International Journal of Mathematics and
Computer Science, **15**(2020), no. 4, 1199–1206

# Turning Near rings into New Near rings

**A. V. Ramakrishna**[1], **T. V. N. Prasanna**[2], **D. V. Lakshmi**[3]

[1]Department of Mathematics
R. V. R and J. C. College of Engineering, Chowdavaram
Guntur-522019, Andhra Pradesh, India

[2]Department of BS&H
Vignan's Nirula Institute of Technology& Science for Women
Guntur-522005, Andhra Pradesh, India

[3]Bapatla Women's Engineering College
Bapatla, Andhra Pradesh, India

email: amathi7@gmail.com, tvnp11@gmail.com, himaja96@gmail.com

(Received August 5, 2020, Accepted September 10, 2020)

**Abstract**

In this paper, we construct new near rings $N_f$ from near rings $N$ through 'semilinear-like' mappings and study how nice properties by one of them are enjoyed by the other. It is gratifying to note that almost all nice properties usually imposed upon one of them are inherited by the other with the imposition of subjectivity on $f$ when needed.

## 1 Introduction

A right near ring is a triple $(N, +, .)$, where $(N, +)$ is a group, $(N, .)$ is a semigroup satisfying the right distributive law: $(a + b)c = ac + bc$ for all $a, b, c \in N$. By a near ring we mean a right near ring.

**Key words and phrases:** near ring, near field, semilinear map.
**AMS (MOS) Subject Classifications**: 16Y30
**ISSN** 1814-0432, 2020, http://ijmcs.future-in-tech.net

Let $(N, +, \cdot)$ be a near ring. When there is no confusion, we write $N$ is a near ring, instead of $(N, +, \cdot)$ is a near ring.

**Definition 1.1.** *In a near ring* $(N, +, .)$,

1. **zero-symmetric part of** $N$*:* $N_0 = \{n \in N \mid n0 = 0\}$*;*

2. $N$ *is called* **zero-symmetric near ring** *if* $N = N_0$*;*

3. **the constant part of** $N$:$N_c := \{n \in N \mid n0 = n\} = \{n \in N \mid nn' = n \text{ for all } n' \in N\}$.

**Definition 1.2.** $N$ *is said to fulfill the* **insertion of factors property (IFP)** *provided that for all* $a, b, n \in N$ *:* $ab = 0$ *implies* $anb = 0$.

**Definition 1.3.** *An element* $x$ *in* $N$ *is called* **distributive** *if* $x(y + z) = xy + xz$*, for all* $y, z$ *in* $N$*.* $N$ *is called a* **distributive near ring** *if each element of* $N$ *is distributive.*

**Definition 1.4.** [1] *A continuous mapping* $\lambda : \mathbb{R}^n \to \mathbb{R}$ *is called a* **semilinear map** *if* $\lambda(\lambda(v)w) = \lambda(v)\lambda(w)$ *for all* $v, w \in \mathbb{R}^n$.

**Definition 1.5.** *We call a mapping* $f : N \to N$ *is* ***semilinear*** *if* $f(f(n_1)n_2) = f(n_1)f(n_2)$ *for all* $n_1, n_2 \in N$.

By modifying the semilinearity condition above, we define a dually semilinear map as follows

**Definition 1.6.** *We call a mapping* $f : N \to N$ *is* ***dually semilinear*** *if* $f(n_1 f(n_2)) = f(n_1)f(n_2)$ *for all* $n_1, n_2 \in N$.

# 2 Construction of New near rings from old near rings through mappings

**Theorem 2.1.** *Let* $f : N \to N$ *be a mapping such that* $f(n_1 f(n_2)) = f(n_1)f(n_2)$ *for all* $n_1, n_2 \in N$*. Define the binary operation* $*$ *on* $N$ *by* $n_1 * n_2 = n_1 f(n_2)$*, for all* $n_1, n_2 \in N$*. Then* $(N, +, *)$ *is a near ring.*

*Proof.* For any $n_1, n_2, n_3 \in N$,

$$\begin{aligned} n_1 * (n_2 * n_3) &= n_1 f(n_2 * n_3) \\ &= n_1 f(n_2 f(n_3)) \\ &= n_1[f(n_2)f(n_3)]. \\ \text{Also } (n_1 * n_2) * n_3 &= (n_1 * n_2) f(n_3) \\ &= [n_1 f(n_2)] f(n_3)] \\ &= n_1[f(n_2)f(n_3)]. \end{aligned}$$

So the binary operation $*$ is associative.

$$\begin{aligned} \text{Now } (n_1 + n_2) * n_3 &= (n_1 + n_2) f(n_3) \\ &= n_1 f(n_3) + n_2 f(n_3) \\ &= n_1 * n_3 + n_2 * n_3. \end{aligned}$$

So the binary operation $*$ is right distributive.
Therefore $(N, +, *)$ is a near ring. □

The near ring $(N, +, *)$ obtained from a near ring $(N, +, \cdot)$ by the means of a semilinear map $f$ from $N$ to $N$ is hereafterwards denoted by $N_f$. Let $N_f$ be the near ring obtained in Theorem 2.1

**Theorem 2.2.** *Suppose $f(0) = 0$, then $N$ is zero-symmetric if and only if $N_f$ is zero-symmetric.*

**Theorem 2.3.** *(a) If $N$ is distributive and $f$ preserves addition, then $N_f$ is distributive.*
*(b) If $f$ preserves addition, $f$ is onto and $N_f$ is distributive, then $N$ is distributive.*

*Proof.* (a) For any $n_1, n_2, n_3 \in N$,

$$\begin{aligned} n_1 * (n_2 + n_3) &= n_1 f(n_2 + n_3) \\ &= n_1[f(n_2 + f(n_3)] \\ &= n_1 f(n_2) + n_1 f(n_3) \\ &= n_1 * n_2 + n_1 * n_3. \end{aligned}$$

Therefore $N_f$ is distributive.
(b) Let $n_1, n_2, n_3 \in N$.

Since $f$ is onto, $n_2 = f(x_2)$ and $n_3 = f(x_3)$ for some $x_2, x_3 \in N$.

$$\begin{aligned} n_1(n_2 + n_3) &= n_1(f(x_2) + f(x_3)) \\ &= n_1 f(x_2 + x_3) \\ &= n_1 * (x_2 + x_3) \\ &= n_1 * x_2 + n_1 * x_3 \\ &= n_1 f(x_2) + n_1 f(x_3) \\ &= n_1 n_2 + n_1 n_3. \end{aligned}$$

Therefore $N$ is distributive. □

**Theorem 2.4.** *Suppose $f$ is onto then $N$ fulfills the insertion of factors property (IFP) if and only if $N_f$ has IFP.*

**Definition 2.5.** *[3] $N$ is said to be* **planar near ring** *if $|N/\equiv| \geq 3$ and if every equation $xa = xb + c$ $(a \not\equiv b)$ has a unique solution in $N$.*

**Theorem 2.6.** *If $N$ is a planar near ring, then $N_f$ is also a planar near ring.*

*Proof.* For any $n_1, n_2, n_3 \in N$,

$$\begin{aligned} x * n_1 &= x * n_2 + n_3 \\ \Rightarrow x f(n_1) &= x f(n_2) + n_3. \end{aligned}$$

By the hypothesis, $N_f$ is a planar near ring. □

**Definition 2.7.** *[2] An* **additive mapping** *$D : N \to N$ is called a* **derivation** *if*

$$D(n_1 n_2) = D(n_1)n_2 + n_1 D(n_2) \text{ for all } n_1, n_2 \in N.$$

**Theorem 2.8.** *Let $D$ be a derivation on $N$. If $D$ commutes with $f$, then $D$ is a derivation on $N_f$.*

*Proof.* For any $n_1, n_2 \in N$,

$$\begin{aligned} D(n_1 * n_2) &= D(n_1 f(n_2)) \\ &= D(n_1) f(n_2) + n_1 D(f(n_2)) \text{ and} \\ D(n_1) * n_2 + n_1 * D(n_2) &= D(n_1) f(n_2) + n_1 f(D(n_2)) \\ &= D(n_1) f(n_2) + n_1 D(f(n_2)). \end{aligned}$$

Therefore $D$ is a derivation on $N_f$. □

**Remark 2.9.** *Even though + is not commutative, the following theorem shows that the terms $n_1 * D(n_2)$ and $D(n_1) * n_2$ commute with respect to addition.*

**Theorem 2.10.** *Let $D$ be a derivation on $N_f$. Then $D(n_1 * n_2) = n_1 * D(n_2) + D(n_1) * n_2$.*

*Proof.* For any $n_1, n_2 \in N$,

$$\begin{aligned} D((n_1 + n_1) * n_2) &= D(n_1 + n_1) * n_2 + (n_1 + n_1) * D(n_2) \\ &= [D(n_1) + D(n_1)] * n_2 + n_1 * D(n_2) + n_1 * D(n_2) \\ &= D(n_1) * n_2 + D(n_1) * n_2 + n_1 * D(n_2) + n_1 * D(n_2) \text{ and} \\ D(n_1 * n_2 + n_1 * n_2) &= D(n_1 * n_2) + D(n_1 * n_2) \\ &= D(n_1) * n_2 + n_1 * D(n_2) + D(n_1) * n_2 + n_1 * D(n_2). \end{aligned}$$

Since $D((n_1 + n_2) * n_2) = D(n_1 * n_2 + n_1 * n_2)$,
it follows that $D(n_1 * n_2) = D(n_1) * n_2 + n_1 * D(n_2) = n_1 * D(n_2) + D(n_1) * n_2$. □

**Theorem 2.11.** *If $D$ is an additive endomorphism of $N$ satisfying $D(n_1 * n_2) = n_1 * D(n_2) + D(n_1) * n_2$ for all $n_1, n_2 \in N$, then $D$ is a derivation on $N_f$.*

*Proof.* For any $n_1, n_2 \in N$,

$$\begin{aligned} D((n_1 + n_1) * n_2) &= (n_1 + n_1) * D(n_2) + D(n_1 + n_1) * n_2 \\ &= n_1 * D(n_2) + n_1 * D(n_2) + D(n_1) * n_2 + D(n_1) * n_2 \text{ and} \\ D(n_1 * n_2 + n_1 * n_2) &= D(n_1 * n_2) + D(n_1 * n_2) \\ &= n_1 * D(n_2) + D(n_1) * n_2 + n_1 * D(n_2) + D(n_1) * n_2. \end{aligned}$$

Since $D((n_1 + n_2) * n_2) = D(n_1 * n_2 + n_1 * n_2)$,
it follows that $D(n_1 * n_2) = n_1 * D(n_2) + D(n_1) * n_2 = D(n_1) * n_2 + n_1 * D(n_2)$. □

**Theorem 2.12.** *If $N_f$ is a near-field and $f(N) = N$ , then $N$ is a near field.*

**Theorem 2.13.** *Let $g : (N, +, *) \to (N, +, *)$ be a mapping such that $g(n_1 * g(n_2)) = g(n_1) * g(n_2)$ for all $n_1, n_2 \in N$. Define $n_1 *_1 n_2 = n_1 * g(n_2)$. Then $((N_f)_g, +, *_1)$ is a near ring. Also $N_{fog} = (N_f)_g$ whenever $fog = gof$.*

*Proof.* First we show that $f \circ g$ is a mapping such that $(f \circ g)(n_1(f \circ g)(n_2)) = (f \circ g)(n_1)(f \circ g)(n_2)$.

Now for any $n_1, n_2 \in N$,

$$\begin{aligned}(f \circ g)(n_1(f \circ g)(n_2)) &= (f \circ g)(n_1)f(g(n_2)) \\ &= f[g(n_1)f(g(n_2))] \\ &= f[g(n_1) * g(f(n_2))] \\ &= f[g(n_1) * g(f(n_2))] \\ &= f[g(n_1)f(g(n_2))] \\ &= f(g(n_1))f(g(n_2)). \\ \text{Now } n_1 *_1 n_2 &= n_1 * g(n_2) \\ &= n_1 f(g(n_2)) \\ &= n_1(f \circ g)(n_2)\end{aligned}$$

and hence $N_{f \circ g} = (N, +, *_1)$ is also a near ring. □

That the condition $f \circ g = g \circ f$ is necessary in the above statement is clear from the following:

**Example 2.14.** *Let $f, g$ be mappings from $\mathbb{R}$ into $\mathbb{R}$ such that $f(x) = -1$ and $g(x) = 1$ for all $x \in \mathbb{R}$. Then $f(x_1 f(x_2)) = f(x_1)f(x_2)$ and $g(x_1 g(x_2)) = g(x_1)g(x_2)$ for all $x_1, x_2 \in \mathbb{R}$.*
*Now for any $x \in \mathbb{R}$*

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(1) = -1 \text{ and} \\ (g \circ f)(x) &= g(-x) = 1.\end{aligned}$$

*So $f \circ g \neq g \circ f$.*
*Now for any $x_1, x_2 \in \mathbb{R}$,*

$$\begin{aligned}(f \circ g)(x_1(f \circ g)(x_2)) &= (f \circ g)(x_1 f(g(x_2))) \\ &= (f \circ g)(x_1 f(1)) \\ &= (f \circ g)(-x_1) \\ &= f(g(-x_1)) \\ &= f(1) = -1 \text{ and} \\ (f \circ g)(x_1)(f \circ g)(x_2) &= (f(g(x_1)))(f(g(x_2))) \\ &= f(1)f(1) = 1.\end{aligned}$$

*So $(f \circ g)(x_1(f \circ g)(x_2)) \neq (f \circ g)(x_1)(f \circ g)(x_2)$.*
*In $\mathbb{R}_f$, $x * y = xf(y) = -xy$,*
*in $\mathbb{R}_{f_g}$, $x *_1 y = x * g(y) = x * 1 = -x$ for all $x, y \in \mathbb{R}$.*
*Let $g \circ f = h$. Then $x *_1 y = xh(y) = x(g(f(y)) = x1 = x$.*

# 3 Ideal correspondence of $N$ and $N_f$

**Theorem 3.1.** (1) *Every right ideal of $N$ is a right ideal of $N_f$.*
(2) *A left ideal $I$ of $N$ is a left ideal of $N_f$ if $f$ is an additive homomorphism such that $f(I) \subseteq I$.*

*Proof.* (1) Let $I$ be a right ideal of $N$, we have $IN \subseteq I$.
To show that $I$ is a right ideal of $N_f$.
Now for any $i \in I$ and $n \in N$,
$i * n = if(n) \in IN \subseteq I \Rightarrow I * N \subseteq I$.
So $I$ is a right ideal of $N_f$.
(2) Since $I$ is a left ideal of $N$, we have $n(n' + i) - nn' \in I$ for all $n, n' \in N$ and $i \in I$.
We show that $I$ is a left ideal of $N_f$.
For any $n, n' \in N$ and $i \in I$,
$n * (n' + i) - n * n' = nf(n' + i) - nf(n') = n[f(n') + f(i)] - nf(n') \in I$.
Therefore $I$ is a left ideal of $N_f$. □

Generation of new near ring multiplications via certain self maps on a left near ring along the lines of Theorem 2.1 can also be done. As the proof is similar, we state this fact without proof:

**Theorem 3.2.** *Let $N$ be a left near ring. Let $f : N \to N$ be a mapping such that $f(f(n_1)n_2) = f(n_1)f(n_2)$ for all $n_1, n_2 \in N$. Define the binary operation $*$ on $N$ by $n_1 * n_2 = f(n_1)n_2$ for all $n_1, n_2 \in N$. Then $(N, +, *)$ is a left near ring.*

The near ring $(N, +, *)$ obtained from a near ring $(N, +, \cdot)$ by means of a semilinear map $f$ from $N$ to $N$ is hereafter denoted by ${}_fN$.
Let $N$ be a left near ring. Let $f : N \to N$ be a mapping such that $f(f(n_1)n_2) = f(n_1)f(n_2)$ for all $n_1, n_2 \in N$. Define the binary operation $*$ on $N$ by $n_1 * n_2 = f(n_1)n_2$ for all $n_1, n_2 \in N$.

**Theorem 3.3.** (1) *Every left ideal of the left near ring $N$ is a left ideal of the left near ring ${}_fN$.*
(2) *A right ideal $I$ of $N$ is a right ideal of ${}_fN$ if $f$ is an additive homomorphism such that $f(I) \subseteq I$.*

*Proof.* (1) Let $I$ be a left ideal of the left near ring $N$. We have $NI \subseteq I$.
To show that $I$ is a left ideal of ${}_fN$, let $i \in I$ and $n \in N$.
$n * i = f(n)i \in NI \subseteq I \Rightarrow N * I \subseteq I$.
So $I$ is a left ideal of ${}_fN$.

(2) Since $I$ is a right ideal of $(N, +, \cdot)$, we have $(n + i)n' - nn' \in I$ for all $n, n' \in N$ and $i \in I$.
To show that $I$ is a right ideal of ${}_fN$, let $n, n' \in N$ and $i \in I$.

$$\begin{aligned}(n + i) * n' - n * n' &= f(n + i)n' - f(n)n' \\ &= [f(n) + f(i)]n' - f(n)n' \in I.\end{aligned}$$

Therefore $I$ is a right ideal of ${}_fN$. □

**Acknowledgment.** The authors thank Professor I. Ramabhadra Sarma for his valuable comments and suggestions.

# References

[1] K. D. Magill, Jr., Topological Nearrings Whose Additive Groups are Euclidean, Mathematik, **119,** (1995), 281–301.

[2] G. Piltz, Near-rings, North-Holland Mathematical Studies, Amsterdam, 1983.

[3] J. R. Clay, Nearrings: Genesis and Applications, Oxford Science Publications, New York, 1992.