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Higher Order Fuzzy Initial Value Problem Through Taylor's Method

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Abstract

Taylor's approach explores the approximate solution of higher order Fuzzy linear differential equations. We may obtain solutions by Strong Generalized Differentiability. The consistency, convergence and stability of the proposed system, are demonstrated by some examples with a triangular fuzzy number.

1 Introduction

Taylor polynomial plays a vital role in understanding the numerical method. For Linear Fuzzy differential equations, modern numerical algorithms are also based on the method of Taylor series. The algorithms of Runge-Kutta and Euler's methods are constructed so that they give an approximate solutions and can be revised to have any specified degree of accuracy. The

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Fuzzy Taylor's method is a single step method and does well up to successive derivatives. It is the standard method among numerical algorithms for solving Fuzzy Initial Value Problems (FIVPs). Abbasbandy et al. [3] studied numerical algorithms to solve FDEs using Taylor's method for a particular order p and in turn followed by complete error analysis. Mikaeilvand et al. [2] initiated the fuzzy Taylor's polynomial pantograph equation to find the values for functional argument along with the Fuzzy Initial Value Problems (FIVPs). Taylor's method has been explained by Allahviranloo et al. [1], the Euler method and its local and global truncation error are explained in such a manner to solve FIVPs in the concept of generalized Hukuhara Differentiability. Toksari [5] analyzed the usage of Taylor series for fuzzy multi objective linear fractional programming problems.

2 Preliminaries

This section contains some basic definitions that are needed throughout this paper.

Definition 2.1 A fuzzy set $\tilde{a}$ defined on the set of real numbers R is said to be a fuzzy number if its membership function ã: R $\rightarrow [0,1]$ has the following:

(i) $\tilde{a}$ is convex; i.e, $\tilde{a}\{\lambda x_1 + (1-\lambda)x_2\} \geq \min\{\tilde{a}(x_1), \tilde{a}(x_2)\}$, for all $x_1, x_2 \in R$ and $\lambda \in [0,1]$

(ii) $\tilde{a}$ is normal; i.e., there exists an $x \in R$ such that $\tilde{a}(x) = 1$

(iii) $\tilde{a}$ is piecewise continuous.

Definition 2.2. A triangular fuzzy number $\tilde{A}$ is a fuzzy number specified by $\tilde{A} = (a_1, a_2, a_3)$ with their membership function

$$\tilde{a}(x) = \begin{cases} 0, & x \leq a, \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2, \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x \leq a_3, \\ 0, & x \geq a_3. \end{cases}$$

2.1 Parametric representation of fuzzy numbers

A fuzzy number $\tilde{a} \in F(\mathrm{R})$ can also be represented as a pair $\tilde{a} = (\underline{a}, \bar{a})$ of functions $\underline{a}(\alpha)$ and $\bar{a}(\alpha)$ for $0 \leq \alpha \leq 1$ which satisfy the following requirements:
(i) $\underline{a}(\alpha)$ is a bounded monotone increasing left continuous function.
(ii) $\overline{a}(\alpha)$ is a bounded monotone decreasing left continuous function.

(iii) $\underline{a}(\alpha) \leq \overline{a}(\alpha)$, $0 \leq \alpha \leq 1$.

According to Zedeh's extension principle, if $u, v \in F(R^n)$ and $\lambda \in R$, then $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$ and $[\lambda u]^\alpha = \lambda [u]^\alpha \forall \alpha \in [0,1]$.

Definition 2.4. Let $u, v \in R$. If there exists $w \in R$ such that $u = v \oplus w$, then w is called the $H-$Difference of u and v and is denoted by $u \Theta v$.

Definition 2.5. (Generalized Fuzzy Derivative) [6] Let $F : (a,b) \to R$ and $t_0 \in (a,b)$. We say that F is strongly generalized differentiable at t_0 if there exists as element $F'(t_0) \in R$ such that For $h > 0$ sufficiently small $\exists F(t_0+h)\Theta F(t_0), F(t_0)\Theta F(t_0-h)$, and the limits satisfy $\lim_{h\to 0} \frac{F(t_0+h)\Theta f(t_0)}{h} = \lim_{h\to 0} \frac{F(t_0)\Theta F(t_0-h)}{h} = F'(t_0)$.

Theorem 2.6:
Let $F' : (a,b) \to R$ and $t_0 \in (a,b)$. We say that F' is strongly generalized differentiable at t_0 if there exists as element $F''(t_0) \in R$ such that For $h > 0$ sufficiently small $\exists F'(t_0+h)\Theta F'(t_0), F'(t_0)\Theta F'(t_0-h)$, and the limits satisfy

$$\lim_{h\to 0} \frac{F'(t_0+h)\Theta F'(t_0)}{h} = \lim_{h\to 0} \frac{F'(t_0)\Theta F'(t_0-h)}{h} = F''(t_0)$$

Theorem 2.7: [1] Let $F_1[t,u,v], G_1[t,\hat{u},\hat{v}]$, $F_2[t,\hat{u},\hat{v}]$, and $G_2[t,\hat{u},\hat{v}]$, belongs to $c^4(k)$ and let the partial derivative of F_1, G_1, F_2, G_2 be bounded over k. Then, for arbitrary fixed α: $0 \leq \alpha \leq 1$, the the approximate solutions converge uniformly in it to the exact solutions.

3 Fuzzy Initial Value Problem

An initial value problem is a system of an ordinary differential equations together with the initial conditions. Consider a function of Nth order fuzzy differential equations with initial conditions

$$\tilde{y}^n(t) = \tilde{f}\left(t, \tilde{y}(t), \tilde{y}'(t), ..., \tilde{y}^{(n-1)}(t)\right)$$
$$\tilde{y}(t_0) = \tilde{x}_0, ..., \tilde{y}^{(n-1)}(t_0) = \tilde{x}_0$$

By using the Zadeh's extension principle, we have the membership function

$$[\tilde{f}(t,y)]^\alpha = \tilde{f}(t,[y]^\alpha) = (\min \tilde{f}(t,[\underline{y_\alpha}, \overline{y_\alpha}], \max \tilde{f}(t,[\underline{y_\alpha}, \overline{y_\alpha}])$$

3.1 Fuzzy Taylor's Method

Case (i): If $\tilde{y}(t)$ and $\tilde{y}'(t), \tilde{y}''(t), \tilde{y}'''(t), \cdots, \tilde{y}^{(n)}(t)$ are (i) differentiable, then the Taylor's method of order p is based on the expansion

$$\tilde{y}(t+h;\alpha) = \sum_{i=0}^{p} \frac{h^i}{i!}\tilde{y}^i(t,\alpha), \tag{3.1}$$

where $\tilde{y}(t,\alpha) = (\underline{y}, \overline{y})$.

$$\text{We define} \quad \begin{cases} \tilde{F}[t,\tilde{y},\alpha] = \sum_{i=0}^{p-1} \dfrac{h^i}{(i+1)!}\underline{y}^i(t,\tilde{y};\alpha), \\ \tilde{G}[t,\tilde{y},\alpha] = \sum_{i=0}^{p-1} \dfrac{h^i}{(i+1)!}\overline{y}^i(t,\tilde{y};\alpha). \end{cases} \tag{3.2}$$

The exact and approximate solutions at $t_n, 0 \leq n \leq N$ are denoted by $[\tilde{Y}(t_n)]_\alpha = [\underline{Y}(t_n;\alpha), \overline{Y}(t_n;\alpha)]$ and $[\tilde{y}(t_n)]_\alpha = [\underline{y}(t_n;\alpha), \overline{y}(t_n;\alpha)]$ respectively. The solution is calculated at the grid points of $a = t_0 \leq t_1 \leq t_2 \leq \cdots t_N = b$ and $h = \dfrac{(b-a)}{N} = t_{i+1} - t_i$.

Case (ii): If $\tilde{y}(t)$ and $\tilde{y}'(t), \tilde{y}''(t), \tilde{y}'''(t), \cdots, \tilde{y}^{(n)}(t)$ are (ii) differentiable, then the Taylor's method of order p is

$$\underline{y}(t_{n+1};\alpha) = \underline{y}(t_n;\alpha) + (-1)^{(n)} h \sum_{i=0}^{p-1} \frac{h^i}{(i+1)!}\underline{y}^i(t,\tilde{y};\alpha)$$

$$\bar{y}(t_{n+1};\alpha) = \bar{y}(t_n;\alpha) + (-1)^{(n)} h \sum_{i=0}^{p-1} \frac{h^i}{(i+1)!}\overline{y}^i(t,\tilde{y};\alpha)$$

4 Numerical Examples

Example 1 Solve the following fourth-order fuzzy initial value problem:

$$\begin{cases} y^{iv}(t) = y(t), \\ y(0) = y'(0) = y''(0) = y'''(0) = (\alpha - 1, 1 - \alpha) \text{ for the range} \\ 0 \le t \le 1 \text{ in the step of } 0.1 \end{cases}$$

Solution:
Case (i): If $y, y', y'', y''', \cdots, y^n$ are $(i)gh$ differentiable fuzzy functions, then the exact solution of the given problem is

$$\underline{Y}(t,\alpha) = (\alpha - 1)e^t$$
$$\overline{Y}(t,\alpha) = (1 - \alpha)e^t$$

Hence, by equation (3.2), the approximate solution of the given problem

Table 1: The values of $\underline{y}(t,\alpha)$ by Fuzzy Taylor's method for $h = 0.1$,$t = 1$ at various α-level set under $(i)gh$ differentiable.

h	$\underline{y}(t,\alpha)$ for $\alpha = 0$	$\underline{y}(t,\alpha)$ for $\alpha = 0.2$	$\underline{y}(t,\alpha)$ for $\alpha = 0.4$	$\underline{y}(t,\alpha)$ for $\alpha = 0.6$	$\underline{y}(t,\alpha)$ for $\alpha = 0.8$	$\underline{y}(t,\alpha)$ for $\alpha = 1$
0.2	-1.2214	-0.9771	-0.7328	-0.4886	-0.2443	0
0.4	-1.4918	-1.1935	-0.8951	-0.5967	-0.2984	0
0.6	-1.8221	-1.4577	-1.0933	-0.7288	-0.3644	0
0.8	-2.2255	-1.7805	-1.3354	-0.8902	-0.4451	0
1	-2.7183	-2.1747	-1.6310	-1.0873	-0.5437	0

Table 2: The values of $\overline{y}(t,\alpha)$ by Fuzzy Taylor's method for $h = 0.1$, $t = 1$ at various α-level set under $(i)gh$ differentiable.

h	$\overline{y}(t,\alpha)$ for $\alpha = 0$	$\overline{y}(t,\alpha)$ for $\alpha = 0.2$	$\overline{y}(t,\alpha)$ for $\alpha = 0.4$	$\overline{y}(t,\alpha)$ for $\alpha = 0.6$	$\overline{y}(t,\alpha)$ for $\alpha = 0.8$	$\overline{y}(t,\alpha)$ for $\alpha = 1$
0.2	1.2214	0.9771	0.7328	0.4886	0.2443	0
0.4	1.4918	1.1935	0.8951	0.5967	0.2984	0
0.6	1.8221	1.4577	1.0933	0.7288	0.3644	0
0.8	2.2255	1.7805	1.3354	0.8902	0.4451	0
1	2.7183	2.1747	1.6310	1.0873	0.5437	0

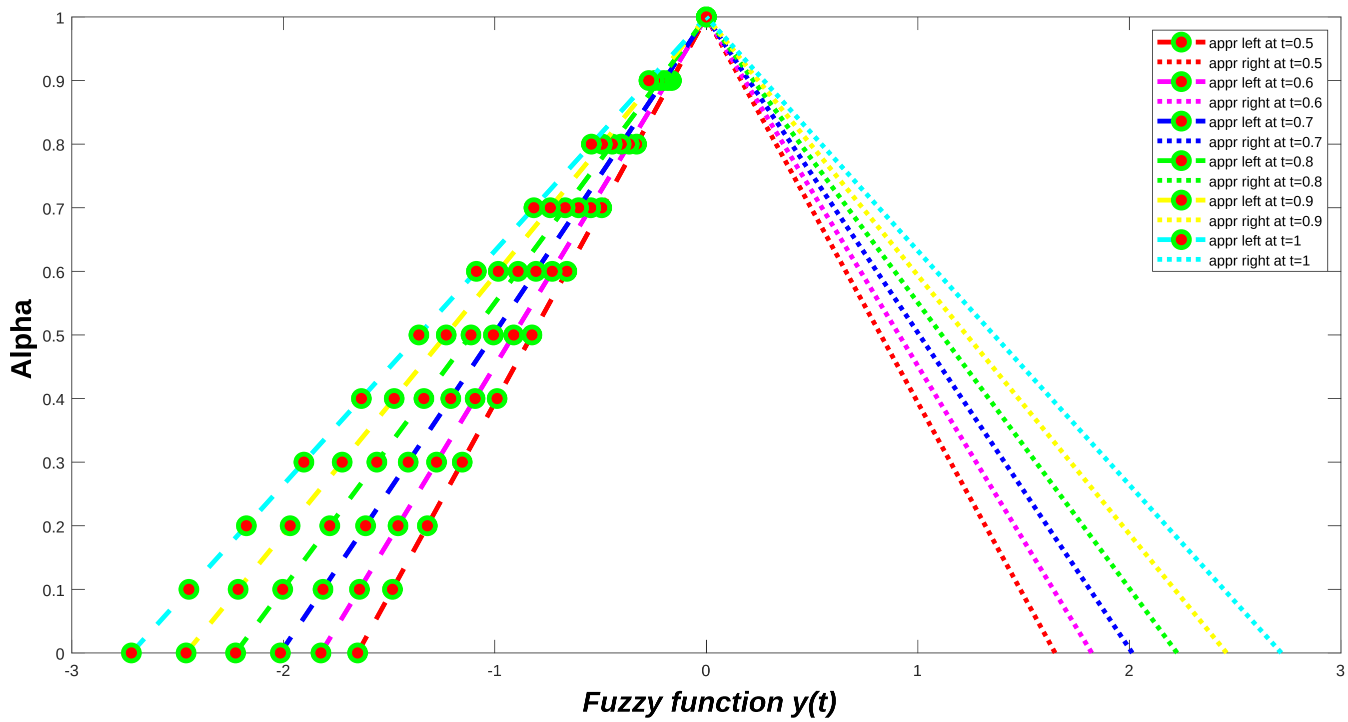

Figure 1: Solution of fourth order linear fuzzy differential equation using Taylor's method at various values of t under $(i)gh$ differentiability.

by Taylor's method of order p is

$$\underline{y}(t_{n+1};\alpha) = \underline{y}(t_n;\alpha) + h\sum_{i=0}^{p-1}\frac{h^i}{(i+1)!}\underline{y}^i(t,y;\alpha)$$

$$\overline{y}(t_{n+1};\alpha) = \overline{y}(t_n;\alpha) + h\sum_{i=0}^{p-1}\frac{h^i}{(i+1)!}\overline{y}^i(t,y;\alpha)$$

Case (ii): If $y, y', y'', y''', \cdots, y^n$ are $(ii)gh$ differentiable fuzzy functions, then the exact solution of (3.2) is given by

$$\underline{Y}(t,\alpha) = (\alpha-1)e^{-t} \quad \text{and} \quad \overline{Y}(t,\alpha) = (1-\alpha)e^{-t}$$

Using the Taylor's method of order p, we have the approximate solution of the given problem as follows:

Table 3: The values of $\underline{y}(t,\alpha)$ by Fuzzy Taylor's method for $h = 0.1, \quad t = 1$ at various α-level set under $(ii)gh$ differentiable.

h	$\underline{y}(t,\alpha)$ for $\alpha = 0$	$\underline{y}(t,\alpha)$ for $\alpha = 0.2$	$\underline{y}(t,\alpha)$ for $\alpha = 0.4$	$\underline{y}(t,\alpha)$ for $\alpha = 0.6$	$\underline{y}(t,\alpha)$ for $\alpha = 0.8$	$\underline{y}(t,\alpha)$ for $\alpha = 1$
0.2	-0.8187	-0.6550	-0.4912	-0.3275	-0.1637	0
0.4	-0.6703	-0.5363	-0.4022	-0.2681	-0.1341	0
0.6	-0.5488	-0.4390	-0.3293	-0.2195	-0.1098	0
0.8	-0.4493	-0.3595	-0.2696	-0.1797	-0.0899	0
1	-0.3679	-0.2943	-0.2207	-0.1471	-0.0736	0

Table 4: The values of $\overline{y}(t,\alpha)$ by Fuzzy Taylor's method for $h = 0.1, \; t = 1$ at various α-level set under $(ii)gh$ differentiable.

h	$\underline{y}(t,\alpha)$ for $\alpha = 0$	$\underline{y}(t,\alpha)$ for $\alpha = 0.2$	$\underline{y}(t,\alpha)$ for $\alpha = 0.4$	$\underline{y}(t,\alpha)$ for $\alpha = 0.6$	$\underline{y}(t,\alpha)$ for $\alpha = 0.8$	$\underline{y}(t,\alpha)$ for $\alpha = 1$
0.2	0.8187	0.6550	0.4912	0.3275	0.1637	0
0.4	0.6703	0.5363	0.4022	0.2681	0.1341	0
0.6	0.5488	0.4390	0.3293	0.2195	0.1098	0
0.8	0.4493	0.3595	0.2696	0.1797	0.0899	0
1	0.3679	0.2943	0.2207	0.1471	0.0736	0

Tables 3 and 4 show the left and right level functions $\underline{y}(t,\alpha)$ and $\overline{y}(t,\alpha)$ by fuzzy Taylor's method of various α level in the step size 0.1 under $(ii)gh$ differentiable.

From the above tables $(3.2), (3.3), (3.4), (3.5)$, we see that for $t = 1$, the approximate solution of (3.2) using a fuzzy version of Taylor's method at $h = 0.1$ associated with the derivative of $(i)gh$ and $(ii)gh$ differentiability.

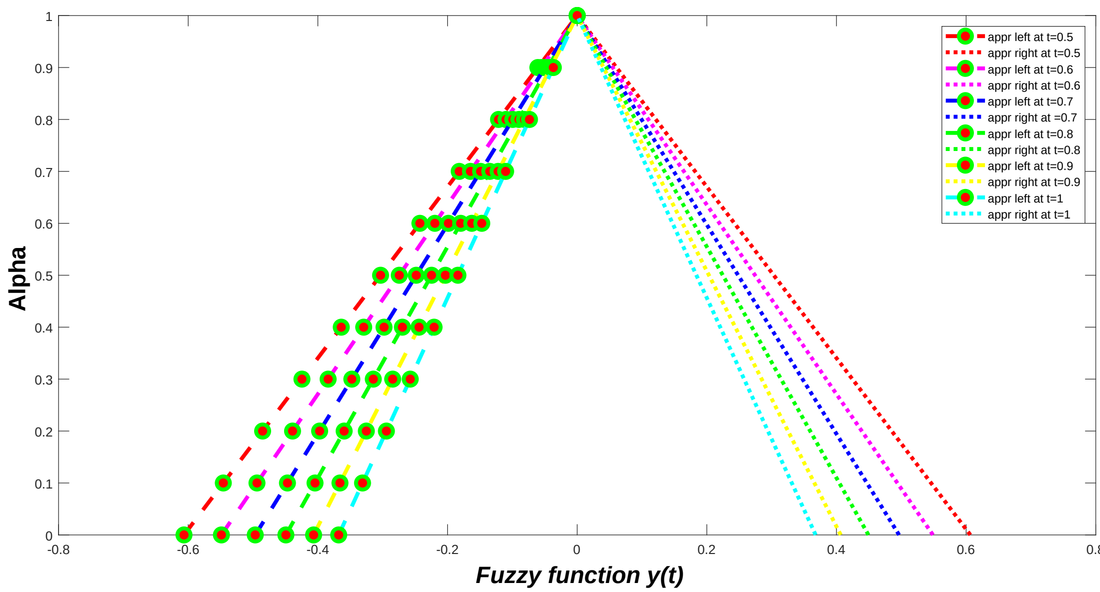

Figure 2: Solution of fourth order linear fuzzy differential equation using Taylor's method at various values of t under $(ii)gh$ differentiability.

Figure 2 shows a graph of the solution of fuzzy Taylor's method at varies values of the under $(ii)gh$ differentiable.

Remark 5.5.1: By summing the above infinite series at any neighboring point $x0 + h$, the value of y can be obtained. The summation has to be terminated in any practical computation after some finite number of terms. After the $H-$th derivative term, if the series has been terminated, then it is of order h. As the polynomials approximate, that formula is known as a Taylor approximation to y and is a suitable one. Such polynomials are easier to "evaluate the value and distinguish and integrate. It gives a simple adaptation of classical calculus to establish the solution as an infinite series". It is an efficient one-step approach if we can easily find the successive derivatives. Where t does not need to be grid point, it is easy to obtain the value for any x. Truncation error is managed by the correct order analysis of the derivatives. The Taylor series approach is a generic one that can be used to test certain approaches.

5 Conclusion

In this paper, Strong Generalized differentiability was applied to find a numerical solution of fuzzy differential equations (FDEs) for a triangular fuzzy number, using the Taylor series method. From the above examples, we saw that the approximate solution of fuzzy differential equations is similar to the exact solution for $h = 0.1$. We shall consider the approaches to higher-order in our future research.

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