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# Complete and horizontal lifts of Metallic structures

**Mohammad Nazrul Islam Khan**

Department of Computer Engineering
College of Computer
Qassim University
Buraydah, Saudi Arabia

email: m.nazrul@qu.edu.sa, mnazrul@rediffmail.com

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**Abstract**

The aim of this paper is to study the complete and horizontal lifts of metallic structure on tangent bundles. Integrability conditions for complete, horizontal lifts and third order tangent bundle are investigated.

## 1 Introduction

Consider the general quadratic equation $x^2 - \alpha x - \beta I = 0$, where $\alpha$ and $\beta$ are positive integers. Its positive solution $\sigma_\alpha^\beta = \frac{1}{2}(\alpha + \sqrt{\alpha^2 + 4\beta})$ is referred to in our context as the Metallic Means Family. Crasmareanu and Hretcanu [1] studied metallic structures on a Riemannian manifold which are polynomial structures satisfying $J^2 - pJ - qI$, where $p$ and $q$ are positive integers .

Using the lift function, it is convenient to generalize to differentiable structures on any manifold $M$ to its tangent bundle. The complete, vertical, horizontal lifts of a tensor field and connections on any manifold $M$ to its tangent bundle $TM$ have been obtained by Yano and Ishihara [7]. Das and the author [4] have been studying almost product structure by means of the complex, vertical and horizontal lifts of an almost $r-$contact structure. The

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main purpose of this paper is to study metallic structures on tangent bundles and establish integrability conditions.

Let $M$ be an $n-$dimensional differentiable manifold of class $C^\infty$. A tensor field $F$ of type (1,1) is called the metallic structure on $M$ if $F$ satisfies the equation [2]

$$F^2 - \alpha F - \beta I = 0, \tag{1.1}$$

where $\alpha$ and $\beta$ are positive integers and $I$ is the unit vector field on $M$ and $F$ is of constant rank $r$ everywhere in $M$.

Let $l$ and $m$ be operators defined as

$$\begin{aligned} (a) \quad l &= \frac{F^2 - \alpha F}{\beta} \\ (b) \quad m &= I - \frac{F^2 - \alpha F}{\beta} \end{aligned} \tag{1.2}$$

The operators $l$ and $m$ defined in equation (1.2) satisfy the following identities:

$$\begin{aligned} l + m &= 0 \\ l^2 = l, \quad m^2 = m, \quad lm &= ml = 0 \\ Fl = lF = F, \quad Fm &= mF = 0. \end{aligned} \tag{1.3}$$

Thus there exist two complementary distributions $D_l$ and $D_m$ corresponding to the projection tensors $l$ and $m$ respectively in $M$. If the rank of $F$ is $r$, then $D_l$ is $r$-dimensional and $D_m$ is $(n-r)$-dimensional, where dim$M = r$.

## 2 The complete lift of $F$ in the tangent bundle $T(M)$

Let $M$ be an $n$-dimensional differentiable manifold of class $C^\infty$ and $T_p(M)$ the tangent space at a point $p$ of $M$. Then $T(M) = \cup_{p\in M} T_p M$ is a tangent bundle over the manifold $M$. The tangent bundle $TM$ of $M$ is a differentiable manifold of dimension $2n$. Let $\Im_s^r$ denote the set of tensor field of class $C^\infty$ and type $(r,s)$ in $M$ and let $\Im_s^r(T(M))$ denote the corresponding set of tensor fields in $T(M)$ [6].

Let $F, G$ be elements of $\Im_1^1(M)$. Then [7]

$$(FG)^C = F^C G^C. \tag{2.4}$$

Putting $F = G$ in equation (2.6), we obtain

$$(F^2)^C = (F^C)^2. \tag{2.5}$$

Also,

$$(F + G)^C = F^C + G^C. \tag{2.6}$$

Operating the complete lifts of both sides of equation (1.1), we get

$$(F^2 - \alpha F - \beta I)^C = 0$$
$$(F^2)^C - (\alpha F)^C - \beta I^C = 0$$

In view of (2.5) and $I^C = I$, we get

$$(F^C)^2 - \alpha F^C - \beta I = 0 \tag{2.7}$$

In view of equations (1.1), (2.7) and [7], we can easily say that the rank of $F^C$ is $2r$ if and only if the rank of $F$ is $r$. Therefore, we have the following theorems:

**Theorem 2.1.** *Let $F \in \Im_1^1$ be a metallic structure in $M$. Then its complete lift $F^C$ is also metallic structure in $TM$.*

**Theorem 2.2.** *The metallic structure $F$ of rank $r$ in $M$ if and only if its complete lift $F^C$ is of rank $2r$ in $TM$.*

Let $F$ be a metallic structure of rank $r$ in $M$. Then the complete lift $l^C$ of $l$ and $m^C$ of $m$ are complementary projection tensors in $TM$. Thus there exist two complementary distributions $D_{l^C}$ and $D_{m^C}$ determined by $l^C$ and $m^C$ respectively in $TM$. The distributions $D_{l^C}$ and $D_{m^C}$ are, respectively, the complete lifts of $D_l^C$ and $D_m^C$ of $D_l$ and $D_m$ [4].

# 3 Integrability conditions of metallic structure in the tangent bundle

Let $F$ be the metallic structure that is $F^2 - \alpha F - \beta I = 0$. Then the Nijenhuis tensor $N$ of $F$ is a tensor of type (1,2) given by [7]

$$N(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y]. \tag{3.8}$$

Let $N^C$ be the Nijenhuis tensor of $F^C$ in $TM$. Then

$$\begin{aligned} N^C(X^C, Y^C) &= [F^C X^C, F^C Y^C] - F^C[F^C X^C, Y^C] \\ &- F^C[X^C, F^C Y^C] + (F^2)^C[X^C, Y^C]. \end{aligned} \tag{3.9}$$

Let $X, Y \in \Im_0^1(M)$ and $F \in \Im_1^1(M)$. We have

$$\begin{aligned} [X^C, Y^C] &= [X, Y]^C \\ (X+Y)^C &= X^C + Y^C \\ F^C X^C &= (FX)^C. \end{aligned} \tag{3.10}$$

Using equations (1.3) and (3.12), we get

$$\begin{aligned} F^C l^C &= (Fl)^C = F^C \\ F^C m^C &= (Fm)^C = 0 \end{aligned} \tag{3.11}$$

**Theorem 3.1.** *The following identities hold:*

$$N^C(m^C X^C, m^C Y^C) = (F^C)^C [m^C X^C, m^C Y^C], \tag{3.12}$$
$$m^C N^C(X^C, Y^C) = m^C [F^C X^C, F^C Y^C], \tag{3.13}$$
$$m^C(l^C X^C, l^C Y^C) = m^C [F^C X^C, F^C Y^C], \tag{3.14}$$
$$m^C N^C((F^2 - \alpha F)^C X^C, (F^2 - \alpha F)^C Y^C) = \beta^2 m^C N^C(l^C X^C, l^C Y^C). \tag{3.15}$$

*Proof:* The proof of equations (3.12) to (3.15) follow by virtue of equations (1.3), (3.11) and (3.8).

**Theorem 3.2.** *Let $X, Y \in \Im_0^1(M)$. The following conditions are equivalent*

$$\begin{aligned} (a) \quad & m^C N^C(X^C, Y^C) = 0 \\ (b) \quad & m^C N^C(l^C X^C, l^C Y^C) = 0 \\ (c) \quad & m^C N^C((F^2 - \alpha F)^C X^C, (F^2 - \alpha F)^C Y^C) = 0. \end{aligned}$$

*Proof:* By equation (3.15), we have

$$N^C(l^C X^C, l^C Y^C) = 0 \leftrightarrow N^C((F^2 - \alpha F)^C X^C, (F^2 - \alpha F)^C Y^C) = 0$$

Now, the right sides of the equations (3.13), (3.14) are equal which, in view of the last equation, shows that conditions (a), (b), and (c) are equivalent.

**Theorem 3.3.** *The complete lift $D_m^C$ in $TM$ of a distribution $D_m$ in $M$ is integral if $D_m$ is integrable in $M$.*

*Proof:* The distribution $D_m$ is integral if and only if [7]

$$l[mX, mY] = 0 \tag{3.16}$$

for all $X, Y \in \Im(M)$, where $l = I - m$. Operating complete lift of both sides and using (3.12), we get

$$l^C[m^C X^C, m^C Y^C] = 0 \tag{3.17}$$

for all $X, Y \in \Im(M)$, where $l^C = (I - m)^C = I - m^C$ is the projection tensor complementary to $m^C$. Thus condition (3.16) implies (3.17).

**Theorem 3.4.** *The complete lift $D_m^C$ in $TM$ of a distribution $D_m$ in $M$ is integral if $l^C N^C(m^C X^C, m^C Y^C) = 0$, or equivalently $N^C(m^C X^C, m^C Y^C) = 0$, for all $X, Y \in \Im(M)$.*

*Proof:* The distribution $D_m$ is integral in $M$ if and only if [7]

$$N(mX, mY) = 0$$

for all $X, Y \in \Im(M)$. By virtue of condition (3.12), we have

$$N^C(m^C X^C, m^C Y^C) = (F^2)^C(m^C X^C, m^C Y^C)$$

Multiplying throughout by $l^C$, we get

$$l^C N^C(m^C X^C, m^C Y^C) = (F^2)^C l^C(m^C X^C, m^C Y^C)$$

In view of (3.17), the above relation becomes

$$l^C N^C(m^C X^C, m^C Y^C) = 0 \tag{3.18}$$

Also, we have

$$m^C N^C(m^C X^C, m^C Y^C) = 0 \tag{3.19}$$

Adding equations (3.18) and (3.19), we get

$$(l^C + m^C)N^C(m^C X^C, m^C Y^C) = 0$$

Since $l^C + m^C = I^C = I$, we have

$$N^C(m^C X^C, m^C Y^C) = 0$$

.

**Theorem 3.5.** *Let the distribution $D_l$ be integrable in $M$; that is, $mN(X, Y) = 0$ for all $X, Y \in \Im_0^1(M)$. Then the distribution $D_l^C$ is integrable in $TM$ if and only if the one of the conditions of Theorem (3.2) is satisfied.*

*Proof:* The distribution $D_l$ is integral in $M$ if and only if

$$mN(lX, lY) = 0$$

Thus, distribution $D_l^C$ is integrable in $TM$ if and only if

$$m^C N^C(l^C X^C, l^C Y^C) = 0.$$

Thus, the theorem follows by making use of equation (3.15).

**Theorem 3.6.** *The complete lift $F^C$ of a metallic structure $F$ in $M$ is partially integrable in $TM$ if and only if $F$ is partially integrable in $M$.*

*Proof:* The metallic structure $F$ in $M$ is partially integrable if and only if

$$N(lX, lY) = 0, \forall X, Y \in \Im_0^1(M). \tag{3.20}$$

In view of equations (1.3) and (3.8), we obtain

$$N^C(l^C X^C, l^C Y^C) = (N(lX, lY))^C$$

which implies

$$N^C(l^C X^C, l^C Y^C) = 0 \Leftrightarrow N(lX, lY) = 0.$$

Also, from Theorem (3.2), $N^C(l^C X^C, l^C Y^C) = 0$ is equivalent to

$$N^C((F^2 - \alpha F)^C, (F^2 - \alpha F)^C X^C, (F^2 - \alpha F)^C, (F^2 - \alpha F)^C Y^C = 0.$$

**Theorem 3.7.** *The complete lift $F^C$ of a metallic structure $F$ in $M$ is partially integrable in $TM$ if and only if $F$ is partially integrable in $M$.*

*Proof:* A necessary and sufficient condition for a metallic structure in $M$ to be integrable is that

$$(N(X, Y)) = 0 \tag{3.21}$$

for all $X, Y \in \Im_0^1(M)$.

In view of equation (3.8), we get

$$N^C(X^C, Y^C) = (N(X, Y))^C.$$

Therefore, with the help of equation (3.21), we obtain the result.

Now, we prove some theorems on horizontal lift of the metallic structure. Suppose that there are tensor fields $S$ and $\nabla_\gamma S$ in $M$ and $TM$ respectively with affine connection $\nabla$ given by

$$S = S^{i\ldots h}_{k\ldots j}\frac{\partial}{\partial x^i} \otimes \ldots \otimes \frac{\partial}{\partial x^h} \otimes dx^k \otimes \ldots \otimes dx^j$$

$$\nabla_\gamma S = y^l \nabla_\gamma S^{i\ldots h}_{k\ldots j}\frac{\partial}{\partial x^i} \otimes \ldots \otimes \frac{\partial}{\partial y^h} \otimes dx^k \otimes \ldots \otimes dx^j$$

corresponding to the induced coordinates $(x^h, y^h)$ in $\pi^{-1}(U)$[7].
Now, we define the horizontal lift $S^H$ of a tensor field $S$ in $M$ to $TM$ by

$$S^H = S^C - \nabla_\gamma S.$$

**Theorem 3.8.** *Let $F \in \Im^1_1$ be a metallic structure in $M$. Then its horizontal lift $F^H$is also metallic structure in $TM$.*

*Proof:* If $P(t)$ is a polynomial in one variable $t$, then we have [7]

$$(P(F))^H = P(F^H) \tag{3.22}$$

for all $F \in \Im^1_1(M)$.

Operating the horizontal lifts of both sides of equation (1.1), we get

$$(F^2 - \alpha F - \beta I)^H = 0$$
$$(F^2)^H - (\alpha F)^H - \beta I^H = 0$$

In view of (3.22) and $I^H = I$, we get

$$(F^H)^2 - \alpha F^H - \beta I = 0 \tag{3.23}$$

which shows that $F^H$ is a metallic structure in $TM$ [6]. In view of equations (1.1) and (3.23), we can easily say that the rank of $F^H$ is $2r$ if and only if the rank of $F$ is $r$. Therefore, we have the following theorem:

**Theorem 3.9.** *Let $I$ be the identity tensor field of type (1,1) in $M$. Then the metallic structure $F$ of rank $r$ in $M$ if and only if its complete lift $F^H$ is of rank $2r$ in $TM$.*

Let $m$ be a projection tensor field of type (1,1) in $M$ defined by (1.3). Then there exists a distribution $D$ in $M$ determined by $m$. Also

$$m^2 = m.$$

In view of (3.22), we get

$$(m^H)^2 = m^H$$

. Thus, $m^H$ is also a projection in $TM$. Hence, there exists a distribution $D^H$ in $TM$ corresponding to $m^H$, which is called the horizontal lift of the distribution $D$.

# 4 Prolongation of a metallic structure in third tangent bundle $T_3M$

Let $T_3M$ be the third order tangent bundle over $M$ and let $F^{III}$ be the third lift on $F$ in $T_3M$. Then, for any $F, G \in \Im_1^1(M)$, we have

$$\begin{aligned}(G^{III}F^{III})X^{III} &= (G^{III}(F^{III}X^{III}) \\ &= (G^{III}(FX)^{III}) \\ &= (G(FX))^{III} \\ &= (GF)^{III}X^{III}\end{aligned} \tag{4.24}$$

for all $X \in \Im_0^1(M)$. Thus we have

$$G^{III}F^{III} = (GF)^{III}$$

If $P(t)$ is a polynomial in one variable $t$, then we have [7]

$$(P(F))^{III} = P(F^{III}) \tag{4.25}$$

for all $F \in \Im_1^1(M)$.

**Theorem 4.1.** *Let $F \in \Im_1^1(M)$ be a metallic structure in $M$. Then the third lift $F^{III}$ is also metallic structure in $T_3M$.*

*Proof:* If $P(t)$ is a polynomial in one variable $t$, then [7]

$$(P(F))^{III} = P(F^{III}) \tag{4.26}$$

for all $F \in \Im_1^1(M)$. Operating the third lifts of both sides of equation (1.1), we get

$$\begin{aligned}(F^2 - \alpha F - \beta I)^{III} = 0 \\ (F^2)^{III} - (\alpha F)^{III} - \beta I^{III} = 0\end{aligned}$$

Using (4.26) and $I^{III} = I$, we get

$$(F^{III})^2 - \alpha F^{III} - \beta I = 0 \tag{4.27}$$

which shows that $F^{III}$ is a metallic structure in $T_3M$.

**Theorem 4.2.** *The third lift $F^{III}$ is integrable in $T_3M$ if and only if $F$ is integrable in $M$.*

*Proof:* Let $N^{III}$ and $N$ be Nijenhuis tensors of $F^{III}$ and $F$ respectively. Then

$$N^{III}(X,Y) = (N(X,Y))^{III}. \tag{4.28}$$

Now, the metallic structure is integrable in $M$ if and only if $N(X,Y) = 0$. Then, from (4.28), we get

$$N^{III}(X,Y) = 0. \tag{4.29}$$

Thus, $F^{III}$ is integrable if and only if $F$ is integrable in $M$.

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