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On the Achromatic Number of Certain Distance Graphs

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Abstract

The achromatic number for a graph $G = (V, E)$ is the largest integer m such that there is a partition of V into disjoint independent sets $(V_1, \ldots, V_m)$ satisfying the condition that for each pair of distinct sets V_i, V_j, $V_i \cup V_j$ is not an independent set in G. In this paper, we present $O(1)$ approximation algorithms to determine the achromatic number of certain distance graphs.

1 Introduction

A proper coloring of a graph G is an assignment of colors to the vertices of G such that adjacent vertices are assigned different colors. A proper coloring of a graph G is said to be *complete* if, for every pair of colors i and j, there are adjacent vertices u and v colored i and j respectively. The *achromatic number* of the graph G is the largest number m such that G has a complete coloring

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with m colors. Thus the achromatic number for a graph $G = (V, E)$ is the largest integer m such that there is a partition of V into disjoint independent sets $(V_1,\ V_2, \ldots,\ V_m)$ such that for each pair of distinct sets $V_i, V_j,\ V_i \cup V_j$ is not an independent set in G.
Graph coloring problem is expected to have wide variety of applications such as scheduling, frequency assignment in cellular networks, timetabling, crew assignment etc. Small Communication Time task systems show that the achromatic number of the co-comparability graph is an upper bound on the minimum number of processors [1].

2 Overview of the paper

Harary, Hedetniemi and Prins introduced the *achromatic number* [6]. The articles by Edwards [4] contain a huge collection of references of research papers related to achromatic problem.

Computing the achromatic number of a general graph was proved to be *NP-complete* by Yannakakis and Gavril [10]. A simple proof of this fact appears in [5]. It was further proved that the achromatic number problem remains *NP*-complete even for connected graphs which are both interval graphs and cographs simultaneously [2]. Cairnie and Edwards [3] and Edwards and McDiarmid show that the problem is *NP*- hard even on trees. Moreover, it is polynomially solvable for paths, cycles [4], union of paths and gives the approximation algorithm for the achromatic number problem on bipartite graphs.

Since the achromatic optimization problem is *NP*-hard, most of the research studies related to achromatic problem focus on approximation algorithms. It is stated in [3] that "for achromatic numbers, there appear to be only a few results on special graphs apart from those for paths and cycles". Geller and Kronk [5] proved that there is almost optimal coloring for families of paths and cycles. This result was extended to bounded degree trees [3]. Roichman [8] gives a lower bound on the achromatic number of Hypercubes . In this paper, we determine an approximation algorithm for the achromatic number of certain distance graphs.

3 Main results

Lemma 3.1. *Any partial complete coloring can be extended to a complete coloring of the entire graph.*

By this lemma, the achromatic number of a subgraph H remains a lower bound for the achromatic number of the given graph. This key observation has motivated us to design efficient approximation algorithms for certain interconnection networks. Throughout the paper our strategy is as follows.

Strategy: We identify an induced subgraph of the given graph whose achromatic number is easily computable. The achromatic number of this subgraph is then used as a lower bound for the achromatic number of the given graph.

Mesh Network The mesh-connected computer, one of the most popular architectures refers to a grid like topology, in which processors are placed in a square or rectangular grid, with each processor being connected by a communication link to its neighbors to four directions. Being planar graphs, mesh connected computers have easy physical layout.

Definition 3.2. *The topology of a network whose components are all connected directly to every other component is called a Mesh Network.*
A mesh is embedded in many graphs. This motivated us to study the achromatic number for a mesh network.

Theorem 3.3. *Let $v_{i,j}$ denote the vertex in the i th row and j th row of $M_{m\times n}$, $1 \leq i \leq m, 1 \leq j \leq n$. Let $l(v)$ denote the label of vertex v. Label the vertices inductively as follows.*

$l(v_i, 1) = i, 1 \leq i \leq m, l(v_{i,j}) = (l(v_{i,j-1}) + j) \mod m,\ 1 \leq i \leq m,$
$2 \leq j \leq \lceil \frac{n}{2} \rceil$

For $1 \leq i \leq m$,

$$l(v_{i,j-1}) = \begin{Bmatrix} i & \text{i is odd} \\ m+1 & otherwise \end{Bmatrix}$$

and

$$l(v_{i,n}) = \begin{Bmatrix} m+1 & \text{i is odd} \\ i & otherwise \end{Bmatrix}$$

It is easy to verify that the labeling determines an achromatic labeling. Then for $m > n, \psi(M_{m,n}) \geq m + 1$.

The proof of the following theorem is straightforward, since $M_{m,n}$ has $2mn - m - n$ edges.

Theorem 3.4. *Let $M_{m,n}$ denote the mesh on m rows and n columns. Then for $m > n, \psi(M_{m,n}) \leq 4m + 1$.*

Theorem 3.5. *Let $M_{m,n}$ denote the mesh on m rows and n columns. Then for $m > n, m + 1 \leq \psi(M_{m,n}) \leq 4m + 1$.*

Proof: Follows from Theorem 3.3 and Theorem 3.4.

4 Distance graphs

In geometric graph theory, a **unit distance graph** is a graph formed from a collection of points in the Euclidean plane by connecting two points by an edge whenever the distance between the two points is exactly one. In general, unit distance graphs are called as distance graphs. In this paper, we consider the distance graphs namely prism graph and stacked prism graph.

4.1 Prism Graph

Definition 4.1. *A prism graph, denoted by Y_n, sometimes also called a circular ladder graph is a graph corresponding to the skeletonof an n - prism. Prism graphs are therefore both planar and polyhedral. An n - prism graph has 2 n vertices and 3 n edges. A prism graph is isomorphic to the graph Cartesian product P_2C_n where P_2 is the path graph on two vertices and C_n is the cycle on n vertices. As a result, it is a unit distance graph.*

Theorem 4.2. *Let $G = Y_n = P_2C_n$ be a prism graph and $|V(G)| = 2n$. Then for $n > 2$, $\psi(G = Y_n) \geq n + 1$.*

Proof: Using the fact that mesh is embedded in prism graph, even after having wraparound, the coloring is proper by Theorem 3.3. Hence, we obtain the required bound for the achromatic number of G.
The number of edges in a prism graph is $3n$ implies the following theorem.

Theorem 4.3. *Let $G = Y_n = P_2C_n$ be a prism graph and $|V(G)| = 2n$. Then for $n > 2$, $\psi(G = Y_n) \leq \frac{1+\sqrt{24n+1}}{2}$.*

Theorem 4.4. *Let $G = Y_n = P_2C_n$ be a prism graph and $|V(G)| = 2n$. Then for $n > 2$, $n + 1 \leq \psi(G) \leq \frac{1+\sqrt{24n+1}}{2}$.*

Theorem 4.5. *There is an O(1)-approximation algorithm to determine the achromatic number of $G = Y_n = P_2C_n$.*

Proof: The expected achromatic number for $G = Y_n = P_2 C_n$ is $\frac{1+\sqrt{24n+1}}{2}$ and the achromatic number realized is $n+1$. Hence $n+1 \leq \psi(G) \leq \frac{1+\sqrt{24n+1}}{2}$. This proves the theorem.

4.2 Stacked Prism Graph

Definition 4.6. *A stacked (or generalized) prism graph $Y_{m,n}$ is a simple graph given by the graph Cartesian product $Y_{m,n} = C_m P_n$ for positive integers m, n with $m \geq n \geq 3$. This graph can therefore be viewed as connecting n concentric cycle graphs C_m along spokes. $Y_{m,n}$ has mn vertices and $m(2n-1)$ edges.*

Theorem 4.7. *Let $G = Y_{m,n} = C_m P_n$ be a stacked (or generalized) prism graph and $|V(G)| = mn$. Then for $m \geq n \geq 3$, $\psi(G = Y_{m,n}) \geq m+1$.*

Proof: Using the fact that a mesh is embedded in a stacked (or generalized) prism graph, even after having wraparound, the coloring is proper by Theorem 3.3. Hence we obtain the required bound for the achromatic number of G.∎

Theorem 4.8. *Let $G = Y_{m,n} = C_m P_n$ be a stacked (or generalized) prism graph and $|V(G)| = mn$. Then for $m \geq n \geq 3$, $\psi(G = Y_{m,n}) \leq \frac{1 \pm \sqrt{1+8m(2n-1)}}{2}$.*

Theorem 4.9. *Let $G = Y_{m,n} = C_m P_n$ be a stacked (or generalized) prism graph and $|V(G)| = mn$. Then for $m \geq n \geq 3$, $m+1 \leq \psi(G) \leq \frac{1 \pm \sqrt{1+8m(2n-1)}}{2}$.*

Theorem 4.10. *There is an O(1)-approximation algorithm to determine the achromatic number of $G = Y_{m,n}$ for $m \geq n \geq 3$.*

Proof: The expected achromatic number for $G = Y_{m,n} = C_m P_n$ is $\frac{1 \pm \sqrt{1+8m(2n-1)}}{2}$ and the achromatic number realized is $m+1$. Hence, $m+1 \leq \psi(G) \leq \frac{1 \pm \sqrt{1+8m(2n-1)}}{2}$. This proves the theorem.

5 Conclusion

In this paper, we presented an $O(1)$-approximation algorithms to determine the achromatic number of certain distance graphs. Finding efficient approximation algorithms to determine achromatic number for other interconnection networks is quite challenging.

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