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# Extraction of Cantor One-Third Set from Stern-Brocot Sequence

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**Abstract**

In this paper, we define and discuss the properties of Cantor one-third set and Stern-Brocot sequence withthe principal aim of extracting the former from the latter.

## 1 Introduction

The Stern-Brocot tree has received much attention recently due to its deep connections with physical chemistry [4]. The Stern-Brocot tree was discovered by Stern [1] in 1858 and, independently, by Brocot [2] in 1861. It was originally used by Brocot to design gear systems with a gear ratio close to some desired value (like the numbers of seconds in a day) by finding a ratio of smooth numbers (number that decomposes into small prime factors) near that value. Since smooth numbers factor into small primes, several

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small gears could be connected in a sequence to generate an effective ratio of the product of their teeth, thus creating a relative small gear train while minimizing its error[3].

The Stern-Brocot tree begins with the numbers $\frac{0}{1}$ and $\frac{1}{0}$. The proceeding levels of the tree are formed by including the mediant fraction $\frac{a+c}{b+d}$ between every pair of neighbor values $\frac{a}{b}$ and $\frac{c}{d}$ and the procedure is repeated. Retracing the tree upward gives a series of progressively worse rational approximations with decreasing denominators. Here we try to extract the Farey sequence from the Stern-Brocot tree [3].

**Definition 1.1.** *(Stern-Brocot Sequence) [5]*
*With* $s_{0,1} = 0$ *and* $s_{0,2} = 1$*, we define for* $n \geq 0$*,* $S_n = \{s_{n,1}, s_{n,2}, s_{n,3} \dots s_{n,2^n+1}\}$ *as the sequence for which, for* $k \geq 1$*,* $n > 0$*,*

$S_{n,2k-1} = s_{n-1,})$ *and* $s_{n,2k} = s_{n-1,k} + s_{n-1,k+1}$

*Similarly, with,* $q_0, 1 = 1, q_0, 2 = 0$*, we define* $Q_n$*. Then the sequence defined by*

$H_n = \{h_{n,1}, h_{n,2}, h_{n,3} \dots h_{n,2^n+1}\}$*, where* $h_{n,i} = \frac{s_{n,i}}{q_{n,i}}$

*is called the Stern-Brocot Sequence of order* $n$*. It represents the sequence containing both the first* $n$ *generations of mediants based on* $H_0$ *and the terms of* $H_0$ *itself[4]*

**Definition 1.2.** *(Levels of the Stern-Brocot Tree) [5]*

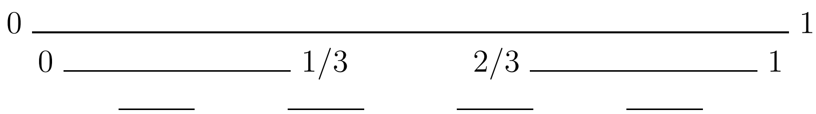
*Let* $H_0$ *be level* $0$ *of the Stern-Brocot Tree. For* $n > 0$*, level* $n$ *of the Stern-Brocot tree is defined as med* $H_{n-1}$*, where*

$Med\, H_{n-1} = \langle (h_{n-1,1} \oplus h_{n-1,2}), (h_{n-1,2} \oplus h_{n-1,3}) \dots\dots\dots\dots (h_{n-1,2^{n-1}} \oplus h_{n-1,2^{n-1}+1}) \rangle$

*and* $\oplus$ *is the child's addition operator where by* $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$

**Cantor Middle One-Third Set:**

As is well known, To construct this set (denoted by $C_3$), we begin with the interval $[0, 1]$ and remove the open set $\left(\frac{1}{3}, \frac{2}{3}\right)$ from the closed interval $[0, 1]$. The set of points that remain after the first step will be called $K_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. In the second step, remove the middle thirds of the two segments of $K_1$; that is, remove $\left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)$ and set $K_2 = \left[0, \frac{1}{9}\right] \cup \left(\frac{2}{9}, \frac{3}{9}\right) \cup \left(\frac{6}{9}, \frac{7}{9}\right) \cup \left(\frac{8}{9}, 1\right)$ be what remains after the first two steps. Delete the middle thirds of the four remaining segments of $K_2$ to get $K_3$. Repeating this process, the limiting set $C_3 = K_\infty$ is called the Cantor middle-$\frac{1}{3}$ set. It is presented below.




Thus, the cantor set is produced by the iterated process of removing the middle third from the previous segments.

**Extraction of Cantor one third Sequence from Stern-Brocot Sequence**

The extraction of Farey Sequence from Stern-Brocot sequence is obtained by the following procedure. Assume that

- $I_N$ denotes the $N^{th}$ level of the Stern-Brocot Sequence; that is, $I_1 = \left\{\frac{0}{1}, \frac{1}{0}\right\}$ , $I_2 = \left\{\frac{0}{1}, \frac{1}{1}\frac{1}{0}\right\}$, etc.
- $F_N$ denotes the $N^{th}$ level of the Non Reducable Farey Subsequence.
- $\gamma(I_N)$ denotes the $N^{th}$ term of the Fibonacci sequence $0, 1, 2, 3, 4, 5, \ldots$
- Let us define $\gamma(I_0) = 0, \gamma(I_1) = 1, \gamma(I_2) = 1, \gamma(I_3) = 2, \gamma(I_4) = 3, \gamma(I_5) = 5$ and so on following the Fibonacci Sequence.
- The starting values of $I$ in the union should be smaller than the ending value $\gamma(I_N)$ otherwise omit the corresponding unions.

**Define the removable term $F_N$ to be for all values of $N \geq 1$ :**

$$F_N = \{I_{N+1}\} - \left[\bigcup_{i=0}^{\gamma(I_{N+1})} \bigcup_{k=i-1}^{\gamma(I_{N+1})} \left[\frac{k+1}{(k+1)-i}\right] \cup \bigcup_{i=1}^{\gamma(I_N)} \bigcup_{K=N}^{\gamma(I_{N+1})} \left[\frac{i+1}{k+1}\right]\right] \quad (1.1)$$

where $\gamma(I_N) = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^{N+1} - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^{N+1}$ follows the Fibonacci sequence terms starting from 0.

**Iteration 1: (N=5)**

We have from the General Case

$$F_5 = \{I_{5+1}\} - \left[\left\{\bigcup_{i=1}^{\gamma(I_6)} \left(\bigcup_{k=i-1}^{\gamma(I_6)} \left[\frac{k+1}{(k+1)-i}\right]\right)\right\} \cup \left\{\bigcup_{i=1}^{\gamma(I_5)} \left(\bigcup_{K=N}^{\gamma(I_6)} \left[\frac{i+1}{k+1}\right]\right)\right\}\right]$$

$$F_5 = \{I_6\} - \left[\left\{\bigcup_{i=1}^{8} \left(\bigcup_{k=i-1}^{8} \left[\frac{k+1}{(k+1)-i}\right]\right)\right\} \cup \left\{\bigcup_{i=1}^{5} \left(\bigcup_{K=5}^{8} \left[\frac{i+1}{k+1}\right]\right)\right\}\right]$$

$$F_5 = \left\{\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{4}, \frac{4}{3}, \frac{7}{5}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, \frac{7}{4}, \frac{2}{1}, \frac{7}{3}, \frac{5}{2}, \frac{8}{3}, \frac{3}{1}, \frac{7}{2}, \frac{4}{1}, \frac{5}{1}, \frac{1}{1}, \frac{1}{0}\right\}$$

$$-\left\{\frac{1}{0},\frac{2}{1},\frac{3}{2},\frac{4}{3},\frac{5}{4},\frac{6}{5},\frac{7}{6},\frac{8}{7},\frac{9}{8},\frac{2}{0},\frac{3}{1},\frac{5}{3},\frac{7}{5},\frac{9}{7},\frac{3}{0},\frac{4}{1},\frac{5}{2},\frac{7}{4},\frac{8}{5},\frac{4}{0},\frac{5}{1},\frac{7}{3},\frac{9}{5},\frac{5}{0},\frac{6}{1},\frac{7}{2},\frac{8}{3},\frac{9}{4},\frac{6}{0},\frac{7}{1},\frac{7}{0},\frac{8}{1},\frac{9}{2},\frac{8}{0},\frac{9}{1},\frac{2}{7},\frac{2}{9},\frac{3}{7},\frac{3}{8},\frac{4}{7},\frac{4}{9},\frac{5}{6},\frac{5}{7},\frac{5}{8},\frac{5}{9},\frac{6}{7}\right\}$$

$F_5 = \left\{\frac{0}{1},\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{2}{3},\frac{3}{4},\frac{4}{5},\frac{1}{1}\right\}$

Also, let $\beta_l$ ,$\beta_l^c$ be removable terms of the iterations $l \geq 2$

$$\beta_l = \left\{\frac{3^{l-1}+1}{3^l}, \frac{3^{l-1}+2}{3^l}, \frac{3^{l-1}+1}{3^l} \ldots, \frac{3^{l-1}+\left(3^{l-1}-1\right)}{3^l}\right\}$$

$$\beta_l^c = \left\{\frac{N-3^{l-1}+1}{3^l}, \frac{N-3^{l-1}+2}{3^l}, \ldots, \frac{N-\left(3^{l-1}+\left(3^{l-1}-1\right)\right)}{3^l}\right\}$$

## Generalized Formula for Cantor One-Third Set from Stern-Brocot Sequence

Here, we define $N = 3^k, \quad k \geq 2$.

In general, the formula is valid only for the powers of 3; that is, $(N = 3^k)$.

The lower limits of the unions of $\beta_l, \beta_l^c$ must always be smaller than the upper limits of the unions.

Removal terms $F_N, \beta_l, \beta_l^c$ as stated above.

$$C_{3^k} = \left[\left(F_N \cap \bigcup_{i=2}^{N}\left\{\frac{i-1}{N}\right\}\right) \cup \{0,1\}\right] - \left[\bigcup_{l=2}^{k}\beta_l \cup \bigcup_{l=2}^{k-1}\beta_l^c\right]$$

## Iteration:2(k=3)

Let $N = 3^k = 3^3 = 27$.

Then,

$$C_{3^k} = \left[\left(F_N \cap \bigcup_{i=2}^{N}\left\{\frac{i}{N}\right\}\right) \cup \{0,1\}\right] - \left[\bigcup_{l=2}^{k}\beta_l \cup \bigcup_{l=2}^{k-1}\beta_l^c\right]$$

$$C_{3^3} = \left[\left(F_{27} \cap \bigcup_{i=2}^{27}\left\{\frac{i}{27}\right\}\right) \cup \{0,1\}\right] - \left[\bigcup_{l=2}^{3}\beta_l \cup \bigcup_{l=2}^{2}\beta_l^c\right]$$

$$C_{27} = \left[\left(F_{27} \cap \bigcup_{i=2}^{27}\left\{\frac{i}{27}\right\}\right) \cup \{0,1\}\right] - \left[\beta_2 \cup \beta_3 \cup \beta_2^c\right] \tag{1.2}$$

Consider,

$$\left[\left(F_{27} \cap \bigcup_{i=2}^{27}\left\{\frac{i}{27}\right\}\right) \cup \{0,1\}\right] = \left[\left(F_{27} \cap \left\{\frac{1}{27},\frac{2}{27},\frac{2}{27},\ldots,\frac{25}{27},\frac{26}{27},\frac{27}{27}\right\}\right) \cup \{0,1\}\right]$$

$$= \left\{\frac{1}{27},\frac{2}{27},\frac{2}{27},\ldots,\frac{25}{27},\frac{26}{27},\frac{27}{27}\right\}$$

Moreover,

$$\begin{aligned} \beta_2 &= \left\{\tfrac{4}{27}, \tfrac{5}{27}\right\} \\ \beta_3 &= \left\{\tfrac{3^{3-1}+1}{3^3}, \tfrac{3^{3-1}+2}{3^3}, \dots \tfrac{3^{3-1}+(3^{3-1}-1)}{3^3}\right\} \\ \beta_3 &= \left\{\tfrac{10}{27}, \tfrac{11}{27}, \dots, \tfrac{17}{27}\right\} \end{aligned}$$

Furthermore,

$$\begin{aligned} \beta_l^c &= \left\{\tfrac{27-(3^{3-1}+1)}{3^3}, \tfrac{27-(3^{3-1}+2)}{3^3}, \dots \tfrac{27-\left(3^{3-1}+(3^{3-1}-1)\right)}{3^3}\right\} \\ \beta_l^c &= \left\{\tfrac{22}{27}, \tfrac{23}{27}\right\} \end{aligned}$$

Equation (1.2) becomes,

$$C_{27} = \left\{\frac{0}{27}, \frac{1}{27}, \frac{2}{27}, \dots, \frac{25}{27}, \frac{26}{27}, \frac{27}{27}\right\} - \left\{\frac{4}{27}, \frac{5}{27}, \frac{6}{27}, \dots, \frac{17}{22}, \frac{22}{27}, \frac{23}{27}\right\}$$

$$C_{27} = \left\{\frac{0}{27}, \frac{1}{27}, \frac{2}{27}, \frac{3}{27}, \frac{4}{27}, \frac{5}{27}, \frac{6}{27}, \frac{7}{27}, \frac{8}{27}, \frac{9}{27}, \frac{18}{27}, \frac{19}{27}, \frac{20}{27}, \frac{21}{27}, \frac{22}{27}, \frac{24}{27}, \frac{25}{27}, \frac{26}{27}, \frac{27}{27}\right\}$$

By grouping this set of elements as intervals, we derive the Cantor one-third set of order 27.

$$C_{27} = \left[\tfrac{0}{27}, \tfrac{1}{27}\right] \left[\tfrac{1}{27}, \tfrac{2}{27}\right] \left[\tfrac{2}{27}, \tfrac{3}{27}\right] \left[\tfrac{3}{27}, \tfrac{6}{27}\right] \left[\tfrac{6}{27}, \tfrac{7}{27}\right] \left[\tfrac{7}{27}, \tfrac{8}{27}\right] \left[\tfrac{8}{27}, \tfrac{9}{27}\right] \left[\tfrac{18}{27}, \tfrac{19}{27}\right] \left[\tfrac{19}{27}, \tfrac{20}{27}\right] \left[\tfrac{20}{27}, \tfrac{21}{27}\right] \left[\tfrac{21}{27}, \tfrac{24}{27}\right] \left[\tfrac{24}{27}, \tfrac{25}{27}\right] \left[\tfrac{25}{27}, \tfrac{26}{27}\right], \left[\tfrac{26}{27}, \tfrac{27}{27}\right]$$

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