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On Classes of Janowski Functions of Complex Order Involving a q-Derivative Operator

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Abstract

In this paper, we introduce a new class of analytic functions of complex order using q differential operator. Moreover, we obtain bounds for the coefficients, a sufficient condition and Fekete-Szegö inequalities for the defined class. Furthermore, we give applications for our main results.

1 Introduction

We start this paper with a very brief introduction on q-calculus and the notations which are required for our study. Quantum calculus, often called

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q-calculus, is based on the idea of finite difference re-scaling. The q-derivative is merely a ratio which is given by

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}.$$

Note that as limit $q \to 1^-$, $D_q f(z) = f'(z)$. Notations and symbols play a very important role in the study of q-calculus. Throughout this paper, we let

$$[n]_q = \sum_{k=1}^{n} q^{k-1}, \quad [0]_q = 0, \quad (q \in \mathbb{C})$$

and the q-shifted factorial by

$$(a;q)_n = \begin{cases} 1, & n = 0 \\ (1-a)(1-aq)\ldots(1-aq^{n-1}), & n = 1, 2, \ldots. \end{cases}$$

Denote by $\mathcal{A}$ the class of functions having a Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} c_n z^n, \quad (z \in \mathcal{U}). \tag{1.1}$$

The Hadamard product or convolution of functions $f(z) = z + \sum_{n=2}^{\infty} c_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} c_n b_n z^n, \quad (z \in \mathcal{U}).$$

Corresponding to a function $\mathcal{G}_{r,s}(a_i,\, b_j;\, q,\, z)$ $(i = 1, 2, \ldots, r;\, j = 1, 2, \ldots, s)$ defined by

$$\mathcal{G}_{r,s}(a_i,\, b_j;\, q,\, z) := z + \sum_{n=2}^{\infty} \frac{(a_1;q)_{n-1}(a_2;q)_{n-1}\ldots(a_r;q)_{n-1}}{(q;q)_{n-1}(b_1;q)_{n-1}\ldots(b_s;q)_{n-1}} z^n, \tag{1.2}$$

we now introduce an operator $\mathcal{J}^m_{\alpha,\beta}(a_1, b_1;\, q)f(z) : \mathcal{U} \longrightarrow \mathcal{U}$ as follows:

$$\mathcal{J}^0_{\alpha,\beta}(a_1, b_1;\, q)f(z) = f(z) * \mathcal{G}_{r,s}(a_i,\, b_j;\, q,\, z).$$

$$\begin{aligned} \mathcal{J}^1_{\alpha,\beta}(a_1, b_1;\, q)f(z) = \alpha\beta z^2 D_q^2[f(z) * \mathcal{G}_{r,s}(a_i,\, b_j;\, q,\, z)] + \\ (\alpha - \beta) z D_q[f(z) * \mathcal{G}_{r,s}(a_i,\, b_j;\, q,\, z)] \\ +(1 - \alpha + \beta)\left[f(z) * \mathcal{G}_{r,s}(a_i,\, b_j;\, q,\, z)\right]. \end{aligned} \tag{1.3}$$

$$\mathcal{J}^m_{\alpha,\beta}(a_1,b_1;\,q)f(z)=\mathcal{J}^m_{\alpha,\beta}(\mathcal{J}^{m-1}_{\alpha,\beta}(a_1,b_1;\,q,)f(z)). \tag{1.4}$$

If $f\in\mathcal{A}$, then from (1.3) and (1.4) we may easily deduce that

$$\mathcal{J}^m_{\alpha,\beta}(a_1,b_1;\,q)f(z)=z+\sum_{n=2}^{\infty}\left(1+(\alpha\beta[n]_q+q(\alpha-\beta))\,[n-1]_q\right)^m\,\kappa_n c_n z^n, \tag{1.5}$$
$$\left(m\in N_0=N\cup\{0\}\text{ and }0\le\beta\le\alpha\right),$$

where

$$\kappa_n=\frac{(a_1;q)_{n-1}(a_2;q)_{n-1}\dots(a_r;q)_{n-1}}{(q;q)_{n-1}(b_1;q)_{n-1}\dots(b_s;q)_{n-1}},\quad(|q|<1)\,.$$

Remark 1.1. *The operator* $\mathcal{J}^m_{\alpha,\beta}(a_1,b_1;\,q)f(z)$ *was motivated by [8]. In this remark, we list some special cases of the operator* $\mathcal{J}^m_{\alpha,\beta}(a_1,b_1;\,q)f$*.*

1. *If we let* $q\to1^-$*, then the operator* $\mathcal{J}^m_{\alpha,\beta}(a_1,b_1;\,q)f$ *reduces to the operator introduced and studied by Karthikeyan et al. [3].*
2. *If we let* $\beta=0$ *in* $\mathcal{J}^m_{\alpha,\beta}(a_1,b_1;\,q)f$*, then we get the operator introduced by Reddy et al. [7].*

Several notable classes of operators can be obtained by specializing the parameters involved in $\mathcal{J}^m_{\alpha,\beta}(a_1,b_1;\,q)f(z)$ *(see [2, 4, 9] and references cited therein.)*

The class $\mathcal{P}$ denotes the class of functions of function of the form $p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots$ that are analytic in $\mathcal{U}$ and such that $Re\,(p(z))>0$ for all z in $\mathcal{U}$. We let $\mathcal{S}^*(\delta)$ and $\mathcal{C}(\delta)$ to denote the familiar classes of starlike of order δ and convex of order δ respectively.

Using the concept of subordination of analytic functions, Ma and Minda [6] introduced the class $\mathcal{S}^*(\phi)$ of functions in $\mathcal{A}$ satisfying $\frac{zf'(z)}{f(z)}\prec\phi$ where $\phi\in\mathcal{P}$ with $\phi'(0)>0$ maps $\mathcal{U}$ onto a region starlike with respect to 1 and symmetric with respect to real axis. This class specializes to several classes of univalent functions for suitable choice of ϕ. For instance, the class $\mathcal{S}^*(\frac{1+Az}{1+Bz})=:\mathcal{S}^*[A;B]$ where $-1\le B<A\le1$ is the class of the Janowski starlike functions.

Using the operator $\mathcal{J}^m_{\alpha,\beta}(a_1,b_1;\,q)f(z)$, we define $\mathcal{P}^{m,q}_{\alpha,\beta}(\gamma;\,a_1,b_1;A,B)$ to be the class of functions $f\in\mathcal{A}_1$ satisfying the inequality

$$1+\frac{1}{\gamma}\left(\frac{\mathcal{J}^{m+1}_{\alpha,\beta}(a_1,b_1;\,q)f(z)}{\mathcal{J}^m_{\alpha,\beta}(a_1,b_1;\,q)f(z)}-1\right)\prec\frac{1+Az}{1+Bz},\quad z\in\mathcal{U}, \tag{1.6}$$

where $\prec$ denotes subordination, $\gamma \in \mathbb{C} \setminus \{0\}$, A and B are arbitrary fixed numbers, $-1 \leq B < A \leq 1$, $m \in \mathbb{N}_0$.

Remark 1.2. *It can be easily seen that several familiar and new subclasses of univalent functions can be obtained as special cases of the class* $\mathcal{P}^{m,q}_{\alpha,\beta}(\gamma;\, a_1, b_1; A, B)$.

Henceforth, we let $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$ ($b_j \neq 0, -1, -2, \ldots;\ j = 1, \ldots, s$) to be real.

2 Main Results

2.1 Coefficient estimates

Theorem 2.1. *Let the function* $f(z) \in \mathcal{A}$ *be in the class* $\mathcal{P}^{m,q}_{\alpha,\beta}(\gamma;\, a_1, b_1; A, B)$ *and let*

$$\Gamma_n = \left| [(A-B)\gamma - B\left(\alpha\beta[n]_q + q(\alpha-\beta)\right)[n-1]_q] \right| - [(\alpha\beta[j]_q + q(\alpha-\beta))\,[j-1]_q]\,.$$

(a) If $\Gamma_2 \leq 0$, *then*

$$|c_j| \leq \frac{(A-B)\mid\gamma\mid}{[1+(\alpha\beta[j]_q + q(\alpha-\beta))\,[j-1]_q]^m\ [(\alpha\beta[j]_q + q(\alpha-\beta))\,[j-1]_q]\,\kappa_j}. \tag{2.7}$$

(b) If $\Gamma_n \geq 0$, *then*

$$\mid c_j \mid \leq \frac{1}{[1+(\alpha\beta[j]_q + q(\alpha-\beta))\,[j-1]_q]^m\,\kappa_n} \prod_{n=2}^{j} \frac{\mid (A-B)\gamma - (\alpha\beta[n-1]_q + q(\alpha-\beta))\,([n-2]_q)B \mid}{\{(\alpha\beta[n]_q + q(\alpha-\beta))\,[n-1]_q\}}. \tag{2.8}$$

(c) If $\Gamma_k \geq 0$ *and* $\Gamma_{k+1} \leq 0$ *for* $k = 2, 3, \ldots, j-2$,

$$\mid c_j \mid \leq \frac{1}{[1+(\alpha\beta[j]_q + q(\alpha-\beta))\,[j-1]_q]^m\,[(\alpha\beta[j]_q + q(\alpha-\beta))\,[j-1]_q]\,\kappa_n} \times \left\{ \prod_{n=2}^{j} \frac{\mid (A-B)\gamma - (\alpha\beta[n-1]_q + q(\alpha-\beta))\,([n-2]_q)B \mid}{\{(\alpha\beta[n]_q + q(\alpha-\beta))\,[n-1]_q\}} \right\}. \tag{2.9}$$

The bounds in (2.7) and (2.8) are sharp for all admissible A, B, $\gamma \in \mathbb{C} \setminus \{0\}$ *and for each* j.

Proof. Let $f(z) \in \mathcal{P}_{\alpha,\beta}^{m,q}(\gamma;\, a_1, b_1; A, B)$. Then there exists an analytic function $w(z)$ in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$1 + \frac{1}{\gamma}\left(\frac{\mathcal{J}_{\alpha,\beta}^{m+1}(a_1, b_1;\, q)f(z)}{\mathcal{J}_{\alpha,\beta}^{m}(a_1, b_1;\, q)f(z)} - 1\right) = \frac{1 + Aw(z)}{1 + Bw(z)} \tag{2.10}$$

Simplifying (2.10), we have

$$\begin{aligned}&\sum_{n=2}^{j}\left(1 + \left(\alpha\beta[n]_q + q(\alpha-\beta)\right)[n-1]_q\right)^m \left[\left(\alpha\beta[n]_q + q(\alpha-\beta)\right)[n-1]_q\right]\kappa_n c_n z^n \\ &+ \sum_{n=j+1}^{\infty} d_n z^n = \Bigg\{(A-B)\gamma z + \sum_{n=2}^{j-1}\left(1 + \left(\alpha\beta[n]_q + q(\alpha-\beta)\right)[n-1]_q\right)^m \\ &\qquad \left[(A-B)\gamma - B\left(\alpha\beta[n]_q + q(\alpha-\beta)\right)[n-1]_q\right]\kappa_n c_n z^n\Bigg\} w(z),\end{aligned} \tag{2.11}$$

for certain coefficients d_n. Let $z = re^{i\theta}$, $r < 1$. Applying the Parseval's formula on both sides of the above inequality and letting $r \to 1^-$, we get

$$\begin{aligned}&\sum_{n=2}^{j}\left(1 + \left(\alpha\beta[n]_q + q(\alpha-\beta)\right)[n-1]_q\right)^{2m} \left[\left(\alpha\beta[n]_q + q(\alpha-\beta)\right)[n-1]_q\right]^2\kappa_n^2 |c_n|^2 \\ &+ \sum_{n=j+1}^{\infty} |d_n|^2 \le (A-B)^2|\gamma|^2 + \sum_{n=2}^{j-1}\left(1 + \left(\alpha\beta[n]_q + q(\alpha-\beta)\right)[n-1]_q\right)^{2m} \\ &\qquad \left|\left[(A-B)\gamma - B\left(\alpha\beta[n]_q + q(\alpha-\beta)\right)[n-1]_q\right]\right|^2\kappa_n^2 |c_n|^2.\end{aligned}$$

By some simplification, we obtain for $j \ge 2$

$$\begin{aligned}&\left[1 + \left(\alpha\beta[j]_q + q(\alpha-\beta)\right)[j-1]_q\right]^{2m} \left[\left(\alpha\beta[j]_q + q(\alpha-\beta)\right)[j-1]_q\right]^2\kappa_j^2 |c_j|^2 \\ &\quad \le (A-B)^2|\gamma|^2 + \sum_{n=2}^{j-1}\left(1 + \left(\alpha\beta[n]_q + q(\alpha-\beta)\right)[n-1]_q\right)^{2m} \\ &\left\{\left|\left[(A-B)\gamma - B\left(\alpha\beta[n]_q + q(\alpha-\beta)\right)[n-1]_q\right]\right|^2 - \left[\left(\alpha\beta[j]_q + q(\alpha-\beta)\right)[j-1]_q\right]^2\right\}\kappa_n^2 |c_n|^2.\end{aligned} \tag{2.12}$$

For $j = 2$, it follows from (2.12) that

$$|c_2| \le \frac{(A-B)\mid\gamma\mid}{\left[1 + \alpha\beta[2]_q + q(\alpha-\beta)\right]^m \left[\alpha\beta[2]_q + q(\alpha-\beta)\right]\kappa_2}. \tag{2.13}$$

Since

$$\big|\,[(A-B)\gamma - B\left(\alpha\beta[n-1]_q + q(\alpha-\beta)\right)[n-2]_q]\,\big| \geq \\ \big|\,[(A-B)\gamma - B\left(\alpha\beta[n]_q + q(\alpha-\beta)\right)[n-1]_q]\,\big| - |B| \geq [n-2]_q,$$

if $\Gamma_n \geq 0$ then $\Gamma_{n-1} \geq 0$ for $n = 2, 3, \ldots$. Again, if $\Gamma_n \leq 0$ the $\Gamma_{n+1} \leq 0$ for $n = 2, 3, \ldots$, because

$$\big|\,[(A-B)\gamma - B\left(\alpha\beta[n+1]_q + q(\alpha-\beta)\right)[n]_q]\,\big| \leq \\ \big|\,[(A-B)\gamma - B\left(\alpha\beta[n]_q + q(\alpha-\beta)\right)[n-1]_q]\,\big| + |B| \leq [n]_q,$$

If $\Gamma_2 \leq 0$, then from the above discussion we can conclude that $\Gamma_n \leq 0$ for all $n > 2$. It follows from (2.13) that

$$|c_j| \leq \frac{(A-B)\mid\gamma\mid}{\left[1 + \left(\alpha\beta[j]_q + q(\alpha-\beta)\right)[j-1]_q\right]^m \left[\left(\alpha\beta[j]_q + q(\alpha-\beta)\right)[j-1]_q\right]\kappa_j}. \tag{2.14}$$

If $\Gamma_{n-1} \geq 0$, then from the above observation $\Gamma_2, \Gamma_3, \ldots, \Gamma_{j-2} \geq 0$. From (2.13), we infer that the inequality (2.8) is true for $j = 2$. We establish (2.8) by mathematical induction. Suppose (2.8) is valid for $n = 2, 3, \ldots, (j-1)$. Then it follows from (2.12) that

$$\left[1 + \left(\alpha\beta[j]_q + q(\alpha-\beta)\right)[j-1]_q\right]^{2m} \left[\left(\alpha\beta[j]_q + q(\alpha-\beta)\right)[j-1]_q\right]^2 \kappa_j^2 |c_j|^2$$

$$\leq (A-B)^2|\gamma|^2 + \sum_{n=2}^{j-1}\left(1 + \left(\alpha\beta[n]_q + q(\alpha-\beta)\right)[n-1]_q\right)^{2m}$$
$$\Big\{\big|\,[(A-B)\gamma - B\left(\alpha\beta[n]_q + q(\alpha-\beta)\right)[n-1]_q]\,\big|^2$$
$$- \left[\left(\alpha\beta[j]_q + q(\alpha-\beta)\right)[j-1]_q\right]^2\Big\}\kappa_n^2|c_n|^2$$
$$\leq (A-B)^2|\gamma|^2 + \sum_{n=2}^{j-1}\left(1 + \left(\alpha\beta[n]_q + q(\alpha-\beta)\right)[n-1]_q\right)^{2m}$$
$$\Big\{\big|\,[(A-B)\gamma - B\left(\alpha\beta[n]_q + q(\alpha-\beta)\right)[n-1]_q]\,\big|^2$$
$$- \left[\left(\alpha\beta[j]_q + q(\alpha-\beta)\right)[j-1]_q\right]^2\Big\}\kappa_n^2 \times \left\{\frac{1}{\left[1 + \left(\alpha\beta[n]_q + q(\alpha-\beta)\right)[n-1]_q\right]^{2m}\kappa_n^2}\right.$$
$$\left.\prod_{j=2}^{n}\frac{\mid(A-B)\gamma - \left(\alpha\beta[j-1]_q + q(\alpha-\beta)\right)([j-2]_q)B\mid^2}{\{\left(\alpha\beta[j]_q + q(\alpha-\beta)\right)[j-1]_q\}^2}\right\}$$

Thus, we get

$$| c_j | \leq \frac{1}{\left[1+\left(\alpha\beta[j]_q+q(\alpha-\beta)\right)[j-1]_q\right]^m \kappa_n} \prod_{n=2}^{j} \frac{| (A-B)\gamma-\left(\alpha\beta[n-1]_q+q(\alpha-\beta)\right)([n-2]_q)B |}{\{(\alpha\beta[n]_q+q(\alpha-\beta))[n-1]_q\}},$$

which completes the proof of (2.8).

Now, assume that $\Gamma_n \geq 0$ and $\Gamma_{n+1} \leq 0$ for $n = 2, 3, \ldots, j-2$. Then $\Gamma_2, \Gamma_3, \ldots, \Gamma_{n-1} \geq 0$ and $\Gamma_{n+2}, \Gamma_{n+3}, \ldots, \Gamma_{j-2} \leq 0$. Then (2.12) gives

$$\begin{aligned}
&\left[1+\left(\alpha\beta[j]_q+q(\alpha-\beta)\right)[j-1]_q\right]^{2m} \left[\left(\alpha\beta[j]_q+q(\alpha-\beta)\right)[j-1]_q\right]^2 \kappa_j^2|c_j|^2 \\
&\leq (A-B)^2|\gamma|^2 + \sum_{n=2}^{l} \left(1+\left(\alpha\beta[n]_q+q(\alpha-\beta)\right)[n-1]_q\right)^{2m} \\
&\left\{\left| \left[(A-B)\gamma - B\left(\alpha\beta[n]_q+q(\alpha-\beta)\right)[n-1]_q\right] \right|^2 - \left[\left(\alpha\beta[j]_q+q(\alpha-\beta)\right)[j-1]_q\right]^2\right\} \kappa_n^2|c_n|^2 \\
&\quad + \sum_{n=l+1}^{j-1} \left(1+\left(\alpha\beta[n]_q+q(\alpha-\beta)\right)[n-1]_q\right)^{2m} \\
&\left\{\left| \left[(A-B)\gamma - B\left(\alpha\beta[n]_q+q(\alpha-\beta)\right)[n-1]_q\right] \right|^2 - \left[\left(\alpha\beta[j]_q+q(\alpha-\beta)\right)[j-1]_q\right]^2\right\} \kappa_n^2|c_n|^2 \\
&\leq (A-B)^2|\gamma|^2 + \sum_{n=2}^{l} \left(1+\left(\alpha\beta[n]_q+q(\alpha-\beta)\right)[n-1]_q\right)^{2m} \\
&\qquad \left\{\left| \left[(A-B)\gamma - B\left(\alpha\beta[n]_q+q(\alpha-\beta)\right)[n-1]_q\right] \right|^2\right\} \kappa_n^2|c_n|^2
\end{aligned}$$

On substituting upper estimates for $c_2, c_3, \ldots, c_l$ obtained above and simplifying, we obtain (2.9).

Also, the bounds in (2.7) are sharp for the functions $f_k(z)$ given by

$$\mathcal{J}_{\alpha,\beta}^{m}(a_1, b_1; q)f_k(z) = \begin{cases} z(1+Bz)^{\frac{(A-B)b}{B\lambda(k-1)}} & if\ B \neq 0, \\ z\, exp\left(\frac{Ab}{\lambda(k-1)}\, z^{k-1}\right) & if\ B = 0. \end{cases}$$

The bounds in (2.8) are sharp for the functions $f(z)$ given by

$$\mathcal{J}_{\alpha,\beta}^{m}(a_1, b_1; q)f = \begin{cases} z(1+Bz)^{\frac{(A-B)b}{B}} & if\ B \neq 0, \\ z\, exp(Abz) & if\ B = 0. \end{cases}$$

□

Remark 2.2. *If we let $r = 2$, $s = 1$, $a_1 = b_1$, $a_2 = q$, $\alpha = 1$, $\beta = 0$, and $q \to 1^-$ in Theorem 2.1, then we get the result due to Attiya [1].*

2.2 Fekete-Szegö Problem.

We use the following lemmas to prove the results in this subsection:

Lemma 2.3. *[5] If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is a function with positive real part, then for each complex number μ*

$$| p_2 - \mu p_1^2 | \leq 2 \max (1, | 2\mu - 1 |) \tag{2.15}$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}.$$

Theorem 2.4. *If $f(z) \in \mathcal{P}_{\alpha,\beta}^{m,q}(\gamma;\, a_1, b_1; A, B)$, then for $\mu \in \mathbb{C}$ we have*

$$\begin{aligned} | c_3 - \mu c_2^2 | \leq & \frac{(A-B) \mid \gamma \mid}{\left[1 + (\alpha\beta[3]_q + q(\alpha-\beta))\,[2]_q\right]^m \left[(\alpha\beta[3]_q + q(\alpha-\beta))\,[2]_q\right] |\kappa_3|} \times \\ & \max\left\{1; \left| B - \frac{2\gamma(A-B)}{[\alpha\beta[2]_q + q(\alpha-\beta)]\ \kappa_2} \right.\right. \\ & \left.\left. + \frac{2(A-B)\gamma\mu\left[1 + (\alpha\beta[3]_q + q(\alpha-\beta))\,[2]_q\right]^m \left[(\alpha\beta[3]_q + q(\alpha-\beta))\,[2]_q\right]\ \kappa_3}{\left[1 + \alpha\beta[2]_q + q(\alpha-\beta)\right]^{2m} \left[\alpha\beta[2]_q + q(\alpha-\beta)\right]^2\ \kappa_2^2} \right| \right\}. \end{aligned} \tag{2.16}$$

Proof. As $f(z) \in \mathcal{P}_{\alpha,\beta}^{m,q}(\gamma;\, a_1, b_1; A, B)$, by (1.6) we have

$$1 + \frac{1}{\gamma}\left(\frac{\mathcal{J}_{\alpha,\beta}^{m+1}(a_1, b_1;\, q) f(z)}{\mathcal{J}_{\alpha,\beta}^{m}(a_1, b_1;\, q) f(z)} - 1 \right) = \frac{1 + Aw(z)}{1 + Bw(z)}.$$

Let

$$h(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + p_1 z + p_2 z^2 + \cdots .$$

Then $Re(h(z)) > 0$ and $h(0) = 1$. Hence,

$$1 + \frac{1}{\gamma}\left(\frac{\mathcal{J}_{\alpha,\beta}^{m+1}(a_1, b_1;\, q) f(z)}{\mathcal{J}_{\alpha,\beta}^{m}(a_1, b_1;\, q) f(z)} - 1 \right) = \frac{1 - A + h(z)(1 + A)}{1 - B + h(z)(1 + B)}. \tag{2.17}$$

From (2.17), we obtain

$$\left(1-\frac{1}{\gamma}\right)+\frac{1}{\gamma}\left\{\left[1+\alpha\beta[2]_q+q(\alpha-\beta)\right]^m\left[\alpha\beta[2]_q+q(\alpha-\beta)\right]\ \kappa_2 c_2\right\} z$$
$$+\frac{1}{\gamma}\left\{\left[1+\left(\alpha\beta[3]_q+q(\alpha-\beta)\right)[2]_q\right]^m\left[\left(\alpha\beta[3]_q+q(\alpha-\beta)\right)[2]_q\right]\ \kappa_3 c_3\right.$$
$$\left.-\left[1+\alpha\beta[2]_q+q(\alpha-\beta)\right]^{2m}\left[\alpha\beta[2]_q+q(\alpha-\beta)\right]\ \kappa_2^2 c_2^2\right\} z^2+\cdots$$
$$=1+\frac{p_1(A-B)}{2}z+\frac{(A-B)}{2}\left(p_2-p_1^2(\frac{1+B}{2})\right)z^2+\cdots. \tag{2.18}$$

Equating the coefficients at z and z^2 on both sides of the above equation, we get

$$c_2=\frac{p_1\gamma(A-B)}{\left[1+\alpha\beta[2]_q+q(\alpha-\beta)\right]^m\left[\alpha\beta[2]_q+q(\alpha-\beta)\right]\ \kappa_2}$$

and

$$c_3=\frac{1}{\left[1+\left(\alpha\beta[3]_q+q(\alpha-\beta)\right)[2]_q\right]^m\left[\left(\alpha\beta[3]_q+q(\alpha-\beta)\right)[2]_q\right]\ \kappa_3}$$
$$\left[\frac{(A-B)\gamma}{2}\left(p_2-p_1^2\left(\frac{1+B}{2}\right)\right)\right]$$
$$+\frac{p_1^2\gamma^2(A-B)^2}{\left[1+\left(\alpha\beta[3]_q+q(\alpha-\beta)\right)[2]_q\right]^m\left[\left(\alpha\beta[3]_q+q(\alpha-\beta)\right)[2]_q\right]\left[\alpha\beta[2]_q+q(\alpha-\beta)\right]\kappa_2\kappa_3}.$$

On computation, we have

$$c_3-\mu c_2^2=\frac{(A-B)\gamma}{2\left[1+\left(\alpha\beta[3]_q+q(\alpha-\beta)\right)[2]_q\right]^m\left[\left(\alpha\beta[3]_q+q(\alpha-\beta)\right)[2]_q\right]\ \kappa_3}\left(p_2-\delta p_1^2\right). \tag{2.19}$$

where

$$\delta=\left(\frac{1+B}{2}-\frac{2\gamma(A-B)}{\left[\alpha\beta[2]_q+q(\alpha-\beta)\right]\ \kappa_2}\right.$$
$$\left.+\frac{2(A-B)\gamma\mu\left[1+\left(\alpha\beta[3]_q+q(\alpha-\beta)\right)[2]_q\right]^m\left[\left(\alpha\beta[3]_q+q(\alpha-\beta)\right)[2]_q\right]\ \kappa_3}{\left[1+\alpha\beta[2]_q+q(\alpha-\beta)\right]^{2m}\left[\alpha\beta[2]_q+q(\alpha-\beta)\right]^2\ \kappa_2^2}\right)$$

On rearranging the terms and taking modulus on both sides, the result follows on the application of the Lemma2.3.

□

2.3 A sufficient condition for a function to be in $\mathcal{P}^{m,q}_{\alpha,\beta}(\gamma;\, a_1, b_1; A, B)$

Theorem 2.5. *Let the function $f(z)$ defined by (1.1) and let*

$$\sum_{n=2}^{\infty} (1 + (\alpha\beta[n]_q + q(\alpha-\beta))\,[n-1]_q)^m \left\{(\alpha\beta[n]_q + q(\alpha-\beta))\,[n-1]_q + |(A-B)\gamma - B\,(\alpha\beta[n]_q + q(\alpha-\beta))\,[n-1]_q|\right\}\kappa_n \mid c_n \mid \leq (A-B) \mid \gamma \mid . \tag{2.20}$$

holds, then $f(z)$ belongs to $\mathcal{P}^{m,q}_{\alpha,\beta}(\gamma;\, a_1, b_1; A, B)$.

Proof. Suppose that the inequality holds. Then we have for $z \in \mathcal{U}$

$$\mid \mathcal{J}^{m+1}_{\alpha,\beta}(a_1, b_1;\, q)f(z) - \mathcal{J}^{m}_{\alpha,\beta}(a_1, b_1;\, q)f(z) \mid - \Big|(A-B)\gamma\mathcal{J}^{m}_{\alpha,\beta}(a_1, b_1;\, q)f(z) - B\big[\mathcal{J}^{m+1}_{\alpha,\beta}(a_1, b_1;\, q)f(z) - \mathcal{J}^{m}_{\alpha,\beta}(a_1, b_1;\, q)f(z)\big]\Big|$$

$$= \left|\sum_{n=2}^{\infty} (1 + (\alpha\beta[n]_q + q(\alpha-\beta))\,[n-1]_q)^m \,\left[(\alpha\beta[n]_q + q(\alpha-\beta))\,[n-1]_q\right]\kappa_n c_n z^n\right| - \left|(A-B)\gamma z + \sum_{n=2}^{\infty} (1 + (\alpha\beta[n]_q + q(\alpha-\beta))\,[n-1]_q)^m \left[(A-B)\gamma - B\,(\alpha\beta[n]_q + q(\alpha-\beta))\,[n-1]_q\right]\kappa_n c_n z^n\right|$$

$$\leq \sum_{k=2}^{\infty} (1 + (\alpha\beta[n]_q + q(\alpha-\beta))\,[n-1]_q)^m \left\{(\alpha\beta[n]_q + q(\alpha-\beta))\,[n-1]_q + |(A-B)\gamma - B\,(\alpha\beta[n]_q + q(\alpha-\beta))\,[n-1]_q|\right\}\kappa_n \mid c_n \mid r^n - (A-B) \mid \gamma \mid r.$$

Letting $r \to 1^-$, we have

$$\mid \mathcal{J}^{m+1}_{\alpha,\beta}(a_1, b_1;\, q)f(z) - \mathcal{J}^{m}_{\alpha,\beta}(a_1, b_1;\, q)f(z) \mid - \Big|(A-B)\gamma\mathcal{J}^{m}_{\alpha,\beta}(a_1, b_1;\, q)f(z) - B\big[\mathcal{J}^{m+1}_{\alpha,\beta}(a_1, b_1;\, q)f(z) - \mathcal{J}^{m}_{\alpha,\beta}(a_1, b_1;\, q)f(z)\big]\Big|$$

$$\leq \sum_{k=2}^{\infty} (1 + (\alpha\beta[n]_q + q(\alpha-\beta))\,[n-1]_q)^m \left\{(\alpha\beta[n]_q + q(\alpha-\beta))\,[n-1]_q + |(A-B)\gamma - B\,(\alpha\beta[n]_q + q(\alpha-\beta))\,[n-1]_q|\right\}\kappa_n \mid c_n \mid - (A-B) \mid \gamma \mid \leq 0.$$

Hence it follows that

$$\frac{\left|\frac{\mathcal{J}^{m+1}_{\alpha,\beta}(a_1,b_1;q)f(z)}{\mathcal{J}^{m}_{\alpha,\beta}(a_1,b_1;q)f(z)}-1\right|}{\left|B\left[\frac{\mathcal{J}^{m+1}_{\alpha,\beta}(a_1,b_1;q)f(z)}{\mathcal{J}^{m}_{\alpha,\beta}(a_1,b_1;q)f(z)}-1\right]-(A-B)\gamma\right|}<1, \quad z\in\mathcal{U}.$$

Letting

$$w(z)=\frac{\frac{\mathcal{J}^{m+1}_{\alpha,\beta}(a_1,b_1;q)f(z)}{\mathcal{J}^{m}_{\alpha,\beta}(a_1,b_1;q)f(z)}-1}{B\left[\frac{\mathcal{J}^{m+1}_{\alpha,\beta}(a_1,b_1;q)f(z)}{\mathcal{J}^{m}_{\alpha,\beta}(a_1,b_1;q)f(z)}-1\right]-(A-B)\gamma},$$

then $w(0)=0$, $w(z)$ is analytic in $|\, z \,|<1$ and $|\, w(z) \,|<1$. Hence we have

$$\frac{\mathcal{J}^{m+1}_{\alpha,\beta}(a_1,b_1;\,q)f(z)}{\mathcal{J}^{m}_{\alpha,\beta}(a_1,b_1;\,q)f(z)}=\frac{1+[B+\gamma(A-B)]w(z)}{1+Bw(z)}$$

which shows that $f(z)$ belongs to $\mathcal{P}^{m,q}_{\alpha,\beta}(\gamma;\,a_1,b_1;A,B)$. □

For $\beta=0$, $\alpha=\lambda$, $a_i=q^{\alpha_i}$, $b_j=q^{\beta_j}$, $\alpha_i,\,\beta_j\in\mathbb{C},\beta_j\neq 0,1,2,\ldots,$ $(i=1,\,\ldots,\,r,\,j=1,\,\ldots,\,s)$ and $q\to 1^-$ in Theorem 2.5, we have the following result.

Corollary 2.6. *[10] Let the function $f(z)$ defined by (1.1) and let*

$$\sum_{n=2}^{\infty}[1+(n-1)\lambda]^m\big\{(n-1)+\,|\,(A-B)\gamma-B(n-1)\,|\,\big\}\lambda\,\Gamma_n\,|\,c_n\,|\leq(A-B)\,|\,\gamma\,|$$

$$\left(\textit{where }\Gamma_n=\frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}\ldots(\alpha_r)_{n-1}}{(\beta_1)_{n-1}\ldots(\beta_s)_{n-1}(n-1)!}\right)$$

holds. Then $f(z)$ belongs to $\mathcal{H}^m_\lambda(b;\,\alpha_1,\beta_1;A,B)$.

If we let $r=2$, $s=1$, $a_1=b_1$, $a_2=q$, $\alpha=1$, $\beta=0$, and $q\to 1^-$ in Theorem 2.5, then we get the following result:

Corollary 2.7. *[1] Let the function $f(z)$ be defined by (1.1) and let*

$$\sum_{n=2}^{\infty}n^m\big\{(n-1)+\,|\,(A-B)\gamma-B(n-1)\,|\,\big\}\,|\,c_n\,|\leq(A-B)\,|\,\gamma\,|$$

hold. Then $f(z)$ belongs to $\mathcal{H}^m(\gamma;A,B)$.

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