International Journal of Mathematics and
Computer Science, **15**(2020), no. 1, 1029–1044

New contraction embedded with simulation function and cyclic (α, β)-admissible in metric-like spaces

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(Received June 2, 2020, Accepted August 27, 2020)

Abstract

In this paper, we introduce the concept of cyclic (α, β)-admissible $\mathcal{Z}$-contraction mapping with respect to ζ. We also establish the existence and uniqueness of fixed points for this class of mappings in metric-like spaces. This work generalizes and extends some theorems in the literature. An example and some consequences are given to support the obtained results.

1 Introduction and preliminaries

The significance of fixed point theory lies in proving the existences and uniqueness of solutions for many problems of Applied Sciences such as Dynamic system, Chemistry, Economics, and Engineering. Over the a long time, Several mathematicians have been formulated and established the famous contractive Banach contraction principle in many different directions, either by generalizing the domain of the mapping or by weakening the contractive condition or in some cases indeed both, see for example ([10]-[16]).

Key words and phrases: Cyclic (α, β)-admissible, $\mathcal{Z}$-contraction mapping(Simulation function), fixed point, metric-like spaces.
AMS (MOS) Subject Classifications: Primary 47H10, Secondary 54H25.
ISSN 1814-0432, 2020, http://ijmcs.future-in-tech.net

First, Harandi [8] introduced a new extension of the concept of partial metric space, called a metric-like space. He established the existence and uniqueness of fixed points in a metric-like space as well as in a partially ordered metric-like space. Clearly, this setting is a generalization of the standard space(metric space). Several authors discussed the existence of fixed and common fixed point in metric-like space (for instance see [4]-[6]).

Also, the notation of $\mathcal{Z}$-contraction was presented in 2015 by Khojasteh et al. [9] . This concept is a type of nonlinear contraction defined by using a specific function, called simulation function. Consequently, they proved the existence and uniqueness of fixed point for $\mathcal{Z}$-contraction mappings (see [9],Theorem 2.8).

Otherwise, Alizadeh et al. [1] introduced the concept of cyclic (α, β)-admissible mapping and proved some new fixed point results which generalize and modify some recent results in the literature.

Definition 1.1. *[8] Let X is a nonempty set. A function $\sigma : X \times X \to [0, \infty)$ is said to be a metric-like space (or a dislocated metric) on X if for any $x, w, y \in X$, the following conditions hold:*

(σ_1) $\sigma(x, y) = 0$ *implies that* $x = y$;

(σ_2) $\sigma(x, y) = \sigma(x, y)$;

(σ_3) $\sigma(x, y) \leq \sigma(x, z) + \sigma(z, y)$.

The pair (X, σ) is called a metric-like space.

It is clear that every metric space and partial metric space is a metric-like space, but the converse is not true.

Example 1.2. *Let $X = \{0, 1\}$ and*

$$\sigma(x, y) = \begin{cases} 2, & \text{if } x = y = 0; \\ 1, & \text{otherwise.} \end{cases}$$

Then (X, σ) is a metric-like space, but it is not a partial metric space. Note that $\sigma(0, 0) \nleq \sigma(0, 1)$.

Example 1.3. *Let $\sigma_k : X \times X \to \mathbb{R}^+$ $(k \in \{1, 2\})$ be a mapping in $X = \mathbb{R}$, then σ_k defined by*

1. $\sigma_1(x, y) = \mid x \mid + \mid y \mid + a,$
2. $\sigma_2(x, y) = x^2 + y^2.$

Then, $\sigma_1(x, y), \sigma_2(x, y)$ *are metric like spaces on* X*, where* $a \geq 0$ *and* $b \in \mathbb{R}$*.*

Moreover, each metric-like σ on X generates a topology τ_σ on X whose base is the family of open σ-balls

$$B_\sigma(x, \epsilon) = \{y \in X : \mid \sigma(x, y) - \sigma(x, x) \mid < \epsilon\},\ for\ all\ x \in X\ and\ \epsilon > 0.$$

The metric-like space (X, σ) is called complete if for each Cauchy sequence $\{X_n\}_\infty^n$, there is some $y \in Y$ such that

$$\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x) = \lim_{n,m\to\infty} \sigma(x_n, x_m).$$

A subset A of a metric-like space (X, σ) is bounded if there is a point $b \in X$ and a positive constant K such that $\sigma(a, b) \leq K$ for all $a \in A$.

Definition 1.4. *([3]) Let* (X, σ) *be a metric like spaces.*

(a) *Any sequence* $\{x_n\}$ *in metric like spaces is a Cauchy sequence if* $\lim_{n,m\to\infty} \sigma(x_n, x_m)$ *exists and is finite.*

(b) (X, σ) *is complete if every Cauchy sequence* $\{x_n\}$ *in metric like spaces converges with regard to* τ_σ *to a point* $x \in X$*; that is,*

$$\lim_{n\to\infty} \sigma(x, x_n) = \sigma(x, x) = \lim_{n,m\to\infty} \sigma(x_n, x_m).$$

Definition 1.5. *([3]) Let* $T : (X, \sigma) \to (X, \sigma)$ *be a mapping in metric like spaces* (X, σ)*, then the mapping* T *is continuous if for any sequence* $\{x_n\}$ *in* X *such that* $\sigma(x_n, x) \to \sigma(x, x)$ *as* $n \to \infty$*, we have* $\sigma(Tx_n, Tx) \to \sigma(Tx, Tx)$ *as* $n \to \infty$*.*

Remark 1.6. *Let* $X = \{0, 1\}$ *be endowed with* $\sigma(x, y) = 1$ *for each* $x, y \in X$*. Take* $x_n = 1$ *for each* $n \in \mathbb{N}$*. It is easy to see that* $x_n \to 0$ *and* $x_n \to 1$*. In metric-like spaces, the limit of a convergent sequence is not necessarily unique.*

The following lemma is known and useful for the rest of paper.

Lemma 1.7. *Let* (X, σ) *be a metric-like space. Let* $\{x_n\}$ *be a sequence in* X *such that* $x_n \to x$ *where* $x \in X$ *and* $\sigma(x, y) = 0$*. Then for all* $y \in X$*, we have* $\lim_{n\to\infty} \sigma(x_n, y) = \sigma(x, y)$*.*

Definition 1.8. *[9] A function $\zeta : [0,\infty)\times[0,\infty) \to \mathbb{R}$ is called a simulation function if ζ satisfies the following conditions:*
(ζ_1) $\zeta(0,0) = 0$,
(ζ_2) $\zeta(t,s) < s - t$ for all $t, s > 0$,
(ζ_3) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0,\infty) > 0$, then

$$\lim_{n\to\infty} \sup \zeta(t_n, s_n) < 0.$$

Definition 1.9. *[1] Let $f : X \to X$ be a mapping and $\alpha, \beta : X \to \mathbb{R}^+$ be two functions. We say that f is a cyclic (α,β)-admissible mapping if*

1. *$\alpha(x) \geq 1$ for some $x \in X \Rightarrow \beta(fx) \geq 1$,*
2. *$\beta(x) \geq 1$ for some $x \in X \Rightarrow \alpha(fx) \geq 1$.*

Example 1.10 ([9],[2]). *For $i \in \{1,2,3,4,5,6,7\}$, let $\phi_i : [0,\infty) \to [0,\infty)$ be a continuous function with $\phi_i(t) = 0$ if and only if $t = 0$. Define the functions $\zeta_i : [0,\infty)\times[0,\infty) \to \mathbb{R}$, $i = 1,2,3,4,5,6,7$ as follows*

1. *$\zeta_1(t,s) = \phi_1(s) - \phi_2(t)$ for all $t, s \in [0,\infty)$, where $\phi_1(t) < t \leq \phi_2(t)$ for all $t > 0$.*
2. *$\zeta_2(t,s) = s - \frac{f(t,s)}{g(t,s)}$ for all $t, s \in [0,\infty)$, where $f, g : [0,\infty)^2 \to (0,\infty)$ are continuous functions with respect to each variable such that $f(t,s) > g(t,s)$ for all $t, s > 0$.*
3. *$\zeta_3(t,s) = s - \phi_3(s) - t$ for all $t, s \in [0,\infty)$.*
4. *If $\varphi : [0,\infty) \to [0,1)$ is a function such that $\lim_{t\to r^+} \sup \varphi(t) < 1$ for all $r > 0$, and we define $\zeta_4(t,s) = s\varphi(s) - t$ for all $t, s \in [0,\infty)$.*

In this paper, we introduce the concept of cyclic (α,β)-admissible $\mathcal{Z}$-contractions with respect to ζ. We also establish the existence of fixed points for this class of mappings in metric-like spaces. Our work generalizes and extends some theorems in the literature.

2 main result

In this section, we present the class of cyclic (α,β)-admissible $\mathcal{Z}$-contraction mapping and prove some fixed point theorems on complete metric-like space.

Theorem 2.1. *Let (X,σ) be a metric-like space and $f: X \to X$ be a cyclic (α,β)-admissible $\mathcal{Z}$-contraction mapping if there exist $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(t) < t$ such that:*

$$\zeta(\psi(\sigma(fx, fy)), \psi(M(x,y))) \geq 0 \tag{2.1}$$

for all $x, y \in X$ satisfying $\alpha(x)\beta(y) \geq 1$ where

$$M(x,y) = \max\{\sigma(x,y), (\sigma(x,fx)+\sigma(y,fy))\times 4^{-1}, (\sigma(fx,y)+\sigma(x,fy))\times 4^{-1}\}$$

Assume that

1. *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,*
2. *f is continuous, or*
3. *if $\{x_n\} \subseteq X$ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$,*

then f has a fixed point $z \in X$ such that $\sigma(z,z) = 0$. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in Fix(f)$, then f has a unique fixed point.

Proof. Since f is a cyclic (α,β)-admissible mapping and $\alpha(x_0) \geq 1$ then $\beta(x_1) = \beta(fx_0) \geq 1$ which implies that $\alpha(fx_1) = \alpha(x_2) \geq 1$. By continuing this method, we have $\alpha(x_{2n}) \geq 1$ and $\beta(x_{2n-1}) \geq 1$ for all $n \in \mathbb{N}$. A gain, since f is a cyclic (α,β)-admissible mapping and $\beta(x_0) \geq 1$, we have $\beta(x_{2n}) \geq 1$ and $\alpha(x_{2n-1}) \geq 1$. Then, we deduce

$$\alpha(x_n) \geq 1 \text{ and } \beta(x_n) \geq 1 \text{ for all } n \in \mathbb{N}_0. \tag{2.2}$$

Equivalently, $\alpha(x_{n-1})\beta(x_n) \geq 1$. Applying (2.1), we obtain

$$\begin{aligned} \zeta(\psi(\sigma(fx_{n-1}, fx_n)), \psi(M(x_{n-1},x_n))) &= \zeta(\psi(\sigma(x_n, x_{n+1})), \psi(M(x_{n-1},x_n))) \\ &\geq 0, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} M(x_{n-1},x_n) &= \max\{\sigma(x_{n-1},x_n), (\sigma(x_{n-1},fx_{n-1}) + \sigma(x_n,fx_n)) \times 4^{-1}, && (2.4) \\ &\quad (\sigma(fx_{n-1},x_n) + \sigma(fx_n,x_{n-1})) \times 4^{-1}\} \\ &= \max\{\sigma(x_{n-1},x_n), (\sigma(x_{n-1},x_n) + \sigma(x_n,x_{n+1})) \times 4^{-1}, && (2.5) \\ &\quad (\sigma(x_n,x_n) + \sigma(x_{n+1},fx_{n-1})) \times 4^{-1}\} \\ &= \max\{\sigma(x_{n-1},x_n), (\sigma(x_{n-1},x_n) + \sigma(x_n,x_{n+1})) \times 4^{-1}, (\sigma(x_{n+1},x_{n-1})) \times 4^{-1}\} \\ &\leq \max\{\sigma(x_{n-1},x_n), (\sigma(x_{n+1},x_n) + \sigma(x_n,x_{n+1})) \times 4^{-1}\} \\ &\leq \max\{\sigma(x_{n-1},x_n), \sigma(x_{n+1},x_n)\} && (2.6) \end{aligned}$$

It follows that

$$\zeta(\psi(\sigma(x_n, x_{n+1})), \psi(\max\{\sigma(x_{n-1}, x_n), \sigma(x_{n+1}, x_n)\}) \geq 0, \tag{2.7}$$

If $\sigma(x_n, x_{n+1}) = 0$ for some n, then $x_n = x_{n+1} = fx_n$, that is, x_n is a fixed point of f and so the proof is finished. Therefore, we suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Now, we shall show that $\sigma(x_n, x_{n+1}) \leq \sigma(x_{n-1}, x_n)$. Arguing by contradiction, we assume $\sigma(x_n, x_{n+1}) > \sigma(x_{n-1}, x_n)$. Therefore, we have two cases.

Case 1:$M(\sigma(x_{n-1}, x_n)) = \sigma(x_n, x_{n+1})$. Then

$$\begin{aligned} 0 &\leq \zeta(\psi(\sigma(x_n, x_{n+1})), \psi(\sigma(x_{n-1}, x_n))) \\ &< \psi(\sigma(x_n, x_{n+1})) - \psi(\sigma(x_n, x_{n+1})), \end{aligned}$$

by using the properties of ψ, we have $\sigma(x_{n+1}, x_n) < \sigma(x_n, x_{n+1})$, which is impossible.

Case 2: $M(\sigma(x_{n-1}, x_n)) = \sigma(x_{n-1}, x_n)$. Then

$$\begin{aligned} 0 &\leq \zeta(\psi(\sigma(x_n, x_{n+1})), \psi(\sigma(x_{n-1}, x_n))) \\ &< \psi(\sigma(x_{n-1}, x_n)) - \psi(\sigma(x_n, x_{n+1})), \end{aligned}$$

by using the properties of ψ, we get

$$\sigma(x_{n+1}, x_n) < \sigma(x_{n-1}, x_n),$$

Hence, we obtain

$$\sigma(x_n, x_{n+1}) \leq \sigma(x_{n-1}, x_n) \text{ for all } n \geq 1, \tag{2.8}$$

which implies that $\{\sigma(x_n, x_{n+1})\}$ is a decreasing sequence of positive real numbers, so there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = r.$$

Suppose that $r > 0$. By the properties of ψ ,(2.7), (2.8) and the condition (ζ_1),

$$0 \leq \limsup_{n\to\infty} \zeta(\psi(\sigma(x_n, x_{n+1})), \psi(\sigma(x_{n-1}, x_n)) < 0,$$

which is a contradiction. Therefore $r = 0$. This implies that

$$\lim_{n,m\to\infty} \sigma(x_n, x_{n+1}) = 0. \tag{2.9}$$

Now, we shall prove that

$$\lim_{n,m\to\infty} \sigma(x_n, x_m) = 0. \tag{2.10}$$

Suppose to the contrary that there exists $\epsilon > 0$, for which we can find subsequences $\{x_m k\}$ and $\{x_n k\}$ of $\{x_n\}$ with $m(k) > n(k) > k$ such that for every k,

$$\sigma(x_{n(k)}, x_{m(k)})) \geq \epsilon. \tag{2.11}$$

This means that

$$\sigma(x_{n(k)}, x_{m(k)-1})) < \epsilon. \tag{2.12}$$

By the triangular inequality and using (2.11) and (2.12), we get

$$\begin{aligned} \epsilon \leq \sigma(x_{n(k)}, x_{m(k)}) &\leq \sigma(x_{n(k)}, x_{m(k)-1}) + \sigma(x_{m(k)-1}, x_{m(k)}) \\ &< \epsilon + \sigma(x_{m(k)-1}, x_{m(k)}) \end{aligned}$$

Letting $n \to \infty$ in the above inequalities and by using (2.10) and (2.11), we have

$$\lim_{n,m\to\infty} \sigma(x_{n(k)}, x_{m(k)}) = \epsilon. \tag{2.13}$$

Since

$$\sigma(x_{n(k)}, x_{m(k)}) \leq \sigma(x_{m(k)}, x_{n(k)+1}) + \sigma(x_{n(k)+1}, x_{n(k)}),$$

and

$$\sigma(x_{n(k)+1}, x_{m(k)+1}) \leq \sigma(x_{m(k)}, x_{m(k)+1}) + \sigma(x_{n(k)+1}, x_{n(k)}),$$

then by letting the limit as $k \to \infty$ in above inequalities and using (2.9) and (2.13), we deduce that

$$\lim_{n,m\to\infty} \sigma(x_{n(k)+1}, x_{m(k)}) = \epsilon. \tag{2.14}$$

similarly, one can easily show that

$$\lim_{n,m\to\infty} \sigma(x_{n(k)+1}, x_{m(k)+1}) = \lim_{n,m\to\infty} \sigma(x_{n(k)}, x_{m(k)+1}) = \epsilon.$$

Again since f is a cyclic (α, β)-admissible $\mathcal{Z}$-contraction mapping and $\alpha(x_n k)\beta(x_m k) \geq 1$, then

$$\begin{aligned} M(x_{n(k)}, x_{m(k)}) &= \max\{\sigma(x_{n(k)}, x_{m(k)}), (\sigma(x_{n(k)}, x_{n(k)+1}) + \sigma(x_{m(k)}, x_{m(k)+1})) \times 4^{-1}, \\ &\quad (\sigma(x_{n(k)+1}, x_{m(k)}) + \sigma(x_{m(k)+1}, x_{n(k)})) \times 4^{-1}\} \end{aligned}$$

letting $n \to \infty$ in (2.15) and using (2.13), (2.14) and (2.9), we have

$$\lim_{n,m\to\infty} M(x_{n(k)}, x_{m(k)}) = \epsilon. \tag{2.15}$$

If $x_n = x_m$ for some $n < m$. then $x_n = fx_n = fx_m = x_{m+1}$ and since $\{\sigma(x_n, x_{n+1})\}$ is decreasing sequence then

$$0 < \sigma(x_n, x_{n+1}) = \sigma(x_m, x_{m+1}) < \sigma(x_{m-1}, x_m) < ... < \sigma(x_n, x_{n+1}),$$

which is a contradiction. Then $x_n \neq x_m$ for all $n < m$. The condition (ζ_2) implies that

$$\limsup_{k\to\infty} \zeta(\psi(\sigma(x_{n(k)}, x_{m(k)})), \psi(M(x_{n(k)}, x_{m(k)}))) < 0,$$

which is a contradiction. So we conclude that $\{x_n\}$ is a Cauchy sequence. Since (X, σ) is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n, u) = \sigma(u, u) = \lim_{n\to\infty} \sigma(x_n, x_m) = 0. \tag{2.16}$$

Now, if f is continuous, we obtain from (2.16) that

$$\lim_{n\to\infty} \sigma(x_{n+1}, fu) = \lim_{n\to\infty} \sigma(fx_{n+1}, fu) = \sigma(fu, fu). \tag{2.17}$$

Otherwise, by Lemma 1.7 and (2.16), we also get

$$\lim_{n\to\infty} \sigma(x_n, fu) = \sigma(u, fu). \tag{2.18}$$

Combining (2.17) and (2.18), we deduce that $\sigma(fu, u) = \sigma(u, fu)$. That is $fu = u$.

Assume that condition (3) is held, that is $\alpha(x_n)\beta(u) \geq 1$. From (2.1) we get

$$0 \leq \zeta(\psi(\sigma(x_{n+1}, fu)), \psi(M(x_n, u))) = \zeta(\psi(\sigma(fx_n, fu)), \psi(M(x_n, u))),$$

where

$$\begin{aligned} M(x_n, u) &= \max\{\sigma(x_n, u), (\sigma(x_n, fx_n) + \sigma(u, fu)) \times 4^{-1}, (\sigma(fx_n, u) + \sigma(x_n, fu)) \times 4^{-1}\} \\ &= \max\{\sigma(x_n, u), (\sigma(x_n, x_{n+1}) + \sigma(u, u)) \times 4^{-1}, (\sigma(x_{n+1}, u) + \sigma(x_n, u)) \times 4^{-1}\}. \end{aligned}$$

By Lemma 1.7 and (2.18)

$$\lim_{k\to\infty} \sigma(x_{n+1}, fu) = \lim_{k\to\infty} M(x_n, u) = \sigma(u, fu) > 0.$$

From (ζ_2)

$$0 \leq \limsup_{k\to\infty} \zeta(\psi(\sigma(x_{n+1}, fu)), \psi(M(x_n, u))) < 0.$$

Since ψ is strictly increasing, we have $\sigma(u, fu) < \sigma(u, fu)$, which is impossible and hence $\sigma(u, fu) = 0$, that is $fu = u$ and so u is a fixed point of f.

Let us now show the uniqueness. Let v be another fixed point of f. Since $\alpha(u)\beta(v) \geq 1$, it follows from (2.1) that

$$\begin{aligned} 0 &\leq \zeta(\psi(\sigma(fu, fv)), \psi(M(u, v))), \\ &= \zeta(\psi(\sigma(u, v)), \psi(\sigma(u, v))), \\ &< \psi(\sigma(u, v)) - \psi(\sigma(u, v)), \end{aligned}$$

Since ψ is strictly increasing, we have $\sigma(u, v) < \sigma(u, v)$, which is a contradiction. Hence $u = v$. □

Corollary 2.2. *Let (X, σ) be a metric space and $f : X \to X$ be a cyclic (α, β)-admissible $\mathcal{Z}$-contraction mapping if there exist $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(t) < t$ such that:*

$$\zeta(\psi(\sigma(fx, fy)), \psi(\sigma(x, y))) \geq 0 \tag{2.19}$$

for all $x, y \in X$ satisfying $\alpha(x)\beta(y) \geq 1$. Assume that

1. *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,*
2. *f is continuous, or*
3. *if $\{x_n\} \subseteq X$ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$.*

Then f has a unique fixed point.

Proof. The rest of proof follows from Theorem 2.1 by considering $M(x, y) = \sigma(x, y)$. □

Corollary 2.3. *Let (X, σ) be a metric like space and $f : X \to X$ be a cyclic (α, β)-admissible $\mathcal{Z}$-contraction mapping if there exist $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(t) < t$ such that:*

$$\zeta(\psi(\alpha(x)\beta(y)\sigma(fx, fy)), \psi(\sigma(x, y))) \geq 0 \tag{2.20}$$

for all $x, y \in X$ satisfying $\alpha(x)\beta(y) \geq 1$. Assume that

1. *there exists* $x_0 \in X$ *such that* $\alpha(x_0) \geq 1$ *and* $\beta(x_0) \geq 1$,

2. f *is continuous, or*

3. *if* $\{x_n\} \subseteq X$ *such that* $x_n \to x$ *and* $\beta(x_n) \geq 1$ *for all* n, *then* $\beta(x) \geq 1$.

Then f *has a unique fixed point.*

Proof. The rest of proof follows from Theorem 2.1 by considering $M(x, y) = \sigma(x, y)$ and $\alpha(x)\beta(y) \geq 1$. □

Corollary 2.4. *Let* (X, σ) *be a metric like space and* $f : X \to X$ *be a cyclic* α*-admissible* $\mathcal{Z}$*-contraction mapping if there exist* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\psi(t) < t$ *such that:*

$$\zeta(\psi(\alpha(x)\alpha(y)\sigma(fx, fy)), \psi(\sigma(x, y))) \geq 0 \tag{2.21}$$

for all $x, y \in X$ *satisfying* $\alpha(x)\alpha(y) \geq 1$. *Assume that*

1. *there exists* $x_0 \in X$ *such that* $\alpha(x_0) \geq 1$ *and* $\alpha(fx_0) \geq 1$,

2. f *is continuous, or*

3. *if* $\{x_n\} \subseteq X$ *such that* $x_n \to x$ *and* $\alpha(x_n) \geq 1$ *for all* n, *then* $\alpha(x) \geq 1$.

Then f *has a unique fixed point.*

Proof. The rest of proof follows from Theorem 2.1 by considering $M(x, y) = \sigma(x, y)$ and $\alpha(x)\beta(y) \geq 1$ and by taking the function $\beta : X \times X \to [0, +\infty)$ to be α. □

Corollary 2.5. *Let* (X, σ) *be a metric like space and* $f : X \to X$ *be a mapping and* $\alpha, \beta : X \times X \to [0, \infty)$ *be two functions. Assume the following conditions hold:*

1. f *is* $(\alpha, \beta)-$*cyclic,*

2. *There exists* $x_0 \in X$ *such that* $\alpha(x_0) \geq 1$ *and* $\beta(x_0) \geq 1$,

3. *There exists* $k \in [0, 1)$ *such that if* $x, y \in X$ *with* $\alpha(x)\beta(y) \geq 1$, *then*

   $$\sigma(Sx, Sy) \leq k \max\{\sigma(x, y), (\sigma(x, fx)+\sigma(y, fy))\times 4^{-1}, (\sigma(fx, y)+\sigma(x, fy))\times 4^{-1}\},$$

4. f *is continuous, or*

5. *if* $\{x_n\} \subseteq X$ *such that* $x_n \to x$ *and* $\alpha(x_n)\beta(x_n) \geq 1$ *for all* n, *then* $\alpha(x)\beta(x) \geq 1$.

Then f *has a unique fixed point.*

Proof. The rest of proof follows from Theorem 2.1, we assume that there exists $k \in [0, 1)$ such that Condition (3) holds. Define the simulation function $\zeta : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ by $\zeta(t, s) = ks - t$. Note that if $x, y \in X$ with $\alpha(x)\beta(y) \geq 1$, then $\zeta(d(fx, fy), \max\{\sigma(x, y), (\sigma(x, fx) + \sigma(y, fy)) \times 4^{-1}, (\sigma(fx, y) + \sigma(x, fy)) \times 4^{-1}\}) \geq 0$. The last inequality together with Conditions (1) ensure that f is generalized $(\alpha, \beta, Z)-$contraction. Thus f satisfies all conditions of Theorem 2.1 and hence f has a fixed point. □

Corollary 2.6. *Let* (X, σ) *be a metric like space and* $f : X \to X$ *be a mapping and* $\alpha, \beta : X \times X \to [0, \infty)$ *be two functions. Assume the following conditions hold:*

1. f *is* $(\alpha, \beta)-$*cyclic,*
2. *There exists* $x_0 \in X$ *such that* $\alpha(x_0) \geq 1$ *and* $\beta(x_0) \geq 1$,
3. *There exist a lower semi-continuous function* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\varphi(t) > 0$ *for all* $t > 0$ *and* $\varphi(0) = 0$ *such that if* $x, y \in X$ *with* $\alpha(x)\beta(y) \geq 1$, *then*
$$\sigma(fx, fy) \leq \sigma(x, y) - \varphi(\sigma(x, y)).$$
4. f *is continuous, or*
5. *if* $\{x_n\} \subseteq X$ *such that* $x_n \to x$ *and* $\alpha(x_n)\beta(x_n) \geq 1$ *for all* n, *then* $\alpha(x)\beta(x) \geq 1$.

Then f *has a unique fixed point.*

Proof. The rest of proof follows from Corollary 2.5 by defining $\zeta : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ via $\zeta(t, s) = s - \varphi(s) - t$. □

Example 2.7. *Let* $X = [0, \infty)$ *endowed with the metric-like* $\sigma(x, y) = x + y$. *Consider* $f : X \to X$ *given by*

$$fx = \begin{cases} \frac{1}{3}x^2 & \textit{if } x \in [0, 1]. \\ x + 2, & \textit{otherwise,} \end{cases}$$

Note that (X,σ) *is a complete metric-like space. Define* $\alpha,\beta : X \to \mathbb{R}^+$ *by*

$$\alpha(x) = \begin{cases} 1 & \text{if } x \in [0,1], \\ 0, & \text{otherwise}, \end{cases} \quad \beta(x) = \begin{cases} 1 & \text{if } x \in [0,1], \\ 0, & \text{otherwise}. \end{cases}$$

Let $\zeta(t,s) = \frac{s}{1+s} - t$ *for all* $s,t \geq 0$ *and* $\psi(t) = t$*. Note that* f *is a cyclic* (α,β)*-admissible. In fact, let* $x,y \in X$ *such that* $\alpha(x) \geq 1$ *and* $\beta(x) \geq 1$*. By definition of* α *and* β*, this implies that* $x,y \in [0,1]$*. Thus,*

$$\beta(fx) \geq 1, \;\; \alpha(fx) \geq 1.$$

Now, if $\{x_n\} \subset X$ *such that* $\beta(x_n) \geq 1$ *and* $x_n \to X$ *as* $n \to \infty$*. Therefore* $x_n \in [0,1]$*, hence* $x \in [0,1]$ *i.e.,* $\beta(x) \geq 1$*.*

Let $\alpha(x)\beta(y) \geq 1$*. Then* $x,y \in [0,1]$ *and so, we have*

$$\begin{aligned} \zeta(\psi(\sigma(fx,fy)),\psi(\sigma(x,y))) &= \frac{\sigma(x,y)}{1+\sigma(x,y)} - \sigma(fx,fy) \\ &= \frac{x+y}{1+x+y} - \sigma(\frac{1}{3}x^2, \frac{1}{3}y^2) \\ &= \frac{x+y}{1+x+y} - \frac{x^2+y^2}{3} \\ &= \frac{3(x+y) - (x^2+y^2)(1+x+y)}{3(1+x+y)} \\ &= \frac{(3-x-x^2-xy-y^2)x + (3y-y^2)y}{3(1+x+y)} \geq 0. \end{aligned}$$

So, the hypotheses of Corollary 2.2 hold and therefore, f *has a unique fixed point* $x = 0$*.*

3 Consequences

Denote by Γ the set of all functions $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

1. γ is Lebesgue-integrable on each compact of $\mathbb{R}^+$;

2. for each $\epsilon > 0$, we have

$$\int_0^{\epsilon} \gamma(z)dz > 0$$

.

Theorem 3.1. *Let (X, σ) be a metric space and $f : X \to X$ be a cyclic (α, β)-admissible $\mathcal{Z}$-contraction mapping if there exist $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(t) < t \leq \phi(t)$ such that:*

$$\zeta(\int_0^{\sigma(fx,fy)} \delta_1(z))dz, \int_0^{\max\{\sigma(x,y),(\sigma(x,fx)+\sigma(y,fy))\times 4^{-1},(\sigma(fx,y)+\sigma(x,fy))\times 4^{-1}\}} \delta_2(z))dz \geq 0.$$

Also, suppose that

1. *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,*
2. *f is continuous.*

Then f possesses a unique fixed point.

Proof. Take $\psi(t) = \int_0^t \delta_1(z))dz$, $\varphi(t) = \int_0^t \delta_2(z))dz$ and $\zeta(t, s) = \psi(t) - \phi(t)$ for all $s, t \geq 0$. Note that ψ is an altering distance function and $\varphi \in \Phi$. Also, f is $(\alpha, \beta)-$cyclic. So S satisfies all the conditions of Theorem 2.1. Therefore, f has a unique fixed point. □

Corollary 3.2. *Let (X, σ) be a metric space and $f : X \to X$ be a cyclic (α, β)-admissible $\mathcal{Z}$-contraction mapping if there exist $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(t) < t \leq \phi(t)$ such that:*

$$\zeta(\int_0^{\sigma(fx,fy)} \delta_1(z))dz, \int_0^{\sigma(x,y),(\sigma(x,fx)} \delta_2(z))dz \geq 0.$$

Also, suppose that

1. *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,*
2. *f is continuous.*

Then f possesses a unique fixed point.

Proof. Take $\psi(t) = \int_0^t \delta_1(z))dz$, $\varphi(t) = \int_0^t \delta_2(z))dz$ and $\zeta(t, s) = \psi(t) - \phi(t)$ for all $s, t \geq 0$. Note that ψ is an altering distance function and $\varphi \in \Phi$. Also, f is $(\alpha, \beta)-$cyclic. So S satisfies all the conditions of Corollary 2.2. Therefore, f has a unique fixed point. □

4 Conclusion

In this paper, we have presented new contraction type mappings expected to hold simulation function and cyclic (α, β)-admissibility on complete metric-like space. Our main result following the continuity of mapping and with dropping the continuity. We obtained Theorem 3.1 which extended and improved our results as a consequences results related with integral equations. Inclosing, we think that the field for applying our result is not restricted to only contractive mappings but extends to apply our results as a nonexpansive mappings.

5 Acknowledgments

The author is grateful for the reviewers for their valuable remarks that led me to revise my paper. The author is also grateful for the support of Al-Zaytoonah University of Jordan.

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