseteq U$. Since A and B are wg$\mathscr{I}$-closed, $\mathrm{Cl}(\mathrm{Int}(A)) - U \in \mathscr{I}$ and $\mathrm{Cl}(\mathrm{Int}(B)) - U \in \mathscr{I}$. Since A and B are open, we have $\mathrm{Cl}(\mathrm{Int}(A \cup B)) - U = [\mathrm{Cl}(\mathrm{Int}(A)) - U] \cup [\mathrm{Cl}(\mathrm{Int}(B)) - U] \in \mathscr{I}$ and hence $A \cup B$ is wg$\mathscr{I}$-closed. □

Proposition 3.6. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A, B \subseteq X$. If A is wg$\mathscr{I}$-closed and $A \subseteq B \subseteq Cl(Int(A))$, then B is wg$\mathscr{I}$-closed.*

Proof. Suppose that A is wg$\mathscr{I}$-closed and $A \subseteq B \subseteq \mathrm{Cl}(\mathrm{Int}(A))$. Let U be an open set and $B \subseteq U$. Then, we have $A \subseteq U$. Since A is wg$\mathscr{I}$-closed, $\mathrm{Cl}(\mathrm{Int}(A)) - U \in \mathscr{I}$. Since $\mathrm{Cl}(\mathrm{Int}(A)) = \mathrm{Cl}(\mathrm{Int}(B))$, we have $\mathrm{Cl}(\mathrm{Int}(B)) - U = \mathrm{Cl}(\mathrm{Int}(A)) - U \in \mathscr{I}$ and hence B is wg$\mathscr{I}$-closed. □

Remark 3.3. *The intersection of two wg$\mathscr{I}$-closed sets need not be wg$\mathscr{I}$-closed as shown by the following example.*

Example 3.7. *Let $X = \{1, 2, 3\}$ with topology $\tau = \{\emptyset, \{1\}, \{1, 2\}, X\}$ and ideal $\mathscr{I} = \{\emptyset, \{3\}\}$. Then $A = \{1, 2\}$ and $B = \{1, 3\}$ are wg$\mathscr{I}$-closed sets, but $A \cap B = \{1\}$ is not wg$\mathscr{I}$-closed.*

Theorem 3.8. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. If A is a wg$\mathscr{I}$-closed set and F is a closed set, then $A \cap F$ wg$\mathscr{I}$-closed.*

Proof. Suppose that A is a wg$\mathscr{I}$-closed set and F is a closed set. Let U be an open set and $A \cap F \subseteq U$. Then, we have $X - U \subseteq X - (A \cap F) = (X - A) \cup (X - F)$ and hence $F \cap (X - U) \subseteq F \cap [(X - A) \cup (X - F)] = F \cap (X - A) \subseteq X - A$. Therefore, $A \subseteq X - [F \cap (X - U)] = U \cup (X - F)$. Since A is wg$\mathscr{I}$-closed and $U \cup (X - F)$ is open, $\mathrm{Cl}(\mathrm{Int}(A)) - [U \cup (X - F)] \in \mathscr{I}$. Since $\mathrm{Cl}(\mathrm{Int}(A \cap F)) \subseteq \mathrm{Cl}(\mathrm{Int}(A)) \cap \mathrm{Int}(F)) \subseteq \mathrm{Cl}(\mathrm{Int}(A)) \cap \mathrm{Cl}(\mathrm{Int}(F)) \subseteq \mathrm{Cl}(\mathrm{Int}(A)) \cap F$,

$$\begin{aligned}
\mathrm{Cl}(\mathrm{Int}(A \cap F)) - U &= \mathrm{Cl}(\mathrm{Int}(A \cap F)) \cap (X - U) \\
&\subseteq [\mathrm{Cl}(\mathrm{Int}(A)) \cap F] \cap (X - U) \\
&= \mathrm{Cl}(\mathrm{Int}(A)) \cap [F \cap (X - U)] \\
&= \mathrm{Cl}(\mathrm{Int}(A)) \cap [X - ((X - F) \cup U)] \\
&= \mathrm{Cl}(\mathrm{Int}(A)) - [(X - F) \cup U]
\end{aligned}$$

and hence $\mathrm{Cl}(\mathrm{Int}(A \cap F)) - U \in \mathscr{I}$. Thus, $A \cap F$ wg$\mathscr{I}$-closed. □

Definition 3.9. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *weakly generalized open with respect to an ideal* (briefly, *wg$\mathscr{I}$-open*) if $X - A$ is wg$\mathscr{I}$-closed.

Theorem 3.10. *A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is wg$\mathscr{I}$-open if and only if $F - U \subseteq Int(Cl(A))$ for some $U \in \mathscr{I}$ whenever $F \subseteq A$ and F is closed.*

Proof. Suppose that A is a wg$\mathscr{I}$-open set. Let F be a closed set and $F \subseteq A$. Then, we have $X - A \subseteq X - F$. Since $X - F$ is open and $X - A$ is wg$\mathscr{I}$-closed, $\mathrm{Cl}(\mathrm{Int}(X - A)) - (X - F) \in \mathscr{I}$. Thus, there exists $U \in \mathscr{I}$ such that $U = \mathrm{Cl}(\mathrm{Int}(X - A)) - (X - F)$ and hence $\mathrm{Cl}(\mathrm{Int}(X - A)) \subseteq (X - F) \cup U$. Consequently, we obtain $F - U = X - [(X - F) \cup U] \subseteq X - \mathrm{Cl}(\mathrm{Int}(X - A)) = \mathrm{Int}(\mathrm{Cl}(A))$.

Conversely, let G be an open set and $X - A \subseteq G$. Then, we have $X - G \subseteq A$. By the hypothesis, $(X - G) - U \subseteq \mathrm{Int}(\mathrm{Cl}(A))$ for some $U \in \mathscr{I}$. Therefore, $X - \mathrm{Int}(\mathrm{Cl}(A)) \subseteq X - [(X - G) - U]$ and hence $\mathrm{Cl}(\mathrm{Int}(X - A)) \subseteq G \cup U$. Since $\mathrm{Cl}(\mathrm{Int}(X - A)) - G = \mathrm{Cl}(\mathrm{Int}(X - A)) \cap (X - G) \subseteq (G \cup U) \cap (X - G) = U \cap (X - G) \subseteq U$, $\mathrm{Cl}(\mathrm{Int}(X - A)) - G \in \mathscr{I}$. Thus, $X - A$ is wg$\mathscr{I}$-closed. This shows that A is wg$\mathscr{I}$-open. □

Recall that the sets A and B are said to be separated if $\mathrm{Cl}(A) \cap B = \emptyset$ and $\mathrm{Cl}(B) \cap A = \emptyset$.

Theorem 3.11. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A, B \subseteq X$. If A and B are separated wg$\mathscr{I}$-open sets, then $A \cup B$ is wg$\mathscr{I}$-open.*

Proof. Suppose that A and B are separated wg$\mathscr{I}$-open sets. Let F be a closed set and $F \subseteq A \cup B$. Then, we have $[F \cap \mathrm{Cl}(A)] \subseteq A$ and $[F \cap \mathrm{Cl}(A)] \subseteq A$. By the hypothesis, $[(F \cap \mathrm{Cl}(A)) - U_1] \subseteq \mathrm{Int}(\mathrm{Cl}(A))$ and $[(F \cap \mathrm{Cl}(B)) - U_2] \subseteq \mathrm{Int}(\mathrm{Cl}(B))$ for some $U_1, U_2 \in \mathscr{I}$. Since

$$F \cap \mathrm{Cl}(A) - \mathrm{Int}(\mathrm{Cl}(A)) \subseteq [F \cap \mathrm{Cl}(A) \cup U_1] \cap [(X - \mathrm{Int}(\mathrm{Cl}(A))) \cup U_1] \subseteq U_1$$

and $F \cap \mathrm{Cl}(B) - \mathrm{Int}(\mathrm{Cl}(B)) \subseteq [F \cap \mathrm{Cl}(A) \cup U_2] \cap [(X - \mathrm{Int}(\mathrm{Cl}(A))) \cup U_2] \subseteq U_2$. Thus, $[(F \cap \mathrm{Cl}(A)) - \mathrm{Int}(\mathrm{Cl}(A))] \in \mathscr{I}$ and $[(F \cap \mathrm{Cl}(B)) - \mathrm{Int}(\mathrm{Cl}(B))] \in \mathscr{I}$. Therefore, $[(F \cap \mathrm{Cl}(A)) - \mathrm{Int}(\mathrm{Cl}(A))] \cup [(F \cap \mathrm{Cl}(B)) - \mathrm{Int}(\mathrm{Cl}(B))] \in \mathscr{I}$. Since

$$\begin{aligned}
&[F \cap [\mathrm{Cl}(A) \cap \mathrm{Cl}(B)]] - [\mathrm{Int}(\mathrm{Cl}(A) \cup \mathrm{Int}(\mathrm{Cl}(B)))] \\
&\subseteq [[F \cap \mathrm{Cl}(A)] - \mathrm{Int}(\mathrm{Cl}(A))] \cup [[F \cap \mathrm{Cl}(B)] - \mathrm{Int}(\mathrm{Cl}(B))],
\end{aligned}$$

$[F \cap [\mathrm{Cl}(A) \cap \mathrm{Cl}(B)]] - [\mathrm{Int}(\mathrm{Cl}(A) \cup \mathrm{Int}(\mathrm{Cl}(B)))] \in \mathscr{I}$. Since $F = F \cap (A \cap B) \subseteq F \cup \mathrm{Cl}(A \cup B)$, we have

$$\begin{aligned}
F - \mathrm{Int}(\mathrm{Cl}(A \cup B)) &\subseteq [F \cap \mathrm{Cl}(A \cup B)] - \mathrm{Int}(\mathrm{Cl}(A \cup B)) \\
&\subseteq [F \cap (\mathrm{Cl}(A \cup B))] - [\mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Int}(\mathrm{Cl}(B))]
\end{aligned}$$

and hence $F - \text{Int}(\text{Cl}(A \cup B)) \in \mathscr{I}$. This implies that $F - U \subseteq \text{Int}(\text{Cl}(A \cup B))$ for some $U \in \mathscr{I}$. Consequently, we obtain $A \cup B$ is wg$\mathscr{I}$-open. □

Corollary 3.12. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A, B \subseteq X$. If A and B are wg$\mathscr{I}$-closed sets such that $X - A$ and $X - B$ are separated, then $A \cap B$ is wg$\mathscr{I}$-closed.*

Proposition 3.13. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A, B \subseteq X$. If A is wg$\mathscr{I}$-open and Int(Cl(A)) $\subseteq B \subseteq A$, then B is wg$\mathscr{I}$-open.*

Proof. Suppose that A is wg$\mathscr{I}$-open and $\text{Int}(\text{Cl}(A)) \subseteq B \subseteq A$. Then, we have $X - A \subseteq X - B \subseteq \text{Cl}(\text{Int}(X - A))$ and by Proposition 3.6, $X - B$ is wg$\mathscr{I}$-closed. Thus, B is wg$\mathscr{I}$-open. □

Theorem 3.14. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A \subseteq X$. Then A is wg$\mathscr{I}$-closed if and only if Cl(Int(A)) $- A$ is wg$\mathscr{I}$-open.*

Proof. Suppose that A is a wg$\mathscr{I}$-closed set. Let F be a closed set and $F \subseteq \text{Cl}(\text{Int}(A)) - A$. By Theorem 3.3, we have $F \in \mathscr{I}$ and there exists $U \in \mathscr{I}$ such that $U = F$. Thus, $F - U \subseteq \text{Int}[\text{Cl}[\text{Cl}(\text{Int}(A) - A)]]$ and by Theorem 3.10, $\text{Cl}(\text{Int}(A)) - A$ is wg$\mathscr{I}$-open.

Conversely, suppose that $\text{Cl}(\text{Int}(A)) - A$ is wg$\mathscr{I}$-open. Let G be an open set and $A \subseteq G$. Then, we have $[\text{Cl}(\text{Int}(A)) \cap (X - G)] \subseteq [\text{Cl}(\text{Int}(A)) \cap (X - A)] = \text{Cl}(\text{Int}(A)) - A$. Since $\text{Cl}(\text{Int}(A)) \cap (X - G)$ is closed and $\text{Cl}(\text{Int}(A)) - A$ is wg$\mathscr{I}$-open, by Theorem 3.10, $[\text{Cl}(\text{Int}(A)) \cap (X - G)] - U \subseteq \text{Cl}(\text{Int}(A)) - A$ for some $U \in \mathscr{I}$. Since

$$\begin{aligned}
[\text{Cl}(\text{Int}(A)) \cap (X - G)] - U &\subseteq \text{Int}[\text{Cl}[\text{Cl}(\text{Int}(A)) - A]] \\
&= \text{Int}[\text{Cl}[\text{Cl}(\text{Int}(A)) \cap (X - A)]] \\
&\subseteq \text{Int}[\text{Cl}(\text{Cl}(\text{Int}(A))) \cap \text{Cl}(X - A)] \\
&= \text{Int}(\text{Cl}(\text{Int}(A))) \cap \text{Int}(\text{Cl}(X - A)) \\
&= \text{Int}(\text{Cl}(\text{Int}(A))) \cap [X - \text{Cl}(\text{Int}(A))] \\
&\subseteq \text{Cl}(\text{Int}(A)) \cap [X - \text{Cl}(\text{Int}(A))] = \emptyset,
\end{aligned}$$

$[\text{Cl}(\text{Int}(A)) \cap (X - G)] \subseteq U$ and hence $\text{Cl}(\text{Int}(A)) - G \in \mathscr{I}$. This shows that A is wg$\mathscr{I}$-closed. □

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