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Generalized Complex Fractional Derivative and Integral operators for the Unified Class of Analytic Functions

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Abstract

In this paper, we introduce a generalization of the Linear fractional differential and integral operator given by Srivastava and Owa [8]. In addition, we study some geometric properties of this new operator for the unified class $U[\Phi, \Psi; \alpha, \beta, \lambda, n]$, which was defined by Darus [2] like growth, distortion and the radii of starlikeness.

1 Introduction

Let A be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

This class consists of functions that are analytic and univalent in the open disk $U = \{z : z \in \mathbb{C}; |z| < 1\}$. Let $S^*(\alpha)$ denote the class consists of the starlike functions in A of order α such that $0 \leq \alpha \leq 1$, where any function $f(z)$ in A belongs to this class if it satisfies the condition $\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha$, for all z in the open unit disk. We will also denote by $C(\alpha)$ the class of

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convex functions in A of order α $(0 \leq \alpha \leq 1)$, and it is well known that $f(z) \in C(\alpha)$, if $zf'(z)$ is starlike of order α.

Silverman [12] defined a subclass of A, which we will denote by T, consisting of all the functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n.$$

In [5, 4], Goodman established the basic concepts of uniformly starlike and uniformly convex functions, denoted by UCV and UST, respectively. After that, Ronning [11] defined a class of starlike functions related to uniformly convex functions which he denoted by S_p, where $f \in S_p$ if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \left|\frac{zf'(z)}{f(z)} - 1\right|.$$

In 1997, Bharati et. al [1] defined a class, called k-uniformly convex function of order α, which they denoted by $k - UCV(\alpha)$, where f belongs to this class if and only if

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} \geq k\left|\frac{zf''(z)}{f'(z)} - 1\right|.$$

Also, they defined the class k-uniformly starlike function of order α, denoted by $k - ST(\alpha)$, where f belongs to this class if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq k\left|\frac{zf'(z)}{f(z)} - 1\right|.$$

Note that $f \in k - UCV$ if and only if $zf \in k - ST(\alpha)$.

We need to mention that, Kanas and Wisniowska [9, 10] also studied similar classes of conic domains with $\alpha = 0$.

After that, Darus [3] defined a new family, denoted by $D(\Phi, \Psi; \alpha, \beta)$, consisting of functions f in A, that satisfy the inequality

$$\mathrm{Re}\left(\frac{f(z) * \Phi(z)}{f(z) * \Psi(x)}\right) > \beta\left|\frac{f(z) * \Phi(z)}{f(z) * \Psi(x)} - 1\right| + \alpha,$$

where $0 \leqslant \alpha < 1, \beta \geq 0, \Phi(z) = z + \sum_{n=2}^{\infty} \varpi_n z^n$, and $\Psi(z) = z + \sum_{n=2}^{\infty} \gamma_n z^n$ are analytic in U such that $f(z) * \Psi(z) \neq 0, \varpi_n > \gamma_n, \varpi_n \geqslant 0$ and $\gamma_n \geqslant 0$. This somehow generalized various subclasses of A.

Before that, Darus [2] unified the subclasses $D_T(\Phi,\Psi;\alpha,\beta) = D(\Phi,\Psi;\alpha,\beta) \cap T$, and $E_T(\Phi,\Psi;\alpha,\beta) = E(\Phi,\Psi;\alpha,\beta) \cap T$, where $f \in E_T$ if and only if $zf' \in D(\Phi,\Psi;\alpha,\beta)$, denoted by $U[\Phi,\Psi;\alpha,\beta,\lambda,n]$, where

$$U[\Phi,\Psi;\alpha,\beta,0,n] = (1-\lambda) D_T(\Phi,\Psi;\alpha,\beta) + \lambda E_T(\Phi,\Psi;\alpha,\beta).$$

As a result, we get $U[\Phi,\Psi;\alpha,\beta,0,n] = D_T(\Phi,\Psi;\alpha,\beta)$and $U[\Phi,\Psi;\alpha,\beta,1,n] = E_T(\Phi,\Psi;\alpha,\beta)$.

2 Generalized Fractional Derivative and Integral Operator

Owa [6] made a huge move in the theory of fractional analysis by introducing the following definitions for fractional differential and integral operators in the complex z-plane:

Definition 2.1. *The fractional derivative operator of order* ν *for* $f(z)$ *is* $D_z^\nu f(z) = \frac{1}{\Gamma(1-\nu)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^\nu} d\xi, 0 \leq \nu < 1,$ *where the function* $f(z)$ *is analytic in a simply-connected region of the complex* $z-$*plane*$, \mathbb{C},$ *containing the origin and the multiplicity of* $(z-\xi)^\nu$ *is removed by assuming that* $\log(z-\xi)$ *to be real when* $z-\xi > 0$.

Definition 2.2. *The fractional integral operator of order* ν *for* $f(z)$ *is* $I_z^\nu f(z) = \frac{1}{\Gamma(\nu)} \frac{d}{dz} \int_0^z f(\xi)(z-\xi)^{\nu-1} d\xi, 0 \leq \nu < 1,$ *where the function* $f(z)$ *is analytic in simply-connected region of the complex* $z-$ *plane*$, \mathbb{C},$ *containing the origin and the multiplicity of* $(z-\xi)^\nu$ *is removed by assuming that* $\log(z-\xi)$ *to be real when* $z-\xi > 0$.

Remark 2.1. *Using Definitions 2.1 and 2.2 one can easily show that*

1. $D_z^0 f(z) = f(0)$.
2. $D_z^\nu \{z^\tau\} = \frac{\Gamma(\tau+1)}{\Gamma(\tau-\nu+1)} \{z^{\tau-\nu}\}, \tau > -1, 0 \leq \nu < 1.$
3. $I_z^\nu \{z^\mu\} = \frac{\Gamma(\mu+1)}{\Gamma(\tau+\nu+1)} \{z^{\tau+\nu}\}, \tau > -1, 0 \leq \nu < 1.$

Using Definitions 2.1 and 2.2, many researchers have introduced new derivative and integral operators. They formed some new classes of univalent functions and studied their properties.

One of the most important fractional linear operator in the complex plane was given by Owa and Srivastava [7]:

$$\Omega^{\nu} f = \Gamma\left(2-\nu\right) z^{\nu} D_z^{\nu} f, \tag{2.1}$$

where $D_z^{\nu} f$ is the fractional derivative operator in Definition 2.1

In this section, we shall generalize the operator given by Owa and Srivastava [7] and introduce a new derivative fractional operator using Definition 2.1. To introduce this operator, we will begin by proving the following lemmas:

Lemma 2.3. *Let $f\left(z\right) = z + \sum_{n=2}^{\infty} a_n z^n$ be a function in the subclass of A. Then the m^{th} fractional derivative for $f\left(z\right)$, denoted by $\left(D_z^{\nu}\right)^{(m)} f\left(z\right)$, is given by*

$$\left(D_z^{\nu}\right)^{(m)} f\left(z\right) = \frac{1}{\Gamma\left(2-m\nu\right)} z^{1-m\nu} + \sum_{n=2}^{\infty} a_n \frac{\Gamma\left(n+1\right)}{\Gamma\left(n-m\nu+1\right)} z^{n-m\nu}.$$

Proof. To prove this lemma, we use mathematical induction. Clearly, the result is true for $m = 1$. Now assume that the result is true for $m = k$; i.e., $\left(D_z^{\nu}\right)^{(k)} f\left(z\right) = \frac{1}{\Gamma(2-k\nu)} z^{1-k\nu} + \sum_{n=2}^{\infty} a_n \frac{\Gamma(n+1)}{\Gamma(n-k\nu+1)} z^{n-k\nu}, kv < 1$. To show that the result is true for $m = k+1$, we write,

$$\begin{aligned}
\left(D_z^{\nu}\right)^{(k+1)} f\left(z\right) &= D_z^{\nu}\left(\left(D_z^{\nu}\right)^{(k)} f\left(z\right)\right) \\
&= \frac{1}{\Gamma\left(2-k\nu\right)} D_z^{\nu}\left(z^{1-k\nu}\right) + \sum_{n=2}^{\infty} a_n \frac{\Gamma\left(n+1\right)}{\Gamma\left(n-k\nu+1\right)} D_z^{\nu}\left(z^{n-k\nu}\right) \\
&= \frac{1}{\Gamma\left(2-k\nu\right)} \frac{\Gamma\left(1-k\nu+1\right)}{\Gamma\left(1-k\nu-\nu+1\right)} z^{1-k\nu-\nu} \\
&+ \sum_{n=2}^{\infty} a_n \frac{\Gamma\left(n+1\right)}{\Gamma\left(n-k\nu+1\right)} \frac{\Gamma\left(n-k\nu+1\right)}{\Gamma\left(n-k\nu-\nu+1\right)} z^{n-k\nu-\nu} \\
&= \frac{1}{\Gamma\left(2-k\nu\right)} \frac{\Gamma\left(2-k\nu\right)}{\Gamma\left(2-\left(k+1\right)\nu\right)} z^{1-(k+1)\nu} \\
&+ \sum_{n=2}^{\infty} a_n \frac{\Gamma\left(n+1\right)}{\Gamma\left(n-k\nu+1\right)} \frac{\Gamma\left(n-k\nu+1\right)}{\Gamma\left(n-\left(k+1\right)\nu+1\right)} z^{n-(k+1)\nu} \\
&= \frac{1}{\Gamma\left(2-\left(k+1\right)\nu\right)} z^{1-(k+1)\nu} + \sum_{n=2}^{\infty} a_n \frac{\Gamma\left(n+1\right)}{\Gamma\left(n-\left(k+1\right)\nu+1\right)} z^{n-(k+1)\nu}.
\end{aligned}$$

Hence the result follows.

We can also prove the following lemma:

Lemma 2.4. *Let* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ *be a function in the subclass of* T. *Then the* m^{th} *fractional derivative for* $f(z)$, *denoted by* $(D_z^{\nu})^{(m)} f(z)$, *is given by*

$$(D_z^{\nu})^{(m)} f(z) = \frac{1}{\Gamma(2-m\nu)} z^{1-m\nu} - \sum_{n=2}^{\infty} a_n \frac{\Gamma(n+1)}{\Gamma(n-m\nu+1)} z^{n-m\nu}.$$

Using those lemmas, we shall introduce the generalized fractional derivative operator as follows:

Definition 2.5. *Let* $f(z)$ *be an element of the class* A *and let* $m = 1, 2, 3, \ldots$. *Then the generalized fractional derivative operator, denoted by* $D_z^{\nu,m}$, *is*

$$D_z^{\nu,m} f = \Gamma(2-mv) z^{m\nu} (D_z^{\nu})^{(m)} f(z), 0 \leq \nu < 1, mv \neq 2, 3, 4, ...$$

.

Remark 2.2. *We notice that* $D_z^{\nu,1} f = \Omega^{\nu} f$.

3 Coefficient inequalities, Growth and Distortion theorems for $D_z^{\nu,m} f$

In this section, we give distortion theorems for the generalized fractional operator $D_z^{\nu,m} f$. However, we do need the following lemmas to establish our new results.

Lemma 3.1. *[3] A function* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ *is in the class* $U[\Phi, \Psi; \alpha, \beta, \lambda, n]$ *if and only if it satisfies the following inequality*

$$\sum_{n=2}^{\infty} (1-\lambda+n\lambda)\left[(1+\beta)\varpi_n - (\alpha+\beta)\gamma_n\right]|a_n| \leq 1-\alpha, \tag{3.2}$$

where $0 \leqslant \alpha < 1, \beta \geq 0$, $\varpi_n > \gamma_n, \varpi_n \geqslant 0$ *and* $\gamma_n \geqslant 0$.

Lemma 3.2. *[3] A function* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ *is in the class* $U[\Phi, \Psi; \alpha, \beta, \lambda, n]$ *if and only if it satisfies the following inequality*

$$|a_n| \leq \frac{1-\alpha}{(1-\lambda+n\lambda)\left[(1+\beta)\varpi_n - (\alpha+\beta)\gamma_n\right]}, n \geq 2. \tag{3.3}$$

where $0 \leqslant \alpha < 1, \beta \geq 0$, $\varpi_n > \gamma_n, \varpi_n \geqslant 0$ *and* $\gamma_n \geqslant 0$.

We now give distortion theorems of $D_z^{\nu,m} f$ in the unified class $U[\Phi,\Psi;\alpha,\beta,\lambda,n]$.

Theorem 3.3. *If* $f(z)=z-\sum_{n=2}^{\infty}a_nz^n$ *is an element in the class* $U[\Phi,\Psi;\alpha,\beta,\lambda,n]$ *,then*

$$\frac{1}{\Gamma(2-\nu)}|z|^{1-\nu}-|z|^{2-\nu}\frac{\Gamma(3)}{\Gamma(3-\nu)}\frac{1-\alpha}{(1+\lambda)[(1+\beta)\varpi_n-(\alpha+\beta)\gamma_n]}$$
$$\leq |D_z^{\nu}f(z)|$$
$$\leq \frac{1}{\Gamma(2-\nu)}|z|^{1-\nu}+|z|^{2-\nu}\frac{\Gamma(3)}{\Gamma(3-\nu)}\frac{1-\alpha}{(1+\lambda)[(1+\beta)\varpi_n-(\alpha+\beta)\gamma_n]}$$

which generalizes the result by Darus [2].

Proof. From Lemma 3.1 we get $f(z)=z-\sum_{n=2}^{\infty}a_nz^n$ belongs to the class $U[\Phi,\Psi;\alpha,\beta,\lambda,n]$ if and only if it satisfies the following inequality

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)[(1+\beta)\varpi_n-(\alpha+\beta)\gamma_n]|a_n|\leq 1-\alpha,$$

and this inequality leads to the following lower bound for $\sum_{n=2}^{\infty}|a_n|$,

$$\sum_{n=2}^{\infty}|a_n|\leq\frac{1-\alpha}{(1+\lambda)[(1+\beta)\varpi_2-(\alpha+\beta)\gamma_2]}.$$

Now,

$$\begin{aligned}|D_z^{\nu}f(z)| &= \left|\frac{1}{\Gamma(2-\nu)}z^{1-\nu}-\sum_{n=2}^{\infty}a_n\frac{\Gamma(n+1)}{\Gamma(n-\nu+1)}z^{n-\nu}\right|\\
&\geq \left|\frac{1}{\Gamma(2-\nu)}z^{1-\nu}\right|-\sum_{n=2}^{\infty}\left|a_n\frac{\Gamma(n+1)}{\Gamma(n-\nu+1)}z^{n-\nu}\right|\\
&\geq \left|\frac{1}{\Gamma(2-\nu)}\right|\left|z^{1-\nu}\right|-\sum_{n=2}^{\infty}|a_n|\left|\frac{\Gamma(n+1)}{\Gamma(n-\nu+1)}\right|\left|z^{n-\nu}\right|\\
&\geq \left|\frac{1}{\Gamma(2-\nu)}\right|\left|z^{1-\nu}\right|-\left|\frac{\Gamma(2+1)}{\Gamma(2-\nu+1)}\right|\left|z^{2-\nu}\right|\sum_{n=2}^{\infty}|a_n|\\
&\geq \frac{1}{\Gamma(2-\nu)}|z|^{1-\nu}-|z|^{2-\nu}\frac{\Gamma(3)}{\Gamma(3-\nu)}\frac{1-\alpha}{(1+\lambda)[(1+\beta)\varpi_n-(\alpha+\beta)\gamma_n.]}\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left|D_z^{\nu} f(z)\right| &= \left|\frac{1}{\Gamma(2-\nu)} z^{1-\nu} - \sum_{n=2}^{\infty} a_n \frac{\Gamma(n+1)}{\Gamma(n-\nu+1)} z^{n-\nu}\right| \\
&\le \left|\frac{1}{\Gamma(2-\nu)} z^{1-\nu}\right| + \sum_{n=2}^{\infty}\left|a_n \frac{\Gamma(n+1)}{\Gamma(n-\nu+1)} z^{n-\nu}\right| \\
&\le \left|\frac{1}{\Gamma(2-\nu)}\right|\left|z^{1-\nu}\right| + \sum_{n=2}^{\infty}|a_n|\left|\frac{\Gamma(n+1)}{\Gamma(n-\nu+1)}\right|\left|z^{n-\nu}\right| \\
&\le \left|\frac{1}{\Gamma(2-\nu)}\right|\left|z^{1-\nu}\right| + \left|\frac{\Gamma(2+1)}{\Gamma(2-\nu+1)}\right|\left|z^{2-\nu}\right|\sum_{n=2}^{\infty}|a_n| \\
&\le \frac{1}{\Gamma(2-\nu)}|z|^{1-\nu} + |z|^{2-\nu}\frac{\Gamma(3)}{\Gamma(3-\nu)}\frac{1-\alpha}{(1+\lambda)\left[(1+\beta)\varpi_n - (\alpha+\beta)\gamma_n,\right]}.
\end{aligned}$$

Hence the result follows.

Next, we find an upper and lower bounds for $\left|\left(D_z^{\nu}\right)^{(m)} f(z)\right|$.

Theorem 3.4. *Let* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ *be an element in the class* $U[\Phi,\Psi;\alpha,\beta,\lambda,n]$. *Then*

$$\begin{aligned}
&\frac{1}{\Gamma(2-m\nu)}|z|^{1-m\nu} - |z|^{2-m\nu}\frac{\Gamma(3)}{\Gamma(3-m\nu)}\frac{1-\alpha}{(1+\lambda)\left[(1+\beta)\varpi_n - (\alpha+\beta)\gamma_n\right]} \\
&\le \left|\left(D_z^{\nu}\right)^{(m)} f(z)\right| \\
&\le \frac{1}{\Gamma(2-m\nu)}|z|^{1-m\nu} + |z|^{2-m\nu}\frac{\Gamma(3)}{\Gamma(3-m\nu)}\frac{1-\alpha}{(1+\lambda)\left[(1+\beta)\varpi_n - (\alpha+\beta)\gamma_n.\right]}
\end{aligned}$$

Proof. As in the proof of the last theorem, we have

$$\begin{aligned}\left|\left(D_z^{\nu}\right)^{(m)} f(z)\right| &= \left|\frac{1}{\Gamma(2-m\nu)} z^{1-m\nu} - \sum_{n=2}^{\infty} a_n \frac{\Gamma(n+1)}{\Gamma(n-m\nu+1)} z^{n-m\nu}\right| \\ &\geq \left|\frac{1}{\Gamma(2-m\nu)} z^{1-m\nu}\right| - \sum_{n=2}^{\infty}\left|a_n \frac{\Gamma(n+1)}{\Gamma(n-m\nu+1)} z^{n-m\nu}\right| \\ &\geq \left|\frac{1}{\Gamma(2-m\nu)}\right|\left|z^{1-m\nu}\right| - \sum_{n=2}^{\infty}|a_n|\left|\frac{\Gamma(n+1)}{\Gamma(n-m\nu+1)}\right|\left|z^{n-m\nu}\right| \\ &\geq \left|\frac{1}{\Gamma(2-m\nu)}\right|\left|z^{1-m\nu}\right| - \left|\frac{\Gamma(2+1)}{\Gamma(2-m\nu+1)}\right|\left|z^{2-m\nu}\right|\sum_{n=2}^{\infty}|a_n| \\ &\geq \frac{1}{\Gamma(2-m\nu)}|z|^{1-m\nu} - |z|^{2-m\nu}\frac{\Gamma(3)}{\Gamma(3-m\nu)}\frac{1-\alpha}{(1+\lambda)\left[(1+\beta)\varpi_n-(\alpha+\beta)\gamma_n\right]}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\left|\left(D_z^{\nu}\right)^{(m)} f(z)\right| &= \left|\frac{1}{\Gamma(2-m\nu)} z^{1-m\nu} - \sum_{n=2}^{\infty} a_n \frac{\Gamma(n+1)}{\Gamma(n-m\nu+1)} z^{n-m\nu}\right| \\ &\leq \left|\frac{1}{\Gamma(2-m\nu)} z^{1-m\nu}\right| + \sum_{n=2}^{\infty}\left|a_n \frac{\Gamma(n+1)}{\Gamma(n-m\nu+1)} z^{n-m\nu}\right| \\ &\leq \left|\frac{1}{\Gamma(2-m\nu)}\right|\left|z^{1-m\nu}\right| + \sum_{n=2}^{\infty}|a_n|\left|\frac{\Gamma(n+1)}{\Gamma(n-m\nu+1)}\right|\left|z^{n-m\nu}\right| \\ &\leq \left|\frac{1}{\Gamma(2-m\nu)}\right|\left|z^{1-m\nu}\right| + \left|\frac{\Gamma(2+1)}{\Gamma(2-m\nu+1)}\right|\left|z^{2-m\nu}\right|\sum_{n=2}^{\infty}|a_n| \\ &\leq \frac{1}{\Gamma(2-m\nu)}|z|^{1-m\nu} + |z|^{2-m\nu}\frac{\Gamma(3)}{\Gamma(3-m\nu)}\frac{1-\alpha}{(1+\lambda)\left[(1+\beta)\varpi_n-(\alpha+\beta)\gamma_n\right]}.\end{aligned}$$

4 New fractional integral operator which is closed under $U[\Phi, \Psi; \alpha, \beta, \lambda, n]$

In this section, we introduce a generalized fractional integral operator. In addition, we will see the closeness of this new operator in the subclass $U[\Phi, \Psi; \alpha, \beta, \lambda, n]$.

Definition 4.1. *Let $f(z) \in A$. Then the generalized fractional integral operator of order ν is defined by*

$$I_n^{\nu,m}(f(z)) = \Gamma(m\nu+2)\, z^{-m\nu}\left(I_n^{\nu}\right)^{(m)} f(z),$$

where $(I_n^{\nu})^{(m)}$ *is the composition for the fractional integral operator given in Definition* 2.2 *for* $m-$ *times.*

As a consequence of applying the operator defined in Definition 2.2 and part (3) Remark 2.1 to the functions $f(z) \in T$ we get the following result:

Let $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$. Then

$$I_n^{\nu,m}(f(z)) = I_n^{\nu,m}\left(z - \sum_{n=2}^{\infty} a_n z^n\right)$$
$$= z - \sum_{n=2}^{\infty} a_n \frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)} z^n$$

In the following theorem, we discuss the closeness of the operator $I_n^{\nu,m}(f(z))$ under the unified subclass $U[\Phi,\Psi;\alpha,\beta,\lambda,n]$.

Theorem 4.2. *Let* $f(z) \in U[\Phi,\Psi;\alpha,\beta,\lambda,n]$. *Then* $I_n^{\nu,m}(f(z))$ *is an element of the subclass* $U[\Phi,\Psi;\alpha,\beta,\lambda,n]$.

Proof. From Lemma 3.1, it is enough to show that

$$\sum_{n=2}^{\infty} \frac{(1-\lambda+n\lambda)[(1+\beta)\varpi_n - (\alpha+\beta)\gamma_n]}{1-\alpha} \frac{\Gamma(n+1)\Gamma(m\nu+2)}{\Gamma(n+1+m\upsilon)} |a_n| \le 1.$$

So let $f(z) \in U[\Phi,\Psi;\alpha,\beta,\lambda,n]$. Then, by Lemma 3.1,

$$\frac{(1-\lambda+n\lambda)[(1+\beta)\varpi_n - (\alpha+\beta)\gamma_n]}{1-\alpha} |a_n| \le 1.$$

Hence it suffices to show that $\frac{\Gamma(n+1)\Gamma(m\nu+2)}{\Gamma(n+1+m\upsilon)} \le 1$. This can be shown by applying the Gamma function properties as follows:

$$\frac{\Gamma(n+1)\Gamma(m\nu+2)}{\Gamma(n+1+m\upsilon)} = \frac{\Gamma(n+1)(m\nu+1)(m\nu)\Gamma(m\nu)}{(m\nu+n)(m\nu+n-1)(m\nu+n-2)\cdots(m\nu+1)m\upsilon\Gamma(m\upsilon)}$$
$$= \frac{(n)(n-1)(n-2)\cdots(2)(1)}{(m\nu+n)(m\nu+n-1)(m\nu+n-2)\cdots(m\nu+2)(1)}$$
$$= \frac{n}{m\nu+n}\frac{n-1}{m\nu+n-1}\cdots\frac{2}{m\nu+2}.\frac{1}{1}$$
$$< 1.$$

Hence the result follows. In the following theorem, we investigate the starlikeness of the operator $I_n^{\nu,m}(f(z))$.

Theorem 4.3. *If $f(z) \in U[\Phi, \Psi; \alpha, \beta, \lambda, n]$, then $I_n^{\nu,m}(f(z))$ is starlike of order $0 \le \rho < 1$ in the disk $|z| < \xi$, where*

$$\xi = \inf_n \left\{ \frac{(1-\rho)(1-\lambda+n\lambda)[(1+\beta)\varpi_n-(\alpha+\beta)\gamma_n]}{C^*(n+2-\rho)(1-\alpha)} |a_n| \right\}^{\dfrac{1}{n-1}}, \text{ and } C^* = \frac{\Gamma(n+1)\Gamma(m\nu+2)}{\Gamma(n+1+m\upsilon)} .$$

Proof. We have to show that $\left| \frac{z\left(I_n^{\nu,m} f(z)\right)'}{I_n^{\nu,m} f(z)} - 1 \right| < 1-\rho.$

To see this, we will start from the left hand side.

$$\begin{aligned}
\left| \frac{z\left(I_n^{\nu,m} f(z)\right)'}{I_n^{\nu,m} f(z)} - 1 \right| &= \left| \frac{z\left(1-\sum_{n=2}^{\infty} n a_n \frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)} z^{n-1}\right)}{z-\sum_{n=2}^{\infty} a_n \frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)} z^{n}} - 1 \right| \\
&= \left| \frac{z\left(1-\sum_{n=2}^{\infty} n a_n \frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)} z^{n-1}\right)}{z\left(1-\sum_{n=2}^{\infty} a_n \frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)} z^{n-1}\right)} - 1 \right| \\
&= \left| \frac{1-\sum_{n=2}^{\infty} n a_n \frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)} z^{n-1}}{1-\sum_{n=2}^{\infty} a_n \frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)} z^{n-1}} - 1 \right| \\
&= \left| \frac{1-\sum_{n=2}^{\infty} n a_n \frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)} z^{n-1} - 1 + \sum_{n=2}^{\infty} a_n \frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)} z^{n-1}}{1-\sum_{n=2}^{\infty} a_n \frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)} z^{n-1}} \right| \\
&= \left| \frac{\sum_{n=2}^{\infty} (n+1) a_n \frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)} z^{n-1}}{1-\sum_{n=2}^{\infty} a_n \frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)} z^{n-1}} \right| \\
&\le \frac{\sum_{n=2}^{\infty} (n+1) |a_n| \frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)} |z|^{n-1'}}{1-\sum_{n=2}^{\infty} |a_n| \frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)} |z|^{n-1}}.
\end{aligned}$$

Then

$$
\begin{aligned}
(1-\rho)-\left|\frac{z\left(I_n^{\nu,m}f(z)\right)'}{I_n^{\nu,m}f(z)}-1\right| &\geq (1-\rho)-\frac{\sum_{n=2}^{\infty}(n+1)\left|a_n\right|\frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)}\left|z\right|^{n-1}}{1-\sum_{n=2}^{\infty}\left|a_n\right|\frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)}\left|z\right|^{n-1}} \\
&= \frac{\begin{array}{c}\{(1-\rho)\left(1-\sum_{n=2}^{\infty}\left|a_n\right|\frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)}\left|z\right|^{n-1}\right)\\ -\sum_{n=2}^{\infty}(n+1)\left|a_n\right|\frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)}\left|z\right|^{n-1}\}\end{array}}{1-\sum_{n=2}^{\infty}\left|a_n\right|\frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)}\left|z\right|^{n-1}} \\
&= \frac{\begin{array}{c}\{\left((1-\rho)-(1-\rho)\sum_{n=2}^{\infty}\left|a_n\right|\frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)}\left|z\right|^{n-1}\right)\\ -\sum_{n=2}^{\infty}(n+1)\left|a_n\right|\frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)}\left|z\right|^{n-1}\}\end{array}}{1-\sum_{n=2}^{\infty}\left|a_n\right|\frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)}\left|z\right|^{n-1}} \\
&= \frac{\{(1-\rho)-\sum_{n=2}^{\infty}(n+2-\rho)\left|a_n\right|\frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)}\left|z\right|^{n-1}}{1-\sum_{n=2}^{\infty}\left|a_n\right|\frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)}\left|z\right|^{n-1}},
\end{aligned}
$$

and finally

$$
|z|<\xi=\inf_n\left\{\frac{(1-\rho)(1-\lambda+n\lambda)\left[(1+\beta)\varpi_n-(\alpha+\beta)\gamma_n\right]}{\frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)}(n+2-\rho)(1-\alpha)}\left|a_n\right|\right\}^{\frac{1}{n-1}},
$$

such that

$$
(1-\rho)-\left|\frac{z\left(I_n^{\nu,m}f(z)\right)'}{I_n^{\nu,m}f(z)}-1\right|>\frac{(1-\rho)-(1-\rho)}{1-\sum_{n=2}^{\infty}\left|a_n\right|\frac{\Gamma(2+m\nu)\Gamma(n+1)}{\Gamma(n+m\nu+1)}\left|z\right|^{n-1}}=0.
$$

Hence the result follows.

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