On the solvability of a third-order p-Laplacian m-point boundary value problem at resonance on the half-line with two dimensional kernel

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Abstract

The solvability of a third-order boundary value problem at resonance on the half-line is considered in this work. By using a semi-projector and the Ge and Ren extension of Mawhin’s coincidence degree theory, existence results are established for the problem, where $\dim \ker L = 2$. An example will be used to illustrate our result.

1 Introduction

Boundary value problems with integral and multi-point boundary conditions in an infinite interval have many real life applications, for instance, in the study of many physical phenomena such as unsteady flow of fluid through a semi-infinite porous media and radially symmetric solutions of nonlinear elliptic equations. They are also found in plasma physics and the study of drain flows see [1].

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If the corresponding homogeneous part of a boundary value problem has a non-trivial solution, then the boundary value problem is said to be at resonance. Resonant problems with both linear and p-Laplacian differential operators have been studied by many authors using Mawhin’s coincidence degree theorem [12] and Ge and Ren [3] extension of the coincidence degree theorem see [14, 9, 6, 5, 7, 2, 13, 10].

However, to the best of our knowledge, only few authors in the literature have considered p-Laplacian boundary value problems on the half-line with two dimensional kernel. Motivated by this, we study the solvability for the following p-Laplacian third-order boundary value problem having integral and m-point boundary conditions at resonance on the half-line with two dimensional kernel:

\[
(\sigma(t)\varphi_p(u''(t)))' + v(t)w(t, u(t), u'(t), u''(t)) = 0, \quad t \in (0, \infty),
\]

\[
u(0) = \sum_{i=1}^{m} \alpha_i \int_{0}^{\xi_i} v(t)u(t)dt, \quad u'(0) = \int_{0}^{\infty} v(t)u'(t)dt, \quad \lim_{t \to \infty} (\sigma(t)\varphi_p(u''(t))) = 0,
\]

where \( w : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R} \) is a \( v \)-Carathéodory function, \( 0 \leq \xi_i < \infty, \xi_i \in \mathbb{R}, i = 1, 2, \ldots, m, \) \( v(t) \in L^1[0, \infty), \) \( v(t) > 0, \) \( \sigma \in C[0, \infty) \cap C^2(0, \infty), \) \( \sigma(t) > 0, \)
\( \varphi_q \left( \frac{1}{\sigma} \right) \in L^1[0, \infty), \) \( \varphi_p(s) = |s|^{p-2}s, \ p > 1 \) and \( \varphi_q = \varphi_q^{-1}. \)

In section 2 of this work, necessary lemmas, theorems and definitions will be given while section 3 will be dedicated to stating and proving the existence result. An example will be given to corroborate the result obtained.

## 2 Preliminaries

**Definition 2.1.** A map \( w : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R} \) is \( v \)-Carathéodory if the following conditions are satisfied:

(i) for each \((d, e, f) \in \mathbb{R}^3, \) the mapping \( t \to w(t, d, e, f) \) is Lebesgue measurable,

(ii) for a.e. \( t \in [0, \infty), \) the mapping \((d, e, f) \to w(t, d, e, f) \) is continuous on \( \mathbb{R}^3, \)

(iii) for each \( k > 0 \) and \( v \in L^1[0, \infty), \) there exists \( \psi_k(t) : [0, \infty) \to [0, \infty) \)

satisfying \( \int_{0}^{\infty} v(t)\psi_k(t)dt < \infty \) such that, for a.e. \( t \in [0, \infty) \) and every \((d, e, f) \in [-k, k], \) we have \( |w(t, d, e, f)| \leq \psi_k(t). \)
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Definition 2.2. [3] Let \((U, \| \cdot \|_U)\) and \((Z, \| \cdot \|_Z)\) be two Banach spaces. The continuous operator \(M : U \cap \text{dom } M \to Z\), is quasi-linear if the following hold

(i) \(\text{Im } M = M(U \cap \text{dom } M)\) is a closed subset of \(Z\);

(ii) \(\ker M = \{u \in U \cap \text{dom } M : Mu = 0\}\) is linearly homeomorphic to \(\mathbb{R}^n\), \(n < \infty\).

Definition 2.3. [4] Let \(U\) be a Banach space and \(U_1 \subset U\) a subspace. Let \(Q : U \to U_1\) be an operator. Then \(Q\) is a semi-projector if

(i) \(Q^2 = Q\),

(ii) \(Q(\lambda u) = \lambda Qu\), where \(u \in U, \lambda \in \mathbb{R}\).

Let \(U_1 = \ker M\) and \(U_2\) be the complement space of \(U_1\) in \(U\), then \(U = U_1 \oplus U_2\). Similarly, if \(Z_1\) is a subspace of \(Z\) and \(Z_2\) is the complement space of \(Z_1\) in \(Z\), then \(Z = Z_1 \oplus Z_2\). Let \(P : U \to U_1\) be a projector, \(Q : Z \to Z_1\) be a semi-projector and \(\Omega \subset U\) an open bounded set with \(\theta \in \Omega\) the origin. Also, let \(N_1\) be denoted by \(N\) and let \(N_\lambda : \overline{\Omega} \to Z\), where \(\lambda \in [0, 1]\) is a continuous operator and \(\Sigma_\lambda = \{u \in \overline{\Omega} : Mu = N_\lambda u\}\).

Definition 2.4. [8] Let \(U\) be the space of all continuous and bounded vector-valued functions on \([0, \infty)\) and \(X \subset U\). Then \(X\) is said to be relatively compact if the following statements hold:

(i) \(X\) is bounded in \(U\),

(ii) all functions from \(X\) are equicontinuous on any compact subinterval of \([0, \infty)\),

(iii) all functions from \(X\) are equiconvergent at \(\infty\); i.e., \(\forall \epsilon > 0, \exists T = T(\epsilon)\) such that \(\|A(t) - A(\infty)\|_{\mathbb{R}^n} < \epsilon \forall t > T\) and \(A \in X\).

Definition 2.5. [3] Let \(N_\lambda : \overline{\Omega} \to Z, \lambda \in [0, 1]\) be a continuous operator. The operator \(N_\lambda\) is said to be \(M\)-compact in \(\overline{\Omega}\) if there exists a vector subspace \(Z_1 \subset Z\) such that \(\dim Z_1 = \dim U_1\) and a compact and continuous operator \(R : \overline{\Omega} \times [0, 1] \to U_2\) such that for \(\lambda \in [0, 1]\), the following hold:

(i) \((I - Q)N_\lambda(\overline{\Omega}) \subset \text{Im } M \subset (I - Q)Z\),

(ii) \(QN_\lambda u = 0 \Leftrightarrow QNu = 0, \lambda \in (0, 1),\)
(iii) $R(\cdot, u)$ is the zero operator and $R(\cdot, \lambda)|_{\Sigma \lambda} = (I - P)|_{\Sigma \lambda}$.

(iv) $M[P + R(\cdot, \lambda)] = (I - Q)N_\lambda$.

Lemma 2.6. [4] The following are properties of the function $\varphi_p : \mathbb{R} \to \mathbb{R}$:

(i) It is continuous, monotonically increasing and invertible with inverse $\varphi_p^{-1} = \varphi_q$, where $q > 1$ and satisfies $\frac{1}{p} + \frac{1}{q} = 1$,

(ii) For any $x, y > 0$,
   
   (a) $\varphi_p(x + y) \leq \varphi_p(x) + \varphi_p(y)$, if $1 < p < 2$,
   
   (b) $\varphi_p(x + y) \leq 2^{p-2}(\varphi_p(x) + \varphi_p(y))$, if $p \geq 2$.

Theorem 2.7. [3] Let $(U, \| \cdot \|_U)$ and $(Z, \| \cdot \|_Z)$ be two Banach spaces and $\Omega \subset U$ an open and bounded set. If the following hold

(B$_1$) the operator $M : U \cap \text{dom } M \to Z$ is a quasi-linear,

(B$_2$) the operator $N_\lambda : \overline{\Omega} \to Z$, $\lambda \in [0, 1]$ is $M$-compact,

(B$_3$) $Mu \neq N_\lambda u$, $\lambda \in [0, 1]$, $u \in \partial \Omega$,

(B$_4$) $\text{deg}\{JQN, \Omega \cap \text{ker } M, 0\} \neq 0$,

then the equation $Mu = Nu$ has at least one solution in $\overline{\Omega}$, where $N = N_1$ and the operator $J : Z_1 \to U_1$ is a homeomorphism with $J(\theta) = \theta$.

Let

$$U = \left\{ u \in C^2[0, \infty) : u, u', \sigma \varphi_p(u'') \in AC[0, \infty), \lim_{t \to \infty} e^{-t}|u^{(i)}(t)| \text{ exist, } i = 0, 1, 2 \right\},$$

with the norm $\|u\| = \max\{|u|, |u'|, |u''|\}$ defined on $U$, where $\|u\| = \sup_{t \in [0, \infty)} e^{-t}|u|$. We claim that the space $(U, \| \cdot \|)$ is a Banach Space.

Let $Y = L^1[0, \infty)$ with the norm $\|y\|_{L^1} = \int_0^\infty |y(t)|dt$ defined on it and $Z = \{ y : [0, \infty) \to \mathbb{R} : \int_0^\infty v(t)|y(t)|dt < \infty \}$ with the norm $\|z\|_Z = \int_0^\infty v(t)|z(v)|dv$. Define $M$ as a continuous operator such that $M : \text{dom } M \subset U \to Z$, where

$$\text{dom } M = \left\{ u \in U : \varphi_p(u''(t)) \in L^1[0, \infty), u(0) = \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} v(t)u(t)dt, \right.$$

$$u'(0) = \int_0^\infty v(t)u'(t)dt, \lim_{t \to \infty} (\sigma(t)\varphi_p(u''(t))) = 0 \right\},$$
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and \(Mu = (\sigma(t)\varphi_p(u''(t)))'\). We will define the operator \(N_\lambda u : \overline{\Omega} \rightarrow Z\) by

\[N_\lambda u = -\lambda f(t, u(t), u'(t), u''(t)) \quad \lambda \in [0, 1], \quad t \in [0, \infty),\]

where \(\Omega \subset U\) is an open and bounded set. Then the boundary value problem (1.1) in abstract form is \(Mu = Nu\).

The following are the assumptions made in this work:

\[(\phi_1) \int_0^\infty v(t)dt = 1, \quad \sum_{i=1}^m \alpha_i \int_0^{\xi_i} v(t)dt = 1, \quad \sum_{i=1}^m \alpha_i \int_0^{\xi_i} tv(t)dt = 0;\]

\[(\phi_2) G = \begin{vmatrix} Q_1e^{-t} & Q_2e^{-t} \\ Q_1te^{-t} & Q_2te^{-t} \end{vmatrix} := \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \quad \text{where}\]

\[Q_1z = \sum_{i=1}^m \alpha_i \int_0^{\xi_i} v(t) \int_0^x \varphi_q \left( \frac{1}{\sigma(s)} \right) \varphi_q \left( \int_s^\infty v(r)z(r)dr \right)dsdxdt \quad \text{and}\]

\[Q_2z = \int_0^\infty v(t) \int_0^t \varphi_q \left( \frac{1}{\sigma(s)} \right) \varphi_q \left( \int_s^\infty v(r)z(r)dr \right)dsdt.\]

By simple calculation, we see that \(\ker M = \{a + bt : a, b \in \mathbb{R}, \quad t \in [0, \infty)\}\) and \(\dim \ker M = 2\).

**Lemma 2.8.** The operator \(M : \text{dom} M \subset U \rightarrow Z\) is quasi-linear.

**Proof.** Clearly, \(\ker M = \{u \in \text{dom} M : u = a + bt, \quad a, \quad b \in \mathbb{R}\}\). Next, we obtain \(\text{Im} M\). Let \(u \in \text{dom} M\) and consider the problem

\[(\sigma(t)\varphi_p(u''(t)))' = -vz, \quad t \in [0, \infty), \quad (2.3)\]

Integrating (2.3) from \(t\) to \(\infty\), we have

\[\sigma(t)\varphi_pu''(t) = \lim_{t \rightarrow \infty} \sigma(t)\varphi_pu''(t) + \int_t^\infty v(r)z(r)dr, \quad (2.4)\]

from (1.2), we obtain

\[u''(t) = \varphi_q \left( \frac{1}{\sigma(t)} \right) \varphi_q \left( \int_t^\infty v(r)z(r)dr \right). \quad (2.5)\]

Integrating (2.5) from 0 to \(t\) yields

\[u'(t) = u'(0) + \int_0^t \varphi_q \left( \frac{1}{\sigma(s)} \right) \varphi_q \left( \int_s^\infty v(r)z(r)dr \right)ds. \quad (2.6)\]

It follows from (2.6) that

\[\int_0^\infty v(t)u'(t)dt = \int_0^\infty v(t)u'(0)dt + \int_0^\infty v(t) \int_0^t \varphi_q \left( \frac{1}{\sigma(s)} \right) \varphi_q \left( \int_s^\infty v(r)z(r)dr \right)dsdt\]
and applying boundary conditions (1.2) gives
\[ u'(0) = u'(0) \int_0^\infty v(t) dt - \int_0^\infty v(t) \int_0^t \varphi_q \left( \frac{1}{\sigma(s)} \right) \varphi_q \left( \int_s^\infty v(r) z(r) dr \right) ds dt. \]

Since \( \int_0^\infty v(t) dt = 1 \), \( \int_0^\infty v(t) \int_0^t \varphi_q \left( \frac{1}{\sigma(s)} \right) \varphi_q \left( \int_s^\infty v(r) z(r) dr \right) ds dt = 0 \).

Integrating (2.6) from 0 to \( t \) gives
\[ u(t) = u(0) + u'(0) t + \int_0^t \int_0^x \varphi_q \left( \frac{1}{\sigma(s)} \right) \varphi_q \left( \int_s^\infty v(r) z(r) dr \right) ds dx dt. \]  

(2.7)

Applying boundary conditions (1.2) gives
\[ u(0) = \sum_{i=1}^m \alpha_i \int_0^{\xi_i} v(t) \left( u(0) + u'(0) t + \int_0^t \int_0^x \varphi_q \left( \frac{1}{\sigma(s)} \right) \varphi_q \left( \int_s^\infty v(r) z(r) dr \right) ds dx \right) dt. \]

Since \( \sum_{i=1}^m \alpha_i \int_0^{\xi_i} v(t) dt = 1 \) and \( \sum_{i=1}^m \alpha_i \int_0^{\xi_i} tv(t) dt = 0 \),

\[ \sum_{i=1}^m \alpha_i \int_0^{\xi_i} v(t) \int_0^t \int_0^x \varphi_q \left( \frac{1}{\sigma(s)} \right) \varphi_q \left( \int_s^\infty v(r) z(r) dr \right) ds dx dt = 0 \]

and
\[ u(t) = a + bt + \int_0^t \int_0^x \varphi_q \left( \frac{1}{\sigma(s)} \right) \varphi_q \left( \int_s^\infty v(r) z(r) dr \right) ds dx dt, \]

where \( a \) and \( b \) are arbitrary constants and \( u(t) \) is a solution to (2.3) satisfying (1.2). So \( \ker M = 2 < \infty \) and \( M \subset (U \cap \text{dom } M) \subset Z \) is closed. Therefore, \( M \) is quasi-linear. \( \square \)

Let the projector \( P : U \to U_1 \) be defined as
\[ Pu(t) = u(0) + u'(0) t, \quad u \in U \]  

(2.8)

and the operators \( \Delta_1, \Delta_2 : Z \to Z_1 \) as
\[ \Delta_1 y = \frac{1}{G}(\delta_{11} Q_1 y + \delta_{12} Q_2 y) e^{-t}, \quad \Delta_2 y = \frac{1}{G}(\delta_{21} Q_1 y + \delta_{22} Q_2 y) e^{-t}, \]

where \( \delta_{ij} \) is the co-factor of \( g_{ij}, i, j = 1, 2 \). Then, the operator \( Q : Z \to Z_1 \) will be defined as
\[ Qy = (\Delta_1 y) + (\Delta_2 y) \cdot t, \]  

(2.9)

where \( Z_1 \) is the complement space of \( \text{Im } M \) in \( Z \).
Lemma 2.9. The operator $Q : Z \to Z_1$ is a semi-projector.

Proof. It can be shown that $\Delta_1((\Delta_1 z)) = (\Delta_1 z)$, $\Delta_1((\Delta_2 z)t) = 0$, $\Delta_2((\Delta_1 z)) = 0$ and $\Delta_2((\Delta_2 z)t) = \Delta_2 z$. Thus, $Q^2 z = Q((\Delta_1 z) + (\Delta_2 z)\cdot t) = (\Delta_1 z) + 0\cdot t + 0\cdot t^2 + (\Delta_2 z)\cdot Q z$ or $Q^2 = Q$. Also, for $\lambda \in \mathbb{R}$, $\Delta_1 \lambda z = \frac{1}{G}((\delta_{11} Q_1 \lambda z + \delta_{12} Q_2 \lambda z)e^{-t} = \lambda \Delta_1 z$ and $\Delta_2 \lambda z = \frac{1}{G}((\delta_{21} Q_1 \lambda z + \delta_{22} Q_2 \lambda z)e^{-t} = \lambda \Delta_2 z$. Hence,

$$Q(\lambda z) = (\Delta_1 \lambda z) + (\Delta_2 \lambda z)\cdot t = \lambda((\Delta_1 z) + (\Delta_2 z)\cdot t) = \lambda Q z.$$  

Therefore, by definition 2.3, $Q : Z \to Z_1$ is a semi-projector. \hfill \square

Let the operator $R : U \times [0, 1] \to U_2$ be defined by

$$R(u, \lambda)(t) = \int_0^t \int_0^x \varphi_q \left( \frac{1}{\sigma(s)} \right) \varphi_q \left( \int_s^\infty (\lambda v(f(r, u(r), u'(r), u''(r))) + Q N u(r)) dr \right) ds dx,$$

where $\varphi_p^{-1} = \varphi_q$, $U_2$ is the complement space of ker $M$ in U.

Lemma 2.10. If $w$ is a $v$-Carathéodory function, then $R : U \times [0, 1] \to U_2$ is $M$-compact.

Proof. Let $\Omega \subset U$ be nonempty, open and bounded. Then, for $u \in \overline{\Omega}$, there exists a constant $k > 0$ such that $\|u\| < k$. Since $w$ is a $v$-Carathéodory function, there exists $\psi_k : [0, \infty) \to [0, \infty)$ satisfying $\int_0^\infty v(t) \psi_k(t) dt < \infty$ such that for a.e. $t \in [0, \infty)$ and $\lambda \in [0, 1]$, we have

$$|N_\lambda u(t)| = |-\lambda w(t, u(t), u'(t), u''(t))| \leq |Nu(t)| = |w(t, u(t), u'(t), u''(t))| \leq \psi_k(t),$$

$$\|N_\lambda u\|_Z = \int_0^\infty v(r)|N_\lambda u(r)| dr \leq \int_0^\infty v(r)|\psi_k(r)| dr \leq \|\psi_k\|_Z,$$

and

$$|QN_\lambda u(t)| = |\lambda QNu(t)| \leq |QN u(t)|,$$

and

$$\|QN_\lambda u\|_Z = \int_0^\infty v(r)|QN_\lambda u(r)| dr \leq \|QN u\|_Z.$$
Now for any $u \in \overline{\Omega}$, $\lambda \in [0, 1]$, we have

$$\|R(u, \lambda)\|_{\infty} = \sup_{t \in [0, \infty)} e^{-t} |R(u, \lambda)(t)| \leq \frac{1}{e} \left\| \phi_q \left( \frac{1}{\sigma} \right) \right\|_{L^1} \phi_q(\|Nu\|_Z + \|QN\|_Z)$$

$$\leq \left\| \phi_q \left( \frac{1}{\sigma} \right) \right\|_Z \phi_q(\|\psi_k\|_Z + \|QN\|_Z) < \infty,$$  

(2.10)

$$\|R'(u, \lambda)\|_{\infty} = \sup_{t \in [0, \infty)} e^{-t} |R'(u, \lambda)(t)| \leq \left\| \phi_q \left( \frac{1}{\sigma} \right) \right\|_{L^1} \phi_q(\|\psi_k\|_Z + \|QN\|_Z) < \infty$$

(2.11)

and

$$\|R''(u, \lambda)\|_{\infty} = \sup_{t \in [0, \infty)} e^{-t} |R''(u, \lambda)(t)| \leq \left\| \phi_q \left( \frac{1}{\sigma} \right) \right\|_{\infty} \phi_q(\|\psi_k\|_Z + \|QN\|_Z) < \infty.$$  

(2.12)

Therefore it follows from (2.10), (2.11) and (2.12) that $R(u, \lambda)\overline{\Omega}$ is uniformly bounded.

Next we show that $R(u, \lambda)\overline{\Omega}$ is equicontinuous in a compact set. Let $u \in \overline{\Omega}$, $\lambda \in [0, 1]$. For any $T \in [0, \infty)$, with $t_1, t_2 \in [0, T]$, where $t_1 < t_2$, we have

$$|R(u, \lambda)(t_2) - R(u, \lambda)(t_1)|$$

$$= \left| \int_0^{t_2} \int_0^x \phi_q \left( \frac{1}{\sigma(s)} \right) \left( \int_s^\infty (\lambda v(r) w(r, u(r), u'(r), u''(r)) + Q N \lambda u(r)) dr \right) ds dx \right.$$  

$$- \int_0^{t_1} \int_0^x \phi_q \left( \frac{1}{\sigma(s)} \right) \left( \int_s^\infty (\lambda v(r) w(r, u(r), u'(r), u''(r)) + Q N \lambda u(r)) dr \right) ds dx \right|  \]

$$\leq \left\| \phi_q \left( \frac{1}{\sigma} \right) \right\|_{L^1} \phi_q(\|\psi_k\|_Z + \|QN\|_Z)(t_2 - t_1) \to 0, \text{ as } t_1 \to t_2,$$  

(2.13)

$$|R'(u, \lambda)(t_2) - R'(u, \lambda)(t_1)|$$

$$= \left| \int_0^{t_2} \phi_q \left( \frac{1}{\sigma(s)} \right) \left( \int_s^\infty (\lambda v(r) w(r, u(r), u'(r), u''(r)) + Q N \lambda u(r)) dr \right) ds \right.$$  

$$- \int_0^{t_1} \phi_q \left( \frac{1}{\sigma(s)} \right) \left( \int_s^\infty (\lambda v(r) w(r, u(r), u'(r), u''(r)) + Q N \lambda u(r)) dr \right) ds \right|  \]

$$\leq \phi_q(\|\psi_k\|_Z + \|QN\|_Z) \int_{t_1}^{t_2} \phi_q \left( \frac{1}{\sigma(s)} \right) ds \to 0, \text{ as } t_1 \to t_2$$  

(2.14)
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and

\[
|R''(u, \lambda)(t_2) - R''(u, \lambda)(t_1)|
\]

\[
= \left| \varphi_q \left( \frac{1}{\sigma(t_2)} \right) \varphi_q \left( \int_{t_2}^{\infty} (\lambda v(r) w(r, u(r), u'(r), u''(r)) + Q Nu(r))dr \right) ds
\]

\[
- \varphi_q \left( \frac{1}{\sigma(t_1)} \right) \varphi_q \left( \int_{t_1}^{\infty} (\lambda v(r) w(r, u(r), u'(r), u''(r)) + Q Nu(r))dr \right) ds
\]

\[
+ \left| \varphi_q \left( \frac{1}{\sigma(t_1)} \right) \varphi_q \left( \int_{t_1}^{t_2} v(r)|Nu(r) + Q Nu(r)|dr \right) \to 0, \text{ as } t_1 \to t_2. \quad (2.15)
\]

Thus, (2.13), (2.14) and (2.15) show that \( R(u, \lambda) \Omega \) is equicontinuous on \([0, T]\).

\[
|R(u, \lambda)(t_2) - R(u, \lambda)(t_1)|
\]

\[
= \left| \int_{0}^{t_2} \int_{0}^{x} \varphi_q \left( \frac{1}{\sigma(s)} \right) \varphi_q \left( \int_{s}^{\infty} (\lambda f(v, u(v), u'(v), u''(v)) + Q Nu(v))dv \right) ds dx
\]

\[
- \int_{0}^{t_1} \int_{0}^{x} \varphi_q \left( \frac{1}{\sigma(s)} \right) \varphi_q \left( \int_{s}^{\infty} (\lambda f(v, u(v), u'(v), u''(v)) + Q Nu(v))dv \right) ds dx
\]

\[
\leq \| \varphi_q \left( \frac{1}{\sigma} \right) \|_{L^1} \varphi_q(\|\psi_k\|_{L^1} + \|Q Nu\|_{L^1})(t_2 - t_1) \to 0, \text{ as } t_1 \to t_2. \quad (2.16)
\]

\[
|R'(u, \lambda)(t_2) - R'(u, \lambda)(t_1)|
\]

\[
= \left| \int_{0}^{t_2} \varphi_q \left( \frac{1}{\sigma(s)} \right) \varphi_q \left( \int_{s}^{\infty} (\lambda f(v, u(v), u'(v), u''(v)) + Q Nu(v))dv \right) ds
\]

\[
- \int_{0}^{t_1} \varphi_q \left( \frac{1}{\sigma(s)} \right) \varphi_q \left( \int_{s}^{\infty} (\lambda f(v, u(v), u'(v), u''(v)) + Q Nu(v))dv \right) ds
\]

\[
\leq \varphi_q(\|\psi_k\|_{L^1} + \|Q Nu\|_{L^1}) \int_{t_1}^{t_2} \left| \varphi_q \left( \frac{1}{\sigma(s)} \right) \right| ds \to 0, \text{ as } t_1 \to t_2 \quad (2.17)
\]
and
\[
|R''(u, \lambda)(t_2) - R''(u, \lambda)(t_1)| \\
= \left| \varphi_q \left( \frac{1}{\sigma(t_2)} \right) \varphi_q \left( \int_{t_2}^{\infty} (\lambda f(v, u(v), u'(v), u''(v)) + QN_\lambda u(v)) dv \right) ds \\
- \varphi_q \left( \frac{1}{\sigma(t_1)} \right) \varphi_q \left( \int_{t_1}^{\infty} (\lambda f(v, u(v), u'(v), u''(v)) + QN_\lambda u(v)) dv \right) ds \right| \\
\leq \varphi_q \left( \frac{1}{\sigma(t_2)} - \frac{1}{\sigma(t_1)} \right) \varphi_q(\|\psi_k\|_{L^1} + \|QN\|_{L^1}) \\
+ \varphi_q \left( \frac{1}{\sigma(t_1)} \right) \varphi_q \left( \int_{t_1}^{t_2} |Nu(v) + QNu(v)| dv \right) \rightarrow 0, \text{ as } t_1 \rightarrow t_2. \\
(2.18)
\]

Thus, (2.16), (2.17) and (2.18) show that \( R(u, \lambda) \) is equicontinuous on \([0, T] \).

We will now prove that \( R(u, \lambda) \) is equiconvergent at \( \infty \). We need
\[
R(u, \lambda)(\infty) = \lim_{t \rightarrow \infty} R(u, \lambda)(t) \\
= \int_0^\infty \int_0^x \varphi_q \left( \frac{1}{\sigma(s)} \right) \varphi_q \left( \int_s^\infty (\lambda v(r, u(r), u'(r), u''(r)) + QN_\lambda u(r)) dr \right) ds dx,
\]
\[
R'(u, \lambda)(\infty) = \lim_{t \rightarrow \infty} R'(u, \lambda)(t) \\
= \int_0^\infty \varphi_q \left( \frac{1}{\sigma(s)} \right) \varphi_q \left( \int_s^\infty (\lambda v(r, u(r), u'(r), u''(r)) + QN_\lambda u(r)) dr \right) ds
\]
and
\[
R''(u, \lambda)(\infty) = \lim_{t \rightarrow \infty} R''(u, \lambda)(t) \\
= \varphi_q \left( \frac{1}{\sigma(s)} \right) \varphi_q \left( \int_s^\infty (\lambda v(r, u(r), u'(r), u''(r)) + QN_\lambda u(r)) dr \right) ds dx = 0.
\]

Then,
\[
|R(u, \lambda)(t) - R(u, \lambda)(\infty)| \\
\leq \left| \varphi_q \left( \frac{1}{\sigma} \right) \varphi_q(\|\psi_k\|_Z + \|QN\|_Z) \int_t^\infty dx \rightarrow 0, \text{ uniformly as } t \rightarrow \infty, \\
(2.19)\right.
\]
Lemma 2.11. The operator $N_\lambda$ is $M$-compact.

Proof. To prove this lemma, we will need to show that the conditions of definition 2.5 hold. Since $w$ is $v$-Carathéodory, $N_\lambda$ is continuous. Let the homeomorphism $J : \text{Im } Q \to \ker M$ be defined by $J(a + bt) = a + bt$. Then $\dim U_1 = \dim Z_1 = 2$.

Since $Q$ is a semi-projector, $Q(I - Q)N_\lambda(\overline{\Omega}) = 0$. Hence, $(I - Q)N_\lambda(\overline{\Omega}) \subset \ker Q = \text{Im } M$. Conversely, let $z \in \text{Im } M$. Then $z = z - Qz = (I - Q)z \in (I - Q)Z$. Hence, condition (i) of definition (2.5) is satisfied. It is easy to show that condition (ii) of definition (2.5) holds as well.

Let $u \in \Sigma_\lambda$, $Mu = N_\lambda u$. Then $QN_\lambda u = (\Delta_1 N_\lambda u) + (\Delta_2 N_\lambda u) \cdot t = 0$ and $R(u, \lambda)(t)$ becomes

$$R(u, \lambda)(t) = \int_0^t \int_0^x \varphi_q \left( \frac{1}{\sigma(s)} \right) \varphi_q \left( \int_s^\infty \lambda v(r)w(r, u(r), u'(r), u''(r))dr \right) dsdx.$$  

Then

$$R(u, 0)(t) = \int_0^t \int_0^x \varphi_q \left( \frac{1}{\sigma(s)} \right) \varphi_q \left( \int_s^\infty 0 \cdot v(r)w(r, u(r), u'(r), u''(r))dr \right) dsdx = 0$$

Therefore $R(u, \lambda)\overline{\Omega}$ is equiconvergent at $\infty$. It then follows from definition 2.4 that $R(u, \lambda)$ is compact. \hfill \Box

Lemma 2.11. The operator $N_\lambda$ is $M$-compact.
and

\[ R(u, \lambda)(t) = \int_0^t \int_0^x \varphi_q \left( \frac{1}{\sigma(s)} \right) \varphi_q \left( \int_s^\infty \lambda v(r)w(r, u(r), u'(r), u''(r))dr \right) dsdx \]

\[ = \int_0^t \int_0^x u''(s)dsdx = u(t) - u(0) - u'(0)t = u(t) - Pu(t) = [(I - P)u](t). \]

(2.22)

Therefore, condition (iii) of definition (2.5) holds.

Let \( u \in \Omega \). Since \( Mu = \frac{1}{v(t)}(\sigma(t)\varphi_p(u''(t)))' \), we have

\[ M[Pu + R(u, \lambda)](t) = \frac{1}{v(t)}(\sigma(t)\varphi_p([Pu + R(u, \lambda)]''(t)))' \]

\[ = \frac{1}{v(t)} \left( (\sigma(t)\varphi_p \left[ u(0) + u'(0)t + \int_0^t \int_0^x \varphi_q \left( \frac{1}{\sigma(s)} \right) \varphi_q \left( \int_s^\infty (\lambda v(r)w(r, u(r), u'(r), u''(r)) \right) dr \right])'[QN_\lambda(r)](r)dr \right)' \]

\[ = -\frac{1}{v(t)} \left( \int_t^\infty v(r)[(-I + Q)N_\lambda](r)dr \right)' \]

i.e., (iv) of definition (2.5) holds. Hence, \( N_\lambda \) is \( M \)-compact in \( \bar{\Omega} \). \( \square \)

3 Existence Result

**Theorem 3.1.** Suppose the following conditions hold:

1. \((H_1)\) there exists functions \( x_1(t), x_2(t), x_3(t), x_4(t) \in L^1[0, \infty) \) such that for all \( (u, v, w) \in \mathbb{R}^3 \) and a.e. \( t \in [0, \infty) \),

\[ |f(t, u, u', u'')| \leq e^{-t}(x_1(t)|u|^{p-1} + x_2(t)|u'|^{p-1} + x_3(t)|u''|^{p-1} + x_4(t), \]

(3.23)

2. \((H_2)\) for \( u \in \text{dom } M \) there exist a constant \( A_0 > 0 \), \( l > 0 \) such that if \( |u(t)| > A_0 \) for \( t \in [0, l] \) or \( |u'(t)| > A_0 \) for \( t \in [0, \infty) \), then either

\[ Q_1 Nu(t) \neq 0 \quad \text{or} \quad Q_2 Nu(t) \neq 0, \quad t \in [0, \infty). \]

(3.24)
(H₃) there exists a constant \( B > 0 \) such that for \( |a| > B \) or \( |b| > B \) either
\[
Q_1N(a+bt) + Q_2N(a+bt) < 0, \quad t \in (0, \infty),
\]

or
\[
Q_1N(a+bt) + Q_2N(a+bt) > 0, \quad t \in (0, \infty),
\]

where \( a, b \in \mathbb{R}, |a| + |b| > B \) and \( t \in [0, \infty) \).

Then the boundary value problem \((1.1)-(1.2)\) has at least one solution provided
\[
\Lambda(\|x_1\|_Z^{q-1} + \|x_2\|_Z^{q-1} + \|x_3\|_Z^{q-1}) < 1 \quad \text{for } p \geq 2,
\]

\[
2^{q-4}\Lambda(\|x_1\|_Z^{q-1} + \|x_2\|_Z^{q-1} + \|x_3\|_Z^{q-1}) < 1 \quad \text{for } 1 < p < 2,
\]

where \( \Lambda = \max \{ (2+l) \|\varphi_q(\frac{1}{\sigma})\|_{L^1}, (1+l) \|\varphi_q(\frac{1}{\sigma})\|_{L^1} + \|\varphi_q(\frac{1}{\sigma})\|_{\infty} \} \).

The following lemmas are also needed to prove our main result.

**Lemma 3.2.** The set \( U_1 = \{ u \in \text{dom} M : Mu = N_\lambda u \text{ for some } \lambda \in (0, 1) \} \) is bounded.

**Proof.** Let \( u \in U_1 \). Then \( N_\lambda u \in \text{Im} M = \ker Q \). Hence, \( QN_\lambda u = 0 \) and \( QNu = 0 \). It follows from \( H_2 \) that there exist \( t_0 \in [0, l] \) and \( t_1 \in [0, \infty) \) such that \( |u(t_0)| \leq A_0, \quad |u'(t_1)| \leq A_0 \). From \( u(0) = u(t_0) + \int_0^{t_0} u'(v)dv, \) we have \( |u(0)| = |u(t_0) - \int_0^{t_0} u'(v)dv| \leq A_0 + d\|u'\|_{\infty}. \) Also, from \( u'(t) = u'(t_1) - \int_{t}^{t_1} u''(v)dv, \) we get \( |u'(t)| = |u(t_1) - \int_{t}^{t_1} u'(v)dv| \leq A_0 + \|u''\|_{L^1}. \) Then
\[
|u'(0)| \leq A_0 + \|u''\|_{L^1}
\]

and
\[
\|u'\|_{\infty} = \sup_{t \in [0, \infty)} e^{-t} |u'(t)| \leq A_0 + \|u''\|_{L^1}.
\]

Hence, from (3.29) and (3.30), we have
\[
|u(0)| \leq 2A_0 + \|u''\|_{L^1}.
\]

Since \( Mu = N_\lambda u \), from (2.5) we have
\[
\|u''\|_{L^1} = \int_0^{\infty} \left| -\varphi_q\left(\frac{1}{\sigma(t)}\right) \varphi_q\left(\int_t^{\infty} \lambda v(r)w(r,u(r),u'(r),u''(r))dr\right) \right| dt
\]

\[
\leq \left\| \varphi_q\left(\frac{1}{\sigma}\right) \right\|_{L^1} \varphi_q(\|Nu\|_Z).
\]
Considering \((H_1)\) and statement (ii) of lemma 2.6, if \(1 < p < 2\), we have

\[
\varphi_q(\|Nu\|_Z) \leq \varphi_q \left( \int_0^\infty v(t) |f(t, u(t), u'(t), u''(t))| \, dt \right) \\
\leq 2^{q-2} [\varphi_q(\|x_1\|_Z \|u\|^{p-1} + \|x_2\|_Z \|u\|^{p-1} + \varphi_q(\|x_3\|_Z \|u\|^{p-1} + \|x_4\|_Z)] \\
\leq 2^{q-4} \|u\| (\|x_1\|_Z^{q-1} + \|x_2\|_Z^{q-1} + \|x_3\|_Z^{q-1} + \|x_4\|_Z^{q-1}) + 2^{2q-4} \|x_4\|_Z^{q-1}.
\]

Similarly, for \(p \geq 2\) we have

\[
\varphi_q(\|Nu\|_Z) \leq \varphi_q \left( \int_0^\infty v(t) |f(t, u, u', u'')| \, dt \right) \\
\leq \|u\| (\|x_1\|_Z^{q-1} + \|x_2\|_Z^{q-1} + \|x_3\|_Z^{q-1} + \|x_4\|_Z^{q-1}).
\]

Since \(QNu = 0\) for \(u \in U_1\) and using (2.10), (2.11) and (2.12), we have

\[
\|R(u, \lambda)\| \leq \max \left\{ \left\| \varphi_q \left( \frac{1}{\sigma} \right) \right\|_{L^1}, \left\| \varphi_q \left( \frac{1}{\sigma} \right) \right\|_{L^\infty} \right\} \varphi_q(\|Nu\|_Z).
\]

Also,

\[
\|Pu\| \leq A_0(2 + l) + (1 + l) \varphi_q(\|Nu\|_Z) \left\| \varphi_q \left( \frac{1}{\sigma} \right) \right\|_{L^1}.
\]

In addition, for \(u \in \Omega_1\), and in view of (2.22), we have

\[
u(t) = Pu(t) + (I - P)u(t) = Pu(t) + R(u, \lambda)u(t).
\]

Therefore,

\[
\|u\| = \|Pu\| + \|R(u, \lambda)\| \leq A_0(2 + l) + (1 + l) \varphi_q(\|Nu\|_Z) \left\| \varphi_q \left( \frac{1}{\sigma} \right) \right\|_{L^1} \\
+ \max \left\{ \left\| \varphi_q \left( \frac{1}{\sigma} \right) \right\|_{L^1}, \left\| \varphi_q \left( \frac{1}{\sigma} \right) \right\|_{L^\infty} \right\} \varphi_q(\|Nu\|_Z) = A_0(2 + l) \\
+ \max \left\{ (2 + l) \left\| \varphi_q \left( \frac{1}{\sigma} \right) \right\|_{L^1}, (1 + l) \left\| \varphi_q \left( \frac{1}{\sigma} \right) \right\|_{L^1} + \left\| \varphi_q \left( \frac{1}{\sigma} \right) \right\|_{L^\infty} \right\} \varphi_q(\|Nu\|_Z).
\]

Set \(\Lambda = \max \left\{ (2 + l) \left\| \varphi_q \left( \frac{1}{\sigma} \right) \right\|_{L^1}, (1 + l) \left\| \varphi_q \left( \frac{1}{\sigma} \right) \right\|_{L^1} + \left\| \varphi_q \left( \frac{1}{\sigma} \right) \right\|_{L^\infty} \right\} \). We have

\[
\|u\| \leq A_0(2 + l) + \Lambda \varphi_q(\|Nu\|_Z).
\]
Lemma 2. We obtain Lemma 3.4.

If $p \geq 2$, then in view of (3.33)

$$
\|u\| \leq \frac{A_0(2 + d) + \Lambda \|x_4\|^{q-1}}{1 - \Lambda(\|x_1\|^{q-1} + \|x_2\|^{q-1} + \|x_3\|^{q-1})},
$$

Therefore $\Omega_1$ is bounded.

Lemma 3.3. Assuming that $(H_3)$ holds, the set $\Omega_2 = \{u \in \ker M : \lambda Nu = 0\}$ is bounded.

Proof. Let $u \in \Omega_2$. Then $u = a + bt$, $a, b \in \mathbb{R}$, $\lambda Nu = 0$ and $Nu \in \text{Im } M$. Hence, from lemma 2.9, $Q_1 N(a + bt) = Q_2 N(a + bt) = 0$. From $(H_3)$ it follows that $|a| < B$ and $|b| < B$. Hence $\|u\| = \max\{|u|, \|u'|, \|u''|\} = \max\{2B, B, 0\} \leq 2B$. So $\Omega_2$ is bounded.

Lemma 3.4. If $\Omega_3 = \{u \in \ker M : -\lambda u + (1 - \lambda)JQNu = 0, \lambda \in [0, 1]\}$, $J : \text{Im } Q \to \ker M$ is a homeomorphism, then $\Omega_3$ is bounded.

Proof. For $a, b \in R$, let $J : \text{Im } Q \to \ker M$ be defined by

$$
J(a + bt) = \frac{1}{G}[\delta_{11}|a| + \delta_{12}|b| + (\delta_{21}|a| + \delta_{22}|b|)t]e^{-t}. \quad (3.36)
$$

If (3.25) holds, for any $u(t) = a + bt \in \Omega_3$, from $-\lambda u + (1 - \lambda)JQNu = 0$, we obtain

$$
\lambda|a| = (1 - \lambda)Q_1 N(a + bt),
\lambda|b| = (1 - \lambda)Q_2 N_2(a + bt). \quad (3.37)
$$

From (3.37), when $\lambda = 1, a = b = 0$. When $\lambda = 0, Q_1 N(a + bt) + Q_2 N(a + bt) = 0$ which contradicts (3.25) and (3.26.) Hence from $(H_3), |a| \leq B$ and $|b| \leq B$. For $\lambda \in (0, 1), a < B, b < B$, from (3.37), we have $\lambda(|a| + |b|) = (1 - \lambda)[Q_1 N(a + bt) + Q_2 N(a + bt)] < 0$, which contradicts $\lambda(|a| + |b|) \geq 0$. Hence, $(H_3), |a| \leq B$ and $|b| \leq B$. Thus $\|u\| \leq 2B$. Therefore $\Omega_3$ is bounded.

Lemma 2.9 shows that condition $(B_1)$ of theorem 2.7 holds. Moreover, Lemma 2.11 implies $(B_2)$. Furthermore, lemmas 3.2 and 3.3 imply $(B_3)$.

We will next prove $(B_4)$. 

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**Theorem 3.5.** Assuming that (H1)-(H3) hold, the boundary value problem (1.1) - (1.2) has at least one solution in $\Omega \cap \partial \Omega$.

**Proof.** Let $\Omega \supset \Omega_1 U \Omega_2 U \Omega_3$ be a nonempty, open and bounded set, $u \in \text{dom} \ M \cap \partial \Omega$ and $H(u, \lambda) = -\lambda u + (1-\lambda)JQNu$, where $J$ is as defined above. Then $H(u, \lambda) \neq 0$ Therefore, by the homotopy property of the Brouwer degree

\[
\deg\{JQN|_{\Omega \cap \ker M}, \Omega \cap \ker M, 0\} = \deg\{H(\cdot, 0), \Omega \cap \ker M, 0\} = \deg\{H(\cdot, 1), \Omega \cap \ker M, 0\} = \deg\{-I, \Omega \cap \ker M, 0\} \neq 0.
\]

Hence, $(B_4)$ of theorem 2.7 holds. \qed

Since all the conditions of theorem 2.7 are satisfied, the abstract equation $Mu = Nu$ has at least one solution in $\Omega \cap \text{dom} \ M$. Hence, (1.1) - (1.2) has at least one solution in $U$.

**Example:** Consider the following boundary value problem

\[
(e^{-5t^2} \varphi_4(u''(t)))' + 2e^{-2t} w(t, u(t), u'(t), u''(t)), \quad t \in (0, \infty), \quad (3.38)
\]

\[
u(0) = 54.9397 \int_0^{1/25} e^{-t} u(t) dt - 32.2679 \int_0^{1/19} e^{-t} u(t) dt,
\]

\[
u'(0) = \int_0^{\infty} 2e^{-2t} u'(t) dt, \quad \lim_{t \to \infty} e^{-5t^2} \varphi_4(u''(t)) = 0,
\]

where

\[
f(t, u, v, w) = \begin{cases} 0, & 0 \leq t \leq 1, \\ e^{-2t^2} \sqrt{u(0)} + e^{-2t^2} \sin \sqrt{u'} + e^{-3t^2} \sin \sqrt{u''} + \frac{1}{2} e^{-2t}, & t > 1. \end{cases}
\]

Here $\sigma(t) = e^{-5t^2 + 2}, p = \frac{4}{3}, q = 4, \alpha_1 = 54.9397, \alpha_2 = -32.2679, \xi_1 = \frac{1}{25}, \xi_2 = \frac{1}{19}, \nu(t) = 2e^{-2t}$. Clearly, $G \neq 0, \sum_{i=1}^{2} \alpha_i \int_0^{\xi_1} v(t) dt = 1, \sum_{i=1}^{2} \alpha_i \int_0^{\xi_2} tv(t) dt = 0$ and $\int_0^{\infty} v(t) dt = 1$. Hence, $(\phi_1)$ and $(\phi_2)$ hold.

\[
\|x_1\|_Z^{-1} = \frac{1}{\xi_1}, \|x_2\|_Z^{-1} = \frac{1}{\sqrt{\xi_1}}, \|x_3\|_Z^{-1} = \frac{8}{125 \xi_1}, \|\varphi_q\left(\frac{1}{\xi_1}\right)\|_{L^1} = \frac{1}{\sqrt{\xi_1}}, \|\varphi_q\left(\frac{1}{\xi_1}\right)\|_{\infty} = 1.
\]

Therefore,

\[
\Lambda = \max \left\{ (2 + 1) \left(\frac{1}{5e^2}\right), (1 + 1) \left(\frac{1}{5e^2}\right) + 1 \right\} = \max\{0.0812, 1.0541\} = 1.0541.
\]
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\[
A(\|x_1\|^3_Z + \|x_2\|^3_Z + \|x_3\|^3_Z) = 1.0541 \left[ \frac{1}{e^3} + \frac{1}{8e^6} + \frac{8}{25e^9} \right] = 0.0529 < 1
\]

Therefore, (3.28) holds. Also (H_2) and (H_3) hold. Since all the conditions of Theorem 2.7 hold, (3.38)-(3.39) has at least one solution.

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**References**


