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Solutions of the Fractional Logistics Equations via the Residual Power Series Method with Adomian Polynomials

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Abstract

In this article, a combined form of the residual power series method with the Adomian polynomial is developed for analytic treatment of the fractional logistic equations. The Caputo operator is used to define the derivative of fractional order. The convergent analysis of solution is proposed. Illustrative examples are examined to support the proposed analysis. The fractional order solutions are compared to the integer order solutions.

1 Introduction

Fractional differential equations are generalizations of differential equations from integer order to non-integer order. The idea about the half order deriva-

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tive is originated Leibniz and LHôpital in 1965 [20]. The topic of fractional calculus has recently considered to be valuable tools in the modeling of many phenomena in real world. Nowadays, various definitions of fractional derivative have arisen. However, the definitions that frequently appeared in most literatures are of Riemann-Liouville type and Caputo type. Many researchers are in an attempt to develop a reliable method to solve fractional differential equations (FDEs), see for example [17, 18]. Nonetheless, all existing computational tools make the computation so cumbersome. Among those tools, residual power series method and Adomian decomposition method will be discussed here.

Residual power series method (RPSM) was proposed in [1]. The solution is written in the form of fractional power series. By means of truncating the series into first terms, the RPSM uses the n-th residual function in determining the n-th coefficients of the fractional power series for iteratively. Many consequence research papers indicate that the RPSM quite suits to nonlinear fractional differential equations with variable coefficients [3, 15, 21, 19, 5, 4, 16].

After introduced by George Adomian in 1980s, the Adomian decomposition method (ADM) becomes an effective tool which researchers commonly used to solve nonlinear differential equations. The key idea of the method is to express the solution in the form of infinite series. The nonlinear term is handled by an infinite series of special polynomials called Adomian polynomials [2]. The Adomain polynomials can be well combined with other series solution method such as differential transformation (DTM), homotopy analysis method (HAM) [7], and variational iteration method (VIM) [9]. The applications of the ADM to fractional differential equations can be found in [6] and the references there in.

Recently theory of fractional differential equations were tackled by many researchers. For the existence of solutions for FDEs, one can see [10, 11, 12, 13, 14] and the references therein. The existence of solutions are the most important qualitative properties of FDEs. The study of the existence of solutions ensures the essential conditions required for a solution of fractional differential equations.

The basic motivation of present study is the extension of RPSM equipped with the Adomian polynomials to tackle the fractional logistic equations. We first introduce some definitions and notations used throughout in this article in Section 2. In Section 3, we show the advantage of integrating the Adomian polynomials to the RPSM in reducing some computational work on algebraic manipulation. The convergence analysis of the proposed scheme is discussed. In Section 4, the nonlinear fractional logistics equation are solved to illustrate the efficiency, applicability and simplicity of RPSM via the Adomian polynomial. The numerical errors versus the exact solutions are also reported for $\alpha = 1$.

2 Preliminaries and Notations

Definition 2.1. *Let n be the smallest integer greater than or equal to α. The Caputo fractional derivative of order $\alpha > 0$ is defined as*

$$D_t^{\alpha}u(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t (t-\tau)^{n-\alpha-1}u^{(n)}(\tau)d\tau, & n-1<\alpha<n \\ u^{(n)}(t), & \alpha = n \in \mathbb{N}, \end{cases}$$

where $\Gamma(\cdot)$ denotes the well-known Gamma function.

Theorem 2.2. *The Caputo fractional derivative of the power function is given by*

$$D_t^{\alpha}t^p = \begin{cases} \dfrac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}t^{p-\alpha}, & n-1<\alpha<n,\ p>n-1,\ p\in\mathbb{R} \\ 0, & n-1<\alpha<n,\ p\leq n-1,\ p\in\mathbb{N}. \end{cases}$$

Definition 2.3. *A power series expansion of the form*

$$\sum_{m=0}^{\infty} c_m(t-t_0)^{m\alpha} = c_0 + c_1(t-t_0)^{\alpha} + c_2(t-t_0)^{2\alpha} + \cdots$$

where $0 \leq n-1 < \alpha$, $t \geq t_0$, is called fractional power series (FPS) about $t = t_0$.

Theorem 2.4. *Suppose that f has a fractional power series represent at $t = t_0$ of the form*

$$f(t) = \sum_{m=0}^{\infty} c_m(t-t_0)^{m\alpha},\ 0 \leq n-1 < \alpha,\ t_0 \leq t < t_0 + R,$$

where R is the radius of convergence. If $D^{m\alpha}f(t)$, $m = 0, 1, 2, \ldots$ are continuous on (t_0, t_0+R), then $c_m = \frac{D^{m\alpha}f(t_0)}{\Gamma(1+m\alpha)}$.

3 Algorithm of RPSM with Adomian Polynomials

Consider the fractional differential equation

$$D_t^{\alpha} u(t) = L(u(t)) + F(u(t)), \quad 0 < t < T, \tag{3.1}$$

subject to the initial condition

$$u(0) = u_0, \tag{3.2}$$

where $L(u(t))$ and $F(u(t))$ are linear term and nonlinear term respectively. D_t^{α} means the Caputo fractional derivative with respect to t of order $0 < \alpha < 1$.

The RPSM assumes the solution $u(t)$, of Eq. (3.1), in a form of fractional power series centered at $t = 0$ as

$$u(t) = \sum_{n=0}^{\infty} \frac{u_n t^{n\alpha}}{\Gamma(1+n\alpha)}. \tag{3.3}$$

Using the initial condition in Eq. (3.2), we approximate $u(t)$ in Eq. (3.3) by

$$u_k(t) = u_0 + \sum_{n=1}^{k} \frac{u_n t^{n\alpha}}{\Gamma(1+n\alpha)}, \quad k = 1, 2, 3, \ldots . \tag{3.4}$$

From Eq. (3.4), the k^{th} residual power series approximation $u_k(t)$ will be obtained by computing the component $u_1, u_2, \ldots, u_k$. Before computing these components, we define the kth residual function

$$Res_k(t) = D_t^{\alpha} u_k(t) - L(u_k(t)) - F(u_k(t)). \tag{3.5}$$

To find the coefficients $u_1, u_2, \ldots, u_k$ of the RPS solution (3.4), we solve equation

$$D_t^{(k-1)\alpha} Res_k(0) = 0, \quad k = 1, 2, 3, ... \tag{3.6}$$

As mentioned above, the RPSM has been used for nonlinear differential equation including the nonlinear fractional differential equation. In this work, the Adomian polynomials are used for the nonlinear term in an easy way. Adomian polynomials decompose a function $u_k(t)$ into a sum of components

$$u_k(t) = \sum_{n=0}^{\infty} v_n(t).$$

A nonlinear operator F of $u(t)$ can be written in the form of

$$F(u(t)) = \sum_{n=0}^{\infty} A_n,$$

where A_n are known as the Adomian polynomials determined formally from the relation

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}[F(\sum_{i=0}^{\infty}\lambda^i v_i)]\right]_{\lambda=0}.$$

Then, the first few polynomials are given by

$$\begin{aligned}
A_0 &= F(v_0),\\
A_1 &= v_1 F'(v_0),\\
A_2 &= v_2 F'(v_0) + \frac{1}{2!}v_1^2 F''(v_0),\\
A_3 &= v_3 F'(v_0) + v_1 v_2 F''(v_0) + \frac{1}{3!}v_1^3 F'''(v_0),\\
A_4 &= v_4 F'(v_0) + \left(\frac{1}{2!}v_2^2 + v_1 v_3\right) F''(v_0) + \frac{1}{2!}v_1^2 v_2 F'''(v_0) + \frac{1}{4!}v_1^4 F'''(v_0).
\end{aligned}$$

Other polynomials can be calculated in similar manner.

4 Applications of RPSM to Fractional Logistics Equation

Consider the fractional logistic equation

$$D_t^{\alpha} u(t) = \rho^{\alpha} u\,(1-u)\,,\ \alpha \in (0,1], \tag{4.7}$$

subject to the initial condition

$$u(0) = u_0, \quad u_0 > 0, \tag{4.8}$$

and $\rho > 0$. The derivative in fractional logistic Eq. (4.7) is in the Caputo sense. According to the RPSM, let $u(t)$ be the solution of fractional logistic equation as a fractional power series about $t = 0$ of the form

$$u(t) = \sum_{n=0}^{\infty} \frac{u_n t^{n\alpha}}{\Gamma(1+n\alpha)}. \tag{4.9}$$

Using the initial condition (4.8), we approximate $u(t)$ in Eq. (4.9) by

$$u_k(t) = u_0 + \sum_{n=1}^{k} \frac{u_n t^{n\alpha}}{\Gamma(1+n\alpha)}, \;\; k = 1, 2, 3, \dots \; . \tag{4.10}$$

To find the values of the RPS coefficient u_n, $n = 1, 2, 3, ...,$ we solve the equation

$$D_t^{(n-1)\alpha} Res_n(0) = 0, \;\; n = 1, 2, 3, \dots \; , \tag{4.11}$$

where $Res_k(t)$ is the kth residual function and it defined by

$$Res_k(t) = D_t^{\alpha} u_k(t) - \rho^{\alpha} \left(u_k(t) - u_k^2(t) \right). \tag{4.12}$$

Since the fractional logistic equation (4.7) is a nonlinear fractional differential equation in term of $u^2(t)$, Adomian polynomials are implemented to calculate nonlinear term of $u^2(t)$. So, the Adomian polynomials and the residual power series method are combined to solve the fractional logistic equation.
We first set

$$u_k(t) = \sum_{i=0}^{k} v_i, \tag{4.13}$$

where $v_0 = u_0$ and

$$v_i = \frac{u_i t^{i\alpha}}{\Gamma(1+i\alpha)}, \;\; i = 1, 2, 3, ..., k. \tag{4.14}$$

Since $F(u_k(t))$ is the nonlinear operator, let

$$F(u_k(t)) = \sum_{n=0}^{\infty} A_n, \tag{4.15}$$

where A_n are called Adomian polynomials determined formally from the relation

$$A_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} [F(\sum_{i=0}^{k} \lambda^i v_i)] \right]_{\lambda=0}. \tag{4.16}$$

From Eq. (4.13), we can rewritten the nonlinear polynomials $u_k^2(t)$ as

$$F(u_k(t)) \;\; = \;\; (v_0 + v_1 + v_2 + v_3 + \cdots + v_k)^2 = \sum_{n=0}^{\infty} A_n.$$

Consequently, the Adomian polynomials for $F(u_k(t)) = u_k^2(t)$ are given by

$$\begin{aligned}
A_0 &= v_0^2 \\
A_1 &= 2v_0v_1 \\
A_2 &= 2v_0v_2 + v_1^2 \\
A_3 &= 2v_0v_3 + 2v_1v_2 \\
A_4 &= v_2^2 + 2v_1v_3 + 2v_0v_4 \\
A_5 &= 2v_2v_3 + 2v_0v_5 + 2v_1v_4 \\
A_6 &= 2v_0v_6 + 2v_1v_5 + 2v_2v_4 + v_3^2 \\
A_7 &= 2v_0v_7 + 2v_2v_5 + 2v_3v_4 + 2v_1v_6 \\
A_8 &= 2v_2v_6 + 2v_3v_5 + v_4^2 + 2v_0v_8 + 2v_1v_7.
\end{aligned}$$

Other polynomials can be calculated by Eq. (4.16).
To find u_1, we substitute the first RPS approximate solution

$$u_1(t) = u_0 + u_1\frac{t^\alpha}{\Gamma(1+\alpha)}$$

into Eq. (4.12) as follows

$$\begin{aligned}
Res_1(t) &= D_t^\alpha u_1(t) - \rho^\alpha\left(u_1(t) - u_1^2(t)\right) \\
&= D_t^\alpha\left(u_0 + u_1\tfrac{t^\alpha}{\Gamma(1+\alpha)}\right) - \rho^\alpha\left(u_0 + u_1\tfrac{t^\alpha}{\Gamma(1+\alpha)}\right) \\
&\quad +\rho^\alpha\left(u_0 + u_1\tfrac{t^\alpha}{\Gamma(1+\alpha)}\right)^2 \\
&= u_1 - \rho^\alpha\left(u_0 + u_1\tfrac{t^\alpha}{\Gamma(1+\alpha)}\right) + \rho^\alpha\left(u_0 + u_1\tfrac{t^\alpha}{\Gamma(1+\alpha)}\right)^2.
\end{aligned}$$

Then, we solve $Res_1(0) = 0$ to get

$$u_1 = \rho^\alpha\left(u_0 - u_0^2\right). \tag{4.17}$$

To find u_2 in Eq. (4.10), the second RPS approximate solution is in form

$$u_2(t) = u_0 + u_1\frac{t^\alpha}{\Gamma(1+\alpha)} + u_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}. \tag{4.18}$$

Using Adomian polynomials and $F(u_2(t)) = u_2^2(t)$, we have

$$\begin{aligned}
F(u_2(t)) &= (v_0 + v_1 + v_2)^2 \\
&= \sum_{n=0}^{\infty} A_n \\
&= v_0^2 + 2v_0v_1 + 2v_0v_2 + v_1^2 + 2v_1v_2 + v_2^2.
\end{aligned}$$

From $v_0 = u_0$ and Eq. (4.14), we have

$$\begin{aligned} u_2^2(t) = F(u_2(t)) &= u_0^2 + 2u_0u_1\tfrac{t^{\alpha}}{\Gamma(1+\alpha)} + 2u_0u_2\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(u_1\tfrac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^2 \\ &\quad +2u_1u_2\tfrac{t^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} + \left(u_2\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^2. \end{aligned} \tag{4.19}$$

Substituting Eq.4.18) and Eq. (4.19) into Eq. (4.12) yields

$$\begin{aligned} Res_2(t) &= D_t^{\alpha}u_2(t) - \rho^{\alpha}\left(u_2(t) - u_2^2(t)\right) \\ &= \left(u_1 + u_2\tfrac{t^{\alpha}}{\Gamma(1+\alpha)}\right) - \rho^{\alpha}\left(u_0 + u_1\tfrac{t^{\alpha}}{\Gamma(1+\alpha)} + u_2\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right) \\ &\quad +\rho^{\alpha}\left[u_0^2 + 2u_0u_1\tfrac{t^{\alpha}}{\Gamma(1+\alpha)} + 2u_0u_2\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(u_1\tfrac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^2\right. \\ &\quad \left.+2u_1u_2\tfrac{t^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} + \left(u_2\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^2\right]. \end{aligned}$$

Applying D_t^{α} on both sides of Eq. (4.20), we obtain

$$\begin{aligned} D_t^{\alpha}Res_2(t) &= u_2 - \rho^{\alpha}\left(u_1 + u_2\tfrac{t^{\alpha}}{\Gamma(1+\alpha)}\right) \\ &\quad +\rho^{\alpha}\left[2u_0u_1 + 2u_0u_2\tfrac{t^{\alpha}}{\Gamma(1+\alpha)} + u_1^2\tfrac{\Gamma(1+2\alpha)t^{\alpha}}{\Gamma^3(1+\alpha)}\right. \\ &\quad \left.+2u_1u_2\tfrac{\Gamma(1+3\alpha)t^{2\alpha}}{\Gamma(1+\alpha)\Gamma^2(1+2\alpha)} + u_2^2\tfrac{\Gamma(1+4\alpha)t^{3\alpha}}{\Gamma^2(1+2\alpha)\Gamma(1+3\alpha)}\right]. \end{aligned}$$

Then, we solve $D_t^{\alpha}Res_2(0) = 0$ to get

$$D^{\alpha}Res_2(0) = u_2 - \rho^{\alpha}\left(u_1 - 2u_0u_1\right) = 0.$$

We have the coefficient u_2 as

$$u_2 = \rho^{\alpha}\left(u_1 - 2u_0u_1\right). \tag{4.20}$$

To find u_3, the third RPS approximate solution is in form

$$u_3(t) = u_0 + u_1\frac{t^{\alpha}}{\Gamma(1+\alpha)} + u_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + u_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}. \tag{4.21}$$

By using Adomian polynomials and $u_3^2(t) = F(u_3(t))$, we have

$$\begin{aligned} F(u_3(t)) &= (v_0 + v_1 + v_2 + v_3)^2 \\ &= \sum_{n=0}^{\infty} A_n \\ &= v_0^2 + 2v_0v_1 + 2v_0v_2 + v_1^2 + +2v_0v_3 + 2v_1v_2 + v_2^2 + 2v_1v_3 \\ &\quad +2v_2v_3 + v_3^2. \end{aligned}$$

From $v_0 = u_0$ and Eq. (4.14), we have

$$\begin{aligned} u_3^2(t) &= F(u_3(t)) \\ &= u_0^2 + 2u_0u_1\tfrac{t^\alpha}{\Gamma(1+\alpha)} + 2u_0u_2\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(u_1\tfrac{t^\alpha}{\Gamma(1+\alpha)}\right)^2 \\ &\quad +2u_0u_3\tfrac{t^{3\alpha}}{\Gamma(1+3\alpha)} + 2u_1u_2\tfrac{t^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} + \left(u_2\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^2 \\ &\quad +2u_1u_3\tfrac{t^{4\alpha}}{\Gamma(1+\alpha)\Gamma(1+3\alpha)} + 2u_2u_3\tfrac{t^{5\alpha}}{\Gamma(1+2\alpha)\Gamma(1+3\alpha)} + \left(u_3\tfrac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right)^2. \end{aligned} \tag{4.22}$$

Substituting Eq. (4.21) and Eq. (4.22) into Eq. (4.12) gives

$$\begin{aligned} Res_3(t) &= D_t^\alpha u_3(t) - \rho^\alpha\left(u_3(t) - u_3^2(t)\right) \\ &= \left(u_1 + u_2\tfrac{t^\alpha}{\Gamma(1+\alpha)} + u_3\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right) \\ &\quad -\rho^\alpha\left(u_0 + u_1\tfrac{t^\alpha}{\Gamma(1+\alpha)} + u_2\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)} + u_3\tfrac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right) \\ &\quad +\rho^\alpha\Bigg[u_0^2 + 2u_0u_1\tfrac{t^\alpha}{\Gamma(1+\alpha)} + 2u_0u_2\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(u_1\tfrac{t^\alpha}{\Gamma(1+\alpha)}\right)^2 \\ &\quad +2u_0u_3\tfrac{t^{3\alpha}}{\Gamma(1+3\alpha)} + 2u_1u_2\tfrac{t^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} + \left(u_2\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^2 \\ &\quad +2u_1u_3\tfrac{t^{4\alpha}}{\Gamma(1+\alpha)\Gamma(1+3\alpha)} + 2u_2u_3\tfrac{t^{5\alpha}}{\Gamma(1+2\alpha)\Gamma(1+3\alpha)} + \left(u_3\tfrac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right)^2\Bigg]. \end{aligned} \tag{4.23}$$

Then, we solve $D_t^{2\alpha}Res_3(0) = 0$ to get

$$u_3 = \rho^\alpha\left(u_2 - 2u_0u_2 - u_1^2\frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\right). \tag{4.24}$$

To find u_4, the fourth RPS approximate solution is in form

$$u_4(t) = u_0 + u_1\frac{t^\alpha}{\Gamma(1+\alpha)} + u_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + u_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + u_4\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}. \tag{4.25}$$

Using the Adomian polynomials and $F(u_4(t)) = u_4^2(t)$, we have

$$\begin{aligned} F(u_4(t)) &= (v_0 + v_1 + v_2 + v_3 + v_4)^2 \\ &= \sum_{n=0}^{\infty} A_n \\ &= v_0^2 + 2v_0v_1 + 2v_0v_2 + v_1^2 + +2v_0v_3 + 2v_1v_2 + v_2^2 + 2v_1v_3 + 2v_0v_4 \\ &\quad +2v_2v_3 + 2v_1v_4 + +2v_2v_4 + v_3^2 + 2v_3v_4 + v_4^2. \end{aligned}$$

From $v_0 = u_0$ and Eq. (4.14), we have

$$\begin{aligned} u_4^2(t) &= u_0^2 + 2u_0u_1\tfrac{t^{\alpha}}{\Gamma(1+\alpha)} + 2u_0u_2\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(u_1\tfrac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^2 \\ &+2u_0u_3\tfrac{t^{3\alpha}}{\Gamma(1+3\alpha)} + 2u_1u_2\tfrac{t^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} + \left(u_2\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^2 \\ &+2u_1u_3\tfrac{t^{4\alpha}}{\Gamma(1+\alpha)\Gamma(1+3\alpha)} + 2u_0u_4\tfrac{t^{4\alpha}}{\Gamma(1+4\alpha)} + 2u_2u_3\tfrac{t^{5\alpha}}{\Gamma(1+2\alpha)\Gamma(1+3\alpha)} \\ &+2u_1u_4\tfrac{t^{5\alpha}}{\Gamma(1+\alpha)\Gamma(1+4\alpha)} + 2u_2u_4\tfrac{t^{6\alpha}}{\Gamma(1+2\alpha)\Gamma(1+4\alpha)} + \left(u_3\tfrac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right)^2 \\ &+2u_3u_4\tfrac{t^{7\alpha}}{\Gamma(1+3\alpha)\Gamma(1+4\alpha)} + \left(u_4\tfrac{t^{4\alpha}}{\Gamma(1+4\alpha)}\right)^2. \end{aligned} \tag{4.26}$$

Proceeding as before, substituting Eq. (4.25) and Eq. (4.26) into Eq. (4.12) gives

$$\begin{aligned} Res_4(t) &= u_1 + u_2\tfrac{t^{\alpha}}{\Gamma(1+\alpha)} + u_3\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)} + u_4\tfrac{t^{3\alpha}}{\Gamma(1+3\alpha)} \\ &-\rho^{\alpha}\left(u_0 + u_1\tfrac{t^{\alpha}}{\Gamma(1+\alpha)} + u_2\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)} + u_3\tfrac{t^{3\alpha}}{\Gamma(1+3\alpha)} + u_4\tfrac{t^{4\alpha}}{\Gamma(1+4\alpha)}\right) \\ &+\rho^{\alpha}\Bigg[u_0^2 + 2u_0u_1\tfrac{t^{\alpha}}{\Gamma(1+\alpha)} + 2u_0u_2\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(u_1\tfrac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^2 \\ &+2u_0u_3\tfrac{t^{3\alpha}}{\Gamma(1+3\alpha)} + 2u_1u_2\tfrac{t^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} + \left(u_2\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^2 \\ &+2u_1u_3\tfrac{t^{4\alpha}}{\Gamma(1+\alpha)\Gamma(1+3\alpha)} + 2u_0u_4\tfrac{t^{4\alpha}}{\Gamma(1+4\alpha)} + 2u_2u_3\tfrac{t^{5\alpha}}{\Gamma(1+2\alpha)\Gamma(1+3\alpha)} \\ &+2u_1u_4\tfrac{t^{5\alpha}}{\Gamma(1+\alpha)\Gamma(1+4\alpha)} + \left(u_3\tfrac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right)^2 + 2u_3u_4\tfrac{t^{7\alpha}}{\Gamma(1+3\alpha)\Gamma(1+4\alpha)} \\ &+\left(u_4\tfrac{t^{4\alpha}}{\Gamma(1+4\alpha)}\right)^2\Bigg]. \end{aligned} \tag{4.27}$$

Thus, we solve $D_t^{3\alpha}Res_4(0) = 0$ to get

$$u_4 = \rho^{\alpha}\left(u_3 - 2u_0u_3 - u_1u_2\frac{\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\right). \tag{4.28}$$

In general to find u_k in Eq. (4.10), we substitute k^{th} RPS approximate solution u_k into

$$Res_k(t) = D_t^{\alpha}u_k(t) - \rho^{\alpha}\left(u_k(t) - u_k^2(t)\right).$$

Then, we solve $D^{(k-1)\alpha}Res_k(0) = 0$ to obtain

$$u_k = \rho^{\alpha}\left(u_{k-1} - \sum_{i=0}^{k-1}\frac{u_iu_{k-1-i}\Gamma(1+(k-1)\alpha)}{\Gamma(1+i\alpha)\Gamma(1+(k-1-i)\alpha)}\right). \tag{4.29}$$

Convergence Analysis

Now, we prove the convergence of the RPSM. We start by the Lemma 4.1.

Lemma 4.1. *The classical power series (CPS) $\sum_{n=0}^{\infty} u_n t^n$, $-\infty < t < \infty$, has a radius of convergence R if and only if the fractional power series (FPS) $\sum_{n=0}^{\infty} u_n t^{n\alpha}$, $t \geq 0$, has a radius of convergence $R^{\frac{1}{\alpha}}$.*

Proof. See [8]. □

Theorem 4.2. *The fractional power series solution of fractional logistic equation (4.7) is expresses as*

$$u(t) = \sum_{n=0}^{\infty} \frac{u_n t^{n\alpha}}{\Gamma(1+n\alpha)}$$

where the coefficients are defined in equation (4.29) has a positive radius of convergence.

Proof. From Eq. (4.29), we can see that

$$\begin{aligned}
&\frac{|u_k|}{\Gamma(1+k\alpha)} \\
\leq\ & |\rho^\alpha| \left(\frac{|u_{k-1}| + \left| \sum_{i=0}^{k-1} \frac{u_i u_{k-1-i} \Gamma(1+(k-1)\alpha)}{\Gamma(1+i\alpha)\Gamma(1+(k-1-i)\alpha)} \right|}{\Gamma(1+k\alpha)} \right) \\
\leq\ & |\rho^\alpha| \left(\frac{|u_{k-1}|}{\Gamma(1+k\alpha)} + \sum_{i=0}^{k-1} \frac{|u_i|\,|u_{k-1-i}|\,\Gamma(1+(k-1)\alpha)}{\Gamma(1+i\alpha)\Gamma(1+(k-1-i)\alpha)\Gamma(1+k\alpha)} \right) \\
\leq\ & |\rho^\alpha| \left(\frac{|u_{k-1}|}{\Gamma(1+k\alpha)} + \max_{\substack{0\leq i\leq k-1 \\ k}} \left\{ \frac{\Gamma(1+(k-1)\alpha)}{\Gamma(1+i\alpha)\Gamma(1+(k-1-i)\alpha)\Gamma(1+k\alpha)} \right\} \sum_{i=0}^{k-1} |u_i|\,|u_{k-1-i}| \right) \\
=\ & A\,|u_{k-1}| + B \sum_{i=0}^{k-1} |u_i|\,|u_{k-1-i}|
\end{aligned}$$

where

$$A = \frac{|\rho^\alpha|}{\Gamma(1+k\alpha)},\ B = \max_{\substack{0\leq i\leq k-1 \\ k}} \left\{ \frac{\Gamma(1+(k-1)\alpha)}{\Gamma(1+i\alpha)\Gamma(1+(k-1-i)\alpha)\Gamma(1+k\alpha)} \right\} |\rho^\alpha|.$$

Let

$$f(t) = \sum_{k=0}^{\infty} a_k t^k \tag{4.30}$$

where $a_0 = |u_0|$ and

$$a_k = A\ a_{k-1} + B\sum_{i=0}^{k-1} a_i a_{k-1-i},\ k = 1, 2, ... \tag{4.31}$$

be the classical power series.
Thus,

$$\begin{aligned} \omega = f(t) &= a_0 + t\sum_{k=0}^{\infty} a_{k+1} t^k \\ &= a_0 + t\left(A\sum_{k=0}^{\infty} a_k t^k + B\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} a_i a_{k-i}\right) t^k\right). \end{aligned}$$

Let

$$F(t,\omega) = \omega - a_0 - t\left(A\omega + B\omega^2\right). \tag{4.32}$$

Then

$$F_\omega(t,\omega) = 1 - t\,(A + 2B\omega)\,.$$

Regarding at point $(0, a_0)$, the function $F(t,\omega)$ is 0 and the partial derivative of the function $F(t,\omega)$ with respect to ω is 1. We can see that $F(t,\omega)$ is an analytic function, so $F(t,\omega)$ has continuous derivatives. By implicit function theorem, there is a neighborhood of $(0, a_0)$ so that whenever t is sufficiently close to 0 there is a unique ω so that $F(t,\omega) = 0$. Then, $f(t)$ is an analytic function in the neighborhood of the point $(0, a_0)$ of the (t,ω)-plane with a positive radius of convergence. From Lemma 4.1, the series in Eq. (4.9) converges. □

5 Numerical Example

In this section, numerical applications of the fractional logistic differential equations are presented. We present three examples to show the efficiency of the RPSM. Comparison with the exact solution of Eq. (4.7) when $\alpha = 1$ is reported in Table 1-3. Let

$$\text{error}(t) = |u_{exact}(t) - u_k(t)|\,,\ t \geq 0, \tag{5.33}$$

where $u_{exact}(t)$ is the exact closed form solution of Eq. (4.7) subject to initial condition (4.8), when $\alpha = 1$, given by

$$u(t) = \frac{u_0}{u_0 + (1 - u_0)e^{-\rho t}}, \ t \geq 0, \tag{5.34}$$

where u_0 is the initial state at the time $t = 0$.
In Examples 5.1 and 5.2, we use $k = 3$. Then

$$u_3(t) = u_0 + u_1\frac{t^\alpha}{\Gamma(1+\alpha)} + u_2\frac{t^2\alpha}{\Gamma(1+2\alpha)} + u_3\frac{t^3\alpha}{\Gamma(1+3\alpha)}, \tag{5.35}$$

where

$$\begin{aligned} u_1 &= \rho^\alpha\left(u_0 - u_0^2\right) \\ u_2 &= \rho^\alpha\left(u_1 - 2u_0u_1\right) \\ u_3 &= \rho^\alpha\left(u_2 - 2u_0u_2 - u_1^2\frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\right). \end{aligned}$$

Example 5.1. *Consider the following fractional logistic equation*

$$D_t^\alpha u(t) = \frac{1}{2^\alpha}u(1-u), \ t > 0, \ 0 < \alpha \leq 1,$$

subject to the initial condition

$$u(0) = 0.85.$$

In Table 1, the numerical error of the 3^{rd} RPS is presented for $\alpha = 1$. The behaviors of approximate solutions are plotted for different values of α, where $\alpha = \{1.0, 0.75, 0.5\}$, in Figure 1.

Table 1: Error when $\alpha = 1$.

t	exact solution	$u_3(t)$	error(t)
0.00	0.850000000000000	0.850000000000000	0
0.02	0.851270542513367	0.851270542493750	1.96169×10^{-11}
0.04	0.852532190262384	0.852532189950000	3.12384×10^{-10}
0.06	0.853784973905210	0.853784972331250	1.57396×10^{-9}
0.08	0.855028924550894	0.855028919600000	4.95089×10^{-9}
0.10	0.856264073748411	0.856264061718750	1.20297×10^{-8}

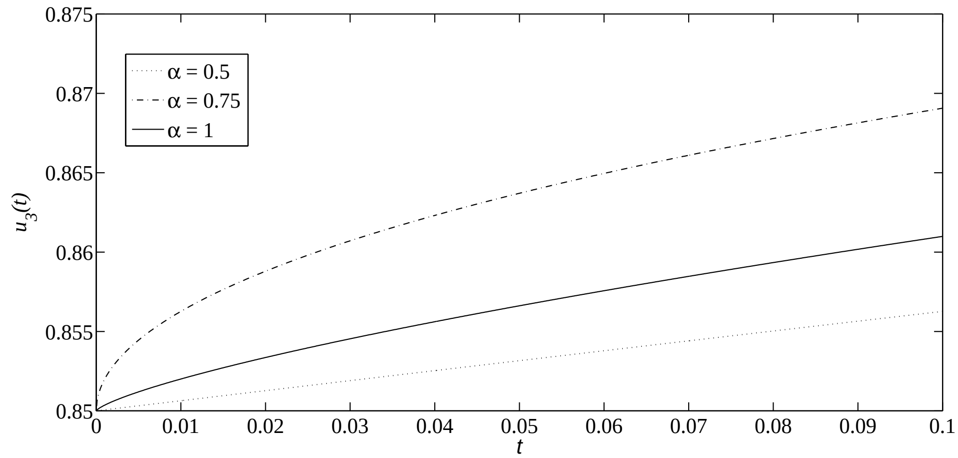

Figure 1: The approximate solution plots of Example 5.1 for different values of α.

Example 5.2. *Consider the following fractional logistic equation*

$$D_t^\alpha u(t) = \frac{1}{4^\alpha} u(1-u),\ t > 0,\ 0 < \alpha \le 1.$$

subject to

$$u(0) = \frac{1}{3}.$$

The numerical error of the 3^{rd} RPS, when $\alpha = 1$, is reported in Table 2. In Figure 2, the behavior of the 3^{rd} RPS approximation is presented with different values of α, where $\alpha = \{1.0, 0.75, 0.5\}$.

Table 2: Error when $\alpha = 1$.

t	exact solution	$u_3(t)$	error(t)
0.00	0.333333333333333	0.333333333333333	0
0.02	0.334445368823947	0.334445368827160	3.21300×10^{-12}
0.04	0.335559246862188	0.335559246913580	5.13920×10^{-11}
0.06	0.336674958073288	0.336674958333333	2.60045×10^{-10}
0.08	0.337792493005687	0.337792493827160	8.21437×10^{-10}
0.10	0.338911842131240	0.338911844135803	2.00456×10^{-9}

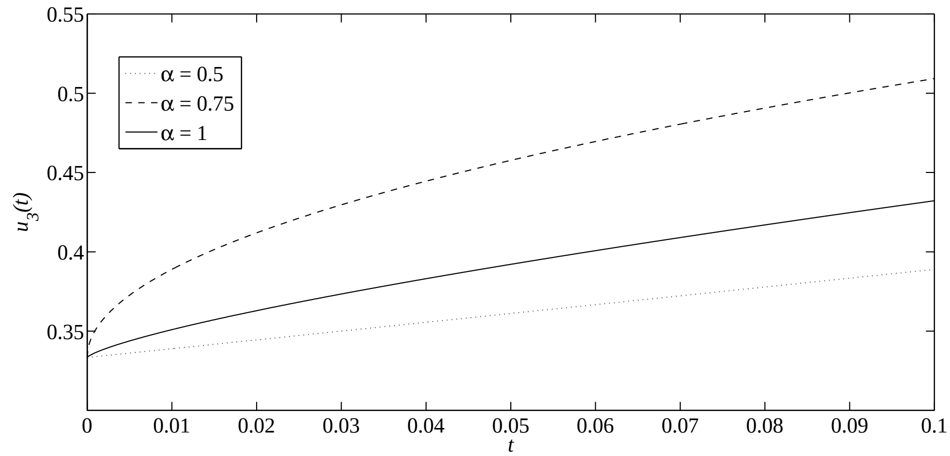

Figure 2: The approximate solution plots of Example 5.2 for different values of α.

In Example 5.3, we use $k = 4$. Then, the approximate solution of fractional logistic equation is given by

$$u_4(t) = u_0 + u_1 \frac{t^\alpha}{\Gamma(1+\alpha)} + u_2 \frac{t^2\alpha}{\Gamma(1+2\alpha)} + u_3 \frac{t^3\alpha}{\Gamma(1+3\alpha)} + u_4 \frac{t^4\alpha}{\Gamma(1+4\alpha)}, \quad (5.36)$$

where

$$\begin{aligned}
u_1 &= \rho^\alpha \left(u_0 - u_0^2\right) \\
u_2 &= \rho^\alpha \left(u_1 - 2u_0 u_1\right) \\
u_3 &= \rho^\alpha \left(u_2 - 2u_0 u_2 - u_1^2 \frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\right). \\
u_4 &= \rho^\alpha \left(u_3 - 2u_0 u_3 - u_1 u_2 \frac{\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\right).
\end{aligned}$$

Let

$$\text{error}(t) = \left|u_{exact}(t) - u_4(t)\right|, \ t \geq 0. \quad (5.37)$$

Example 5.3. *Consider the following fractional logistic equation*

$$D_t^\alpha u(t) = \frac{1}{3^\alpha} u(1-u), \ t > 0, \ 0 < \alpha \leq 1.$$

with the initial condition

$$u(0) = \frac{3}{4}.$$

Table 3 shows the numerical error of the 4^{th} RPS when $\alpha = 1$. Figure 3 shows the representation of the 4^{th} approximate solution with different values of α such that $\alpha = \{1.0, 0.75, 0.5\}$.

Table 3: Error when $\alpha = 1$.

t	exact solution	$u_4(t)$	error(t)
0.00	0.750000000000000	0.750000000000000	0
0.02	0.751247915518896	0.751247915514564	4.33198×10^{-12}
0.04	0.752491657561459	0.752491657492284	6.91750×10^{-11}
0.06	0.753731219529199	0.753731219179688	3.49511×10^{-10}
0.08	0.754966595053057	0.754966593950617	1.10244×10^{-9}
0.10	0.756197777992358	0.756197775306231	2.68613×10^{-9}

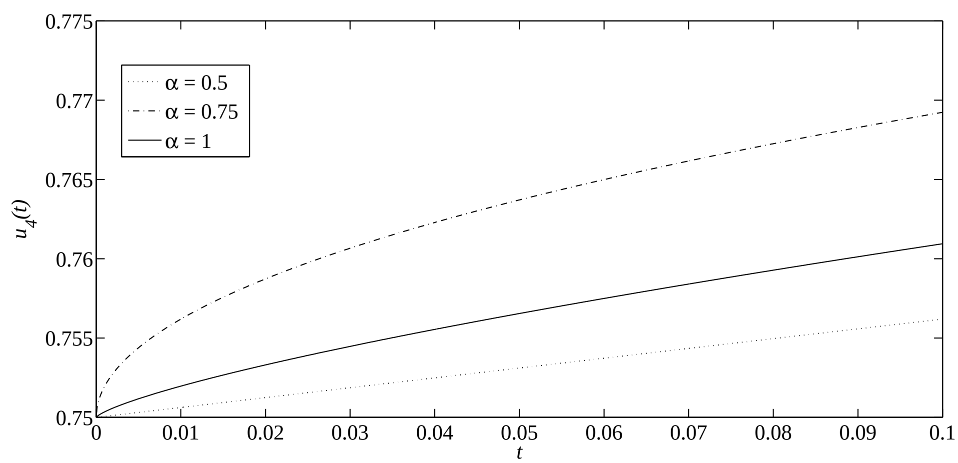

Figure 3: The approximate solution plots of Example 5.3 for different values of α.

6 Conclusions

We have presented a combined form of the residual power series method with the Adomian polynomial for obtaining solutions of fractional logistic equations. We have applied the method to different examples. Comparison with the exact solution when $\alpha = 1$ is reported. We obtained accurate results using a few terms. Therefore, we can see that the RPSM with Adomian polynomial is very powerful for finding the approximate as well as analytical solution of many nonlinear models in science and engineering.

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References

[1] O. Abu Arqub, Series solution of fuzzy differential equations under strongly generalized differentiability, Journal of Advanced Research in Applied Mathematics, **5**, no. 1, (2013), 31-52.

[2] G. Adomian, A review of the decomposition method in applied mathematics, Journal of Mathematical Analysis and Applications, **135**, no. 2, (1988), 501–544.

[3] M. Alquran, Analytical solutions of fractional foam drainage equation by residual power series method, Mathematical Sciences, **8**, no. 4, (2014), 153-160.

[4] S. Alshammari, M. Al-Smadi, M. Al Shammari, M., I. Hashim, M. A. Alias, Advanced analytical treatment of fractional logistic equations based on residual error functions, International Journal of Differential Equations, (2019) (Article ID 7609879), https://doi.org/10.1155/2019/7609879.

[5] B. Chen, L. Qin, F. Xu, J. Zu, Applications of general residual power series method to differential equations with variable coefficients, Discrete Dynamics in Nature and Society, (2018) (Article ID 2394735), https://doi.org/10.1155/2018/2394735.

[6] J. S. Duan, R. Rach, D. Baleanu, A. M. Wazwaz, A review of the Adomian decomposition method and its applications to fractional differential equations, Communications in Fractional Calculus, **3**, no. 2, (2012), 73–99.

[7] A. Elsaid, Adomian polynomials: a powerful tool for iterative methods of series solution of nonlinear equations. Journal of Applied Analysis and Computation, **2**, no. 4, (2012), 381–394.

[8] A. El-Ajou, O. Abu Arqub, Z. Al zhour, S. Momani, New results on fractional power series: Theories and applications, Entropy, **15**, (2013) 5305–5323.

[9] S. A. El-Wakil, New applications of variational iteration method using Adomian polynomials, Nonlinear Dynamics, **52**, nos. 1-2, (2008), 1–49.

[10] H. Khan, F. Jarad, T. Abdeljawad, A. Khan, A singular ABC-fractional differential equation with p-Laplacian operator, Chaos, Solitons and Fractals, (2019), 56–61.

[11] H. Khan, A. Khan, F. Jarad, A. Shah, Existence and data dependence theorems for solutions of an ABC-fractional order impulsive system, Chaos, Solitons and Fractals, **131**, (2020).

[12] A. Khan, H. Khan, J. F. Gmez-Aguilar, T. Abdeljawad, Existence and Hyers-Ulam stability for a nonlinear singular fractional differential equations with Mittag-Leffler kernel, Chaos, Solitons a Fractals, (2019), 422–427.

[13] H. Khan, C. Tunç, W. Chen, A. Khan, Existence theorems and Hyers-Ulam stability for a class of hybrid fractional differential equations with P-Laplacian operator, Journal of Applied Analysis and Computation, **8**, no. 4, (2018), 1211–1226.

[14] H. Khan, C. Tunç, A. Khan, Green functions properties and existence theorems for nonlinear singular delay fractional differential equations, Discrete and Continuous Dynamical System Series S., (2020).

[15] A. Kumar, S. Kumar, M. Singh, Residual power series method for fractional Sharma-Tasso-Olever equation, Communications in Numerical Analysis, no. 1, (2016), 1–10.

[16] D. G. Prakasha, P. Veeresha, H. M. Baskonus, Residual power series method for fractional SwiftHohenberg Equation, Fractal and Fractional, **3**, no. 1, (2019) 9, https://doi.org/10.3390/fractalfract3010009.

[17] I. Talib, F. B. Belgacem, H. Khalil, C. Tunc, Nonlinear fractional partial coupled systems approximate solutions through operational matrices approach, Nonlinear Studied, **26**, no. 4, (2019), 955–971.

[18] I. Talib, , C. Tunc, Z. A. Noor, New operational matrices of orthogonal Legendre polynomials and their Operational, Journal of Taibah University for Science, **13**, no. 1, (2019), 377–389.

[19] H. Tariq, G. Akram, Residual power series method for solving time-space-fractional Benney-Lin equation arising in falling film problems, Journal of Applied Mathematics and Computing, **55**, nos. 1-2, (2017), 683–708.

[20] P. J. Torvik, R. L. Bagley, On the appearance of the fractional derivative in the behavior of real materials, Journal of Applied Mechanics, **51**, no. 2, (1984), 294–298.

[21] Y. Zhang, A. Kumar, S. Kumar, D. Baleanu, X. Yang, Residual power series method for time-fractional Schrödinger equations, Journal of Nonlinear Science and Applications, **9**, no. 11, (2016), 5821–5829.