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Sum of powers of the natural numbers via Stirling numbers of the second kind

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Abstract

In this paper, many properties of several sequences such as the falling numbers and Stirling numbers of the second kind are discussed. We use these properties to find recurrence relations of the sums of powers $\sigma_m(n) = \sum_{k=1}^{n} k^m$.

1 Introduction

Assume that $\{\sigma_m(n)\}_{m=0}^{\infty}$ is the sequence of sums of the m-th power of the first n positive integers defined as

$$\sigma_m(n) = \sum_{k=1}^{n} k^m.$$

For example, $\sigma_1(n) = \frac{n(n+1)}{2}, \sigma_2(n) = \frac{n(n+1)(2n+1)}{6}, \sigma_3(n) = \left[\frac{n(n+1)}{2}\right]^2$, and $\sigma_4(n) = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$. It is clear that if $a = \sigma_1(n)$, then $\sigma_3(n) = a^2$. It has further been proved $\sigma_5(n) = \frac{4a^3-a^2}{3}, \sigma_7(n) = \frac{6a^4-4a^3+a^2}{3}$. Many interesting questions arise here. For example, given $\sigma_m(n)$, can we get $\sigma_{m+1}(n)$? Can we get $\sigma_{m+2}(n)$ using $\sigma_m(n)$ and $\sigma_{m+1}(n)$? For further properties of $\{\sigma_m(n)\}_{m=0}^{\infty}$, see [6], [8], and [9].

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1.1 The falling numbers

Let $\mathbb{N}_0$ be the set of nonnegative integers. For $n \in \mathbb{N}_0$, the falling factorial numbers $(x)_n$ are given as

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1) = \frac{\Gamma(x+1)}{\Gamma(x+1-n)}, \tag{1.1}$$

where $(x)_0 = 1$. it is clear that $(x)_n = \frac{\Gamma(x+1)}{\Gamma(x+1-n)}$. In particular, for $m \geq n$, we get $(m)_n = \frac{m!}{(m-n)!}$.

Proposition 1.1. *For $m \in \mathbb{N}_0$, the falling numbers $(x)_n$ satisfy*

$$(x+m)_m(x)_j = (x+m)_{m+j}.$$

In particular,

$$(x+1)(x)_j = (x+1)_{j+1}. \tag{1.2}$$

Proof.

$$\begin{aligned}(x+m)_m(x)_j &= \frac{\Gamma(x+m+1)}{\Gamma(x+1)}\frac{\Gamma(x+1)}{\Gamma(x-j+1)} \\ &= \frac{\Gamma(x+m+1)}{\Gamma(x+m-(j+m)+1)} = (x+m)_{m+j}.\end{aligned}$$

□

For further properties of these numbers, see [7] and [13].

1.2 The forward difference operator

Definition 1.2. *Let $\mathcal{S}(\mathbb{N})$ be the set of complex valued sequences over $\mathbb{N}$. For $u \in \mathcal{S}(\mathbb{N})$, the forward difference operators $\Delta : \mathcal{S}(\mathbb{N}) \longrightarrow \mathcal{S}(\mathbb{N}) : u \mapsto \Delta u$ is defined as $(\Delta u)(k) = u(k+1) - u(k)$.*

It is easy to show that the forward difference operator satisfies

$$\sum_{k=m}^{n-1} (\triangle f)(k) = f(n) - f(m). \tag{1.3}$$

By substituting $f(n) = (\Delta^{-1}F)(n)$, we get

$$\sum_{k=m}^{n-1} F(k) = \sum_{k=m}^{n-1} (\triangle\Delta^{-1}F)(k) = (\Delta^{-1}F)(n) - (\Delta^{-1}F)(m). \tag{1.4}$$

This equation defines the inverse difference operator as

$$(\Delta^{-1}F)(n) = C + \sum_{k=m}^{n-1} F(k). \tag{1.5}$$

As an application for this definition and using $\Gamma(x+1) = x\Gamma(x)$, we get

$$\begin{aligned}
(k+s+1)_n - (k+s)_n &= \frac{\Gamma(k+s+2)}{\Gamma(k+s+2-n)} - \frac{\Gamma(k+s+1)}{\Gamma(k+s+1-n)} \\
&= \frac{\Gamma(k+s+1)}{\Gamma(k+s+1-n)}\Big(\frac{k+s+1}{k+s+1-n} - 1\Big) \\
&= \frac{\Gamma(k+s+1)}{\Gamma(k+s+1-n)}\frac{n}{k+s+1-n} \\
&= n\frac{\Gamma(k+s+1)}{\Gamma(k+s+1-(n-1))} = n(k+s)_{n-1}.
\end{aligned}$$

Now, using (1.3), we get

$$\sum_{j=1}^{k-1}(j+s)_n = \frac{1}{n+1}\sum_{j=1}^{k-1}\Delta\Big((j+s)_{n+1}\Big) = \frac{1}{n+1}\Big((k+s)_{n+1} - (1+s)_{n+1}\Big) = \frac{(k+s)_{n+1}}{n+1} + C$$

Therefore, we have proven the following proposition

Proposition 1.3. *If $u(k) = (k+s)_n$, then*

1. $(\Delta u)(k) = n(k+s)_{n-1}$.
2. $(\Delta^{-1}u)(k) = \frac{(k+s)_{n+1}}{n+1} + c$.

For further properties for difference operators and their applications, see [10], [1], [2], [3], [4] and [5].

1.3 Stirling numbers

For $n \in \mathbb{N}_0$, the sequence $\{{n \atop k}\}, k = 0, 1, 2 \cdots n$, which satisfies

$$x^n = \sum_{j=0}^{n} \left\{ {n \atop j} \right\} (x)_j \tag{1.6}$$

is called the Stirling numbers of the second kind.

Example 1.4. *For $k \in \mathbb{N}$ and using*

$$\begin{aligned} x^3 &= x + 3x(x-1) + x(x-1)(x-2) \\ &= (x)_1 + 3(x)_2 + (x)_3 \\ &= \left\{ {3 \atop 0} \right\} (x)_0 + \left\{ {3 \atop 1} \right\} (x)_1 + \left\{ {3 \atop 2} \right\} (x)_2 + \left\{ {3 \atop 3} \right\} (x)_3, \end{aligned}$$

we get that $\left\{ {3 \atop 0} \right\} = 0$, $\left\{ {3 \atop 1} \right\} = 1$, $\left\{ {3 \atop 2} \right\} = 3$ *and* $\left\{ {3 \atop 3} \right\} = 1$.

In [11], the following explicit formula is given to calculate these numbers

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (k-j)^n. \tag{1.7}$$

This equation implies that

$$\left\{ {n \atop 0} \right\} = \delta_{n,0} = \begin{cases} 1, & n = 0; \\ 0, & n > 0. \end{cases} . \tag{1.8}$$

Clearly, for $k, m \in \mathbb{N}$ and using (1.8), we have

$$k^m = \sum_{j=1}^{m} \left\{ {m \atop j} \right\} (k)_j. \tag{1.9}$$

Taking the sum for both sides of (1.9) from $k = 1$ to n and using Proposition 1.3, we get

$$\sigma_m(n) = \sum_{j=1}^{m} \left\{ {m \atop j} \right\} \frac{(n+1)_{j+1}}{j+1}. \tag{1.10}$$

Now, interchanging the sums technique gives a result which is needed later.

$$\begin{aligned} \sum_{n=1}^{N} \sigma_m(n) &= \sum_{n=1}^{N} \sum_{k=1}^{n} k^m = \sum_{k=1}^{N} \sum_{n=k}^{N} k^m \\ &= \sum_{k=1}^{N} (N - k + 1) k^m \\ &= (N+1)\sigma_m(n) - \sigma_{m+1}(n). \end{aligned}$$

Therefore,

$$\sum_{n=1}^{N} \sigma_m(n) = (N+1)\sigma_m(N) - \sigma_{m+1}(N). \tag{1.11}$$

For further properties of the Stirling numbers, see [12].

2 Main Results

This section is devoted to construct recurrence relations for $\sigma_m(n)$. To begin with, taking the sum for both sides of (1.10) from $n = 1$ to $n = N$, using (1.11) and Proposition 1.3 we get

$$(N+1)\sigma_m(N) - \sigma_{m+1}(N) = \sum_{j=1}^{m} \left\{ {m \atop j} \right\} \frac{(N+2)_{j+2}}{(j+1)(j+2)}.$$

Therefore, we have the following recurrence relation

Proposition 2.1. *for* $m \in \mathbb{N}$,

$$\sigma_{m+1}(n) = (n+1)\sigma_m(n) - \sum_{j=1}^{m} \left\{ {m \atop j} \right\} \frac{(n+2)_{j+2}}{(j+1)(j+2)}.$$

Example 2.2.

$$\begin{aligned}
\sigma_4(n) &= (n+1)\sigma_3(n) - \sum_{j=1}^{3} \left\{ {3 \atop j} \right\} \frac{(n+2)_{j+2}}{(j+1)(j+2)} \\
&= (n+1)(\frac{n(n+1)}{2})^2 - \frac{(n+2)_3}{(2)(3)} - 3\frac{(n+2)_4}{(3)(4)} - \frac{(n+2)_5}{(4)(5)} \\
&= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.
\end{aligned}$$

We have other recurrence relations that can be generated after proving the following result

Proposition 2.3. *For* $m, k \in \mathbb{N}$, *the following are true*

1. $k^{m+1} + k^m = \sum_{j=1}^{m} \left\{ {m \atop j} \right\} (k+1)_{j+1}$.

2. $k^{m+2} + 3k^{m+1} + 2k^m = \sum_{j=1}^{m} \left\{ {m \atop j} \right\} (k+2)_{j+2}$.

3. $k^{m+3} + 6k^{m+2} + 11k^{m+1} + 6k^m = \sum_{j=1}^{m} \left\{ {m \atop j} \right\} (k+3)_{j+3}$.

Proof. Multiply (1.6) by $k+1$ and use Proposition 1.1 to get

$$\begin{aligned}
k^{m+1} + k^m = (k+1)k^m &= \sum_{j=1}^{m} \left\{ {m \atop j} \right\} (k+1)_1 (k)_j \\
&= \sum_{j=1}^{m} \left\{ {m \atop j} \right\} (k+1)_{j+1}.
\end{aligned}$$

Now, use Proposition 1.1 to get

$$k^{m+2} + 3k^{m+1} + 2k^m = (k+2)(k+1)k^m = \sum_{j=1}^{m} \left\{ {m \atop j} \right\} (k+2)_2 (k)_j$$
$$= \sum_{j=1}^{m} \left\{ {m \atop j} \right\} (k+2)_{j+2}.$$

Again, use Proposition 1.1 to get

$$k^{m+3} + 6k^{m+2} + 11k^{m+1} + 6k^m = (k+3)(k+2)(k+1)k^m$$
$$= \sum_{j=1}^{m} \left\{ {m \atop j} \right\} (k+3)_3 (k)_j$$
$$= \sum_{j=1}^{m} \left\{ {m \atop j} \right\} (k+3)_{j+3}.$$

□

Now, one can easily verify that Proposition 2.3 implies that the following are true.

Proposition 2.4. *For $m, k \in \mathbb{N}$*

1. $k^{m+1} = \sum_{j=1}^{m} \left\{ {m \atop j} \right\} ((k+1)_{j+1} - (k)_j)$.

2. $k^{m+2} = \sum_{j=1}^{m} \left\{ {m \atop j} \right\} ((k+2)_{j+2} - 3(k+1)_{j+1} + (k)_j)$.

3. $k^{m+3} = \sum_{j=1}^{m} \left\{ {m \atop j} \right\} ((k+3)_{j+3} - 6(k+2)_{j+2} + 7(k+1)_{j+1} - (k)_j)$.

Taking the sum for the results in Proposition 2.3 from $k = 1$ to n and using Proposition 1.3 give the following recurrence relations for $\sigma_m(n)$.

Theorem 2.5. *For $m \in \mathbb{N}$*

1. $\sigma_{m+1}(n) = \sum_{j=1}^{m} \left\{ {m \atop j} \right\} \frac{(n+2)_{j+2}}{j+2} - \sigma_m(n)$.

2. $\sigma_{m+2}(n) = \sum_{j=1}^{m} \left\{ {m \atop j} \right\} \frac{(n+3)_{j+3}}{j+3} - 3\sigma_{m+1}(n) - 2\sigma_m(n)$

3. $\sigma_{m+3}(n) = \sum_{j=1}^{m} \left\{ {m \atop j} \right\} \frac{(n+4)_{j+4}}{j+4} - 6\sigma_{m+2}(n) - 11\sigma_{m+1}(n) - 6\sigma_m(n)$.

Also, taking the sum for the results in Proposition will give explicit formulas for $\sigma_{m+1}(n)$,$\sigma_{m+2}(n)$, and $\sigma_{m+2}(n)$ respectively.

Theorem 2.6. *For $m \in \mathbb{N}$, the Stirling numbers satisfy*

1. $\sigma_{m+1}(n) = \sum_{j=1}^{m} \left\{ {m \atop j} \right\} \left(\frac{(n+2)_{j+2}}{j+2} - \frac{(n+1)_{j+1}}{j+1} \right)$.

2. $\sigma_{m+2}(n) = \sum_{j=1}^{m} \left\{ {m \atop j} \right\} \left(\frac{(n+3)_{j+3}}{j+3} - 3\frac{(n+2)_{j+2}}{j+2} + \frac{(n+1)_{j+1}}{j+1} \right)$.

3. $\sigma_{m+3}(n) = \sum_{j=1}^{m} \left\{ {m \atop j} \right\} \left(\frac{(n+4)_{j+4}}{j+4} - 6\frac{(n+3)_{j+3}}{j+3} + 7\frac{(n+2)_{j+2}}{j+2} - \frac{(n+1)_{j+1}}{j+1} \right)$.

Example 2.7. *Using $\left\{ {2 \atop j} \right\} = 1$ for $j = 1, 2$, we get*

$$\begin{aligned}
\sum_{k=1}^{n} k^5 = \sigma_5(n) &= \sum_{j=1}^{2} \left\{ {2 \atop j} \right\} \frac{(n+4)_{j+4}}{j+4} - 6\sigma_4(n) - 11\sigma_3(n) - 6\sigma_2(n) \\
&= \frac{(n+4)_5}{5} + \frac{(n+4)_6}{6} \\
&- \frac{6}{30} n(n+1)(2n+1)(3n^2+3n-1) - 11(\frac{n(n+1)}{2})^2 - 6\frac{n(n+1)(2n+1)}{6} \\
&= \frac{(n+4)(n+3)(n+2)(n+1)n}{5} + \frac{(n+4)(n+3)(n+2)(n+1)n(n-1)}{6} \\
&- \frac{6}{30} n(n+1)(2n+1)(3n^2+3n-1) - 11(\frac{n(n+1)}{2})^2 - 6\frac{n(n+1)(2n+1)}{6} \\
&= \frac{[n(n+1)]^2(2n^2+2n-1)}{12}.
\end{aligned}$$

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