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Distribution of Wythoff Sequences Modulo One

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Abstract

Let α be the golden ratio and $\beta\alpha = -1$. In the study of sumsets associated with Wythoff sequences, it is important to prove the inequality $0 < \{b\alpha\} + \beta^n < 1$ for integers b and n in a certain range. In this article, we continue the investigation by replacing $\{b\alpha\} + \beta^n$ by $\sqrt{5}\beta^{n-1} - \{b\alpha\}$.

1 Introduction

Wythoff sequences arise very often in combinatorics and combinatorial game theory. As a result, many of their combinatorial properties have been extensively studied (see, for example, the works of Fraenkel [1, 2], Kimberling [5], Pitman [8], and Wythoff [10]). However, as far as we know, there are only a few number theoretic results concerning the sumsets associated with Wythoff sequences. In order to describe the structure of such sumsets, it is

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important to prove the inequality [4, Theorem 3.3]:

$$0 < \{b\alpha\} + \beta^n < 1 \text{ for all integers } n \geq 5 \text{ and } 1 \leq b \leq F_{n+1} \text{ with } b \neq F_n. \tag{1.1}$$

Here and throughout this article, $\alpha = (1+\sqrt{5})/2$ is the golden ratio, $\beta\alpha = -1$, x is a real number, a, b, m, n are integers, $\lfloor x \rfloor$ is the largest integer less than or equal to x, $\{x\} = x - \lfloor x \rfloor$, F_n and L_n are the nth Fibonacci number and the nth Lucas number which are defined by $F_n = F_{n-1} + F_{n-2}$, $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ with the initial values $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, and $L_1 = 1$. Moreover, if P is a mathematical statement, then the Iverson notation $[P]$ is defined by

$$[P] = \begin{cases} 1, & \text{if } P \text{ holds;} \\ 0, & \text{otherwise.} \end{cases}$$

In this article, we replace $\{b\alpha\} + \beta^n$ in (1.1) by $\sqrt{5}\beta^{n-1} - \{b\alpha\}$. Our interest is that we have an application in mind. Indeed, it is useful in the study of sumsets associated with Wythoff sequences and Lucas numbers. For a short discussion on the sumsets associated with some Beatty sequences generated by a real number $x > 1$ with $x^2 - ax - b = 0$ for some $a, b \in \mathbb{Z}$ see the last section of [4].

2 Preliminaries and Lemmas

We often use the following facts:
Let $-1 < \beta < 0$ and $(|\beta^n|)_{n \geq 1}$ is strictly decreasing. If $a_1 > a_2 > \cdots > a_r$ are even positive integers, then $0 < \beta^{a_1} < \beta^{a_2} < \cdots < \beta^{a_r}$. If $b_1 > b_2 > \cdots > b_r$ are odd positive integers, then $0 > \beta^{b_1} > \beta^{b_2} > \cdots > \beta^{b_r}$.
In addition, let α and β are roots of the equation $x^2 - x - 1 = 0$. So, for instance, $\alpha\beta = -1$, $\beta^2 = \beta + 1$, $\sqrt{5}\beta + \beta = -2$, $\sqrt{5}\beta^2 + 1 = -3\beta$, and $\beta^n + \sqrt{5}\beta^{n-1} + \beta^{n-2} = 0$ for all $n \geq 2$.
Moreover, it is useful to have the following numerical approximations:
$-0.619 < \beta < -0.618$, $-0.237 < \beta^3 < -0.236$, $0.854 < \sqrt{5}\beta^2 < 0.855$, $-0.528 < \sqrt{5}\beta^3 < -0.527$, $0.326 < \sqrt{5}\beta^4 < 0.327$.
The following results are also applied throughout this article sometimes without reference.

Lemma 2.1. *For $n \in \mathbb{Z}$ and $x, y \in \mathbb{R}$, the following statements hold:*

(i) $\lfloor n + x \rfloor = n + \lfloor x \rfloor$.

(ii) $\{n + x\} = \{x\}$.

(iii) $0 \le \{x\} < 1$.

(iv) $\lfloor x + y \rfloor = \begin{cases} \lfloor x \rfloor + \lfloor y \rfloor, & \text{if } \{x\} + \{y\} < 1; \\ \lfloor x \rfloor + \lfloor y \rfloor + 1, & \text{if } \{x\} + \{y\} \ge 1. \end{cases}$

Proof. These are well-known and can be proved easily. For more details, see [3, Chapter 3]. We also refer the reader to [7] and [9, Proof of Lemma 2.6] for a nice application of these properties. □

Lemma 2.2. *The following statements hold for all* $n \in \mathbb{N} \cup \{0\}$:

(i) (Binet's formula) $L_n = \alpha^n + \beta^n$.

(ii) $\beta L_{n+1} + L_n = -\sqrt{5}\beta^{n+1}$.

(iii) $L_n\alpha = L_{n+1} + \sqrt{5}\beta^n$.

Proof. The formula (i) is well-known. Multiplying (ii) by α, we obtain (iii). The formula (ii) follows a straightforward calculation:
$\beta L_{n+1} + L_n$ is equal to

$$\beta\alpha^{n+1} + \beta^{n+2} + \alpha^n + \beta^n = \beta^{n+2} + \beta^n = \beta^n(-\sqrt{5}\beta) = -\sqrt{5}\beta^{n+1}.$$

□

Lemma 2.3. (Zeckendorf's theorem [11]) *For each* $n \in \mathbb{N}$, $n = F_{a_1} + F_{a_2} + \cdots + F_{a_\ell}$ *where* F_{a_1} *is the largest Fibonacci number not exceeding* n, $a_{i-1} - a_i \ge 2$ *for every* $i = 2, 3, \ldots, \ell$, *and* $a_\ell \ge 2$.

Proof. This is well-known and can be proved by using the greedy algorithm. See also [6] for a more general result. □

Lemma 2.4. [4, Lemma 2.4] *If* $x_1, x_2, \ldots, x_n \in \mathbb{R}$, *then*

$$\{x_1 + x_2 + \cdots + x_n\} = \{\{x_1\} + \{x_2\} + \cdots + \{x_n\}\}.$$

Lemma 2.5. *Let* $n \ge 2$. *Then the following statements hold:*

(i) $\lfloor F_n\alpha \rfloor = F_{n+1} - [n \equiv 0 \pmod 2]$.

(ii) $\lfloor F_n\alpha^2 \rfloor = F_{n+2} - [n \equiv 0 \pmod 2]$.

(iii) $\{F_n\alpha\} = -\beta^n + [n \equiv 0 \pmod 2]$.

(iv) $\{F_n\alpha^2\} = \{F_n\alpha\}$.

(v) $\lfloor L_n\alpha \rfloor = L_{n+1} - [n \equiv 1 \pmod 2]$.

(vi) $\{L_n\alpha\} = \sqrt{5}\beta^n + [n \equiv 1 \pmod 2]$.

(vii) $\lfloor L_n\alpha^2 \rfloor = L_{n+2} - [n \equiv 1 \pmod 2]$.

(viii) $\{L_n\alpha^2\} = \{L_n\alpha\}$.

Proof. The proofs of (i) to (iv) can be found in [4, Lemma 2.5]. By Lemma 2.2(iii), we obtain $\lfloor L_n\alpha \rfloor = L_{n+1} + \lfloor \sqrt{5}\beta^n \rfloor$. If n is even, then $0 < \sqrt{5}\beta^n \leq \sqrt{5}\beta^2 < 1$, and so $\lfloor \sqrt{5}\beta^n \rfloor = 0$. If n is odd, then $-1 < \sqrt{5}\beta^3 \leq \sqrt{5}\beta^n < 0$ and thus $\lfloor \sqrt{5}\beta^n \rfloor = -1$. This implies (v). Then (vi) is a consequence of (v) and Lemma 2.2(iii). By writing $\alpha^2 = \alpha + 1$, we obtain (vii) from (v), and (viii) from Lemma 2.1(ii). This completes the proof. □

3 Main results

The proof of the following theorem is similar to that of [4, Theroem 3.3]. In fact, applying Theorem 3.3 of [4] leads to our main theorem but with a smaller range of b, which is not enough in our application. Therefore, we still need to adjust the proof from [4] to obtain the following theorem:

Theorem 3.1. *Let $n \geq 5$ and $1 \leq b \leq F_{n+1}$. Then the following statements hold:*

(i) *If $b = L_{n-1}$, then $\sqrt{5}\beta^{n-1} - \{b\alpha\} = -[n \equiv 0 \pmod 2]$.*

(ii) *If $b \in \{F_{n-2}, F_n\}$, then $0 < \sqrt{5}\beta^{n-1} - \{b\alpha\} + 2[n \equiv 0 \pmod 2] < 1$.*

(iii) *If $b \notin \{F_{n-2}, F_n, L_{n-1}\}$, then $-1 < \sqrt{5}\beta^{n-1} - \{b\alpha\} < 0$.*

Proof. The statement (i) follows immediately from Lemma 2.5(vi). For (ii), let $b \in \{F_{n-2}, F_n\}$ and $A = \sqrt{5}\beta^{n-1} - \{b\alpha\} + 2[n \equiv 0 \pmod 2]$. Since $\beta^n + \sqrt{5}\beta^{n-1} + \beta^{n-2} = 0$, we obtain by Lemma 2.5(iii) that if $b = F_n$, then

$$A = \sqrt{5}\beta^{n-1} + \beta^n + [n \equiv 0 \pmod 2] = -\beta^{n-2} + [n \equiv 0 \pmod 2],$$

if $b = F_{n-2}$, then

$$A = \sqrt{5}\beta^{n-1} + \beta^{n-2} + [n \equiv 0 \pmod 2] = -\beta^n + [n \equiv 0 \pmod 2].$$

By calculating A according to the parity of n, it is not difficult to see that $0 < A < 1$. This proves (ii). For (iii), if $b = F_{n+1}$, then we apply Lemma 2.5(iii) to obtain

$$\begin{aligned}\sqrt{5}\beta^{n-1} - \{b\alpha\} &= \sqrt{5}\beta^{n-1} + \beta^{n+1} - [n \equiv 1 \pmod 2] \\ &= \beta^{n-3} - [n \equiv 1 \pmod 2],\end{aligned}$$

which is in the interval $(-1, 0)$. Next, let $B = \sqrt{5}\beta^{n-1} - \{b\alpha\} + 1$, where b is not equal to any of F_{n-2}, F_n, L_{n-1}, F_{n+1}. We need to show that $0 < B < 1$.
Case 1 $b = F_k$ where $2 \le k \le n-3$ or $k = n-1$.
Case 1.1 $b = F_2$. Then by Lemma 2.5, $B = \sqrt{5}\beta^{n-1} + \beta^2$. Therefore, $B \le \sqrt{5}\beta^4 + \beta^2 = \beta^2(-3\beta) = -3\beta^3 < 1$. If n is odd, then it is obvious that $B > 0$. If n is even, then $n \ge 6$, and $B \ge \sqrt{5}\beta^5 + \beta^2 = \beta^2(\sqrt{5}\beta^3 + 1) > 0$.
Case 1.2 $b = F_{n-1}$. Then by Lemma 2.5,

$$B = \sqrt{5}\beta^{n-1} + \beta^{n-1} - [n \equiv 1 \pmod 2] + 1.$$

If n is even, then $B < 1$ and $B \ge 1 + \beta^5 + \sqrt{5}\beta^5 = 1 - 2\beta^4 > 0$. If n is odd, then $B > 0$ and $B \le \sqrt{5}\beta^4 + \beta^4 = -2\beta^3 < 1$.
Case 1.3 $b = F_k$ and $3 \le k \le n-3$. This case occurs only when $n \ge 6$. By Lemma 2.5,

$$B = \sqrt{5}\beta^{n-1} + \beta^k - [k \equiv 0 \pmod 2] + 1.$$

We first consider the case that k is even. Then $B = \sqrt{5}\beta^{n-1} + \beta^k$. If n is odd, then $B > 0$ and $B \le \sqrt{5}\beta^4 + \beta^4 = -2\beta^3 < 1$. If n is even, then $B < \beta^k \le \beta^4 < 1$, $k \le n-4$, and $B \ge \sqrt{5}\beta^{n-1} + \beta^{n-4} = \beta^{n-4}(\sqrt{5}\beta^3 + 1) > 0$. Next, suppose k is odd. Then $B = \sqrt{5}\beta^{n-1} + \beta^k + 1$. If n is even, then $B < 1$ and $B \ge \sqrt{5}\beta^5 + \beta^3 + 1 = 1 - 3\beta^4 > 0$. If n is odd, then $k \le n-4$, $B > \sqrt{5}\beta^{n-1} > 0$ and $B \le \sqrt{5}\beta^{n-1} + \beta^{n-4} + 1 < 1$.
Case 2 $F_k < b < F_{k+1}$ for some $k \in \{4, 5, \ldots, n\}$. We apply Lemma 2.5 without further reference. By Zeckendorf's theorem, we can write

$$b = F_{a_1} + F_{a_2} + \cdots + F_{a_\ell},$$

where $\ell \ge 2$, $k = a_1 > a_2 > \cdots > a_\ell \ge 2$ and $a_{i-1} - a_i \ge 2$ for every $i = 2, 3, \ldots, \ell$. Then by Lemma 2.4, we obtain

$$\{b\alpha\} = \{\{F_{a_1}\alpha\} + \{F_{a_2}\alpha\} + \cdots + \{F_{a_\ell}\alpha\}\},$$

which is equal to

$$\{(1 - \beta^{b_1} + 1 - \beta^{b_2} + \cdots + 1 - \beta^{b_r}) + (-\beta^{c_1} - \beta^{c_2} - \cdots - \beta^{c_s})\},$$

where $\{b_1, b_2, \ldots, b_r\} \cup \{c_1, c_2, \ldots, c_s\} = \{a_1, a_2, \ldots, a_\ell\}$, $b_1 > b_2 > \cdots > b_r$ are even numbers, and $c_1 > c_2 > \cdots > c_s$ are odd numbers. Notice that one of the sets $\{b_1, b_2, \ldots, b_r\}$ and $\{c_1, c_2, \ldots, c_s\}$ may be empty. In this case, such a set disappears from the subsequence calculation. Also, for convenience, we let $A = \beta^{b_1} + \beta^{b_2} + \cdots + \beta^{b_r} + \beta^{c_1} + \beta^{c_2} + \cdots + \beta^{c_s}$. Then. by Lemma 2.1, $\{b\alpha\} = \{-A\}$. To show that $0 < B < 1$, it is enough to prove

$$\sqrt{5}\beta^{n-1} < \{b\alpha\} < 1 + \sqrt{5}\beta^{n-1}.$$

Case 2.1 $\{b_1, b_2, \ldots, b_r\}$ is empty. Then

$$A = \beta^{c_1} + \beta^{c_2} + \cdots + \beta^{c_s} > \beta^3 + \beta^5 + \cdots = \frac{\beta^3}{1-\beta^2} = -\beta^2.$$

Therefore $0 < -A < \beta^2 < 1$ and so $\{b\alpha\} = \{-A\} = -A$. If n is even, then obviously $\{b\alpha\} > 0 > \sqrt{5}\beta^{n-1}$ and $\{b\alpha\} = -A < \beta^2 < 1 + \sqrt{5}\beta^3 < 1 + \sqrt{5}\beta^{n-1}$. So assume that n is odd. Then $\{b\alpha\} = -A < \beta^2 < 1 + \sqrt{5}\beta^{n-1}$, and $\{b\alpha\} = -A = |\beta|^{c_1} + |\beta|^{c_2} + \cdots + |\beta|^{c_s}$. If $\ell \geq 3$, then $s \geq 3$, and so

$$\{b\alpha\} \geq |\beta|^{c_1} + |\beta|^{c_2} + |\beta|^{c_3} \geq |\beta|^n + |\beta|^{n-2} + |\beta|^{n-4} > |\beta|^n + |\beta|^{n-2} = \sqrt{5}\beta^{n-1}.$$

Suppose $\ell = 2$. Then $s = 2$ and $\{b\alpha\} = |\beta|^{c_1} + |\beta|^{c_2}$. If $c_1 \neq n$, then

$$|\beta|^{c_1} + |\beta|^{c_2} \geq |\beta|^{n-2} + |\beta|^{n-4} > |\beta|^{n-2} + |\beta|^n = -(\beta^n + \beta^{n-2}) = \sqrt{5}\beta^{n-1}.$$

Since $L_{n-1} = F_n + F_{n-2}$ and $b \neq L_{n-1}$, we see that $\{c_1, c_2\} \neq \{n, n-2\}$. Therefore, if $c_1 = n$, then $c_2 \neq n-2$, and so $c_2 \leq n-4$

$$|\beta|^{c_1} + |\beta|^{c_2} \geq |\beta|^n + |\beta|^{n-4} > |\beta|^n + |\beta|^{n-2} = \sqrt{5}\beta^{n-1}.$$

In any case, $\{b\alpha\} > \sqrt{5}\beta^{n-1}$, as required.
Case 2.2 $\{c_1, c_2, \ldots, c_s\}$ is empty. Then

$$A = \beta^{b_1} + \beta^{b_2} + \cdots + \beta^{b_r} < \beta^2 + \beta^4 + \cdots = \frac{\beta^2}{1-\beta^2} = -\beta.$$

Therefore $-1 < \beta < -A < 0$ and $\{b\alpha\} = \{-A\} = 1 - A$. Suppose n is even. Then $\{b\alpha\} > 0 > \sqrt{5}\beta^{n-1}$ and $\{b\alpha\} = 1 - A = 1 - \beta^{b_1} - \beta^{b_2} - \cdots - \beta^{b_r}$. As in the proof of Case 2.1, if $\ell \geq 3$, then $r \geq 3$ and

$$\{b\alpha\} \leq 1 - \beta^{b_1} - \beta^{b_2} - \beta^{b_3} \leq 1 - \beta^n - \beta^{n-2} - \beta^{n-4} < 1 - \beta^n - \beta^{n-2} = 1 + \sqrt{5}\beta^{n-1}.$$

If $\ell = 2$ and $b_1 \neq n$, then

$$\{b\alpha\} = 1 - \beta^{b_1} - \beta^{b_2} \leq 1 - \beta^{n-2} - \beta^{n-4} < 1 - \beta^{n-2} - \beta^{n} = 1 + \sqrt{5}\beta^{n-1}.$$

If $\ell = 2$ and $b_1 = n$, then $b_2 \leq n-4$ and

$$\{b\alpha\} = 1 - \beta^{b_1} - \beta^{b_2} \leq 1 - \beta^{n} - \beta^{n-4} < 1 - \beta^{n} - \beta^{n-2} = 1 + \sqrt{5}\beta^{n-1}.$$

If n is odd, then $\{b\alpha\} < 1 < 1 + \sqrt{5}\beta^{n-1}$ and $\{b\alpha\} = 1 - A > 1 + \beta = \beta^2 \geq \beta^{n-3} \geq \sqrt{5}\beta^{n-1}$.

Case 2.3 $\{b_1, b_2, \ldots, b_r\}$ and $\{c_1, c_2, \ldots, c_s\}$ are not empty. Then there is some cancellation in the sum defining A. As in Cases 2.1 and 2.2, we have $A < \beta^{b_1} + \beta^{b_2} + \cdots + \beta^{b_r} < -\beta$ and $A > \beta^{c_1} + \beta^{c_2} + \cdots + \beta^{c_s} > -\beta^2$.

Case 2.3.1 A is positive. Then $-1 < \beta < -A < 0$, and so $\{b\alpha\} = \{-A\} = 1 - A$. If n is odd, then $\{b\alpha\} < 1 + \sqrt{5}\beta^{n-1}$ and

$$\{b\alpha\} = 1 - A > 1 + \beta > \sqrt{5}\beta^4 \geq \sqrt{5}\beta^{n-1}.$$

Assume that n is even. Then $\{b\alpha\} > 0 > \sqrt{5}\beta^{n-1}$. It remains to show that $\{b\alpha\} < 1 + \sqrt{5}\beta^{n-1}$. Let $u = \min\{b_1, b_2, \ldots, b_r\}$ and $v = \min\{c_1, c_2, \ldots, c_s\}$. Since $a_{i-1} - a_i \geq 2$ for all $i = 2, 3, \ldots, \ell$ and $a_1 = k \leq n$, we obtain that $u \leq n$ and $|v - u| \geq 3$. Then

$$\beta^u \leq \beta^{b_1} + \beta^{b_2} + \cdots + \beta^{b_r} < \beta^u + \beta^{u+2} + \beta^{u+4} + \cdots = \frac{\beta^u}{1 - \beta^2} = -\beta^{u-1}, \tag{3.2}$$

$$\beta^v \geq \beta^{c_1} + \beta^{c_2} + \cdots + \beta^{c_s} > \beta^v + \beta^{v+2} + \beta^{v+4} + \cdots = \frac{\beta^v}{1 - \beta^2} = -\beta^{v-1}. \tag{3.3}$$

By (3.2) and (3.3), we obtain $\beta^u - \beta^{v-1} < A < \beta^v - \beta^{u-1}$. Since $|v - u| \geq 3$, we see that either $v - u \geq 3$ or $v - u \leq -3$. Suppose for a contradiction that $v - u \leq -3$. Since $v \leq u - 3$ and both v and $u - 3$ are odd, we have $\beta^v \leq \beta^{u-3}$. Thus $A < \beta^v - \beta^{u-1} \leq \beta^{u-3} - \beta^{u-1} = \beta^{u-3}(1 - \beta^2) = -\beta^{u-2} < 0$, which contradicts the assumption that A is positive. Hence $v - u \geq 3$. Since $v - 1 \geq u + 2$ and both $v - 1$ and $u + 2$ are even, $\beta^{v-1} \leq \beta^{u+2}$. So $A > \beta^u - \beta^{u+2} = \beta^u(1 - \beta^2) = -\beta^{u+1}$. We have $u \leq v - 3 \leq n - 3$. Thus $u + 1 \leq n - 2$. Since $n - 2$ is even and $u + 1$ is odd, we have $u + 1 \leq n - 3$. Then $\{b\alpha\} = 1 - A < 1 + \beta^{u+1} \leq 1 + \beta^{n-3}$. Since $\sqrt{5}\beta^2 < 1$ and $n - 3$ is odd, $\sqrt{5}\beta^{n-1} > \beta^{n-3}$. Therefore $\{b\alpha\} < 1 + \beta^{n-3} < 1 + \sqrt{5}\beta^{n-1}$, as required.

Case 2.3.2 A is negative. Then $0 < -A < \beta^2 < 1$. Then $\{b\alpha\} = \{-A\} = -A$. We first show that $\{b\alpha\} < 1 + \sqrt{5}\beta^{n-1}$. If n is odd, then $\{b\alpha\} < 1 < 1+$

$\sqrt{5}\beta^{n-1}$. If n is even, then $\{b\alpha\} = -A < \beta^2 < 1+\sqrt{5}\beta^3 < 1+\sqrt{5}\beta^{n-1}$. Next, we show that $\{b\alpha\} > \sqrt{5}\beta^{n-1}$. If n is even, then $\sqrt{5}\beta^{n-1} < 0 < \{b\alpha\}$. So assume that n is odd. Let $u = \min\{b_1, b_2, \ldots, b_r\}$ and $v = \min\{c_1, c_2, \ldots, c_s\}$. As in Case 2.3.1, we have $u \le n$, $|v-u| \ge 3$, the equalities (3.2) and (3.3) hold, and $\beta^u - \beta^{v-1} < A < \beta^v - \beta^{u-1}$. Since $|v-u| \ge 3$, we see that either $v-u \ge 3$ or $v-u \le -3$. If $v-u \ge 3$, then $\beta^{v-1} \le \beta^{u+2}$ and $A > \beta^u - \beta^{v-1} \ge \beta^u - \beta^{u+2} = -\beta^{u+1} > 0$, which contradicts the assumption that $A < 0$. Thus $v-u \le -3$, and so $A < \beta^{u-3} - \beta^{u-1} = -\beta^{u-2}$. Since $u \le n$, u is even and n is odd, we have $u-2 \le n-3$. Then $-A > \beta^{u-2} \ge \beta^{n-3} > \sqrt{5}\beta^{n-1}$. Therefore $\{b\alpha\} = -A > \sqrt{5}\beta^{n-1}$ as desired. This completes the proof. □

Theorem 3.1 leads to a short proof of [4, Theorem 3.3].

Corollary 3.2. [4, Theorem 3.3] *Let $n \ge 5$, $1 \le b \le F_{n+1}$, and $b \ne F_n$. Then $0 < \{b\alpha\} + \beta^n < 1$.*

Proof. If $b = F_{n-2}$ or $b = L_{n-1}$, we can apply Lemma 2.5 to obtain the desired result. So suppose that $b \ne F_{n-2}$ and $b \ne L_{n-1}$. We first consider the case n is odd. Then it is obvious that $\{b\alpha\}+\beta^n < 1$. For the other inequality, we apply Theorem 3.1 to obtain $\{b\alpha\} > \sqrt{5}\beta^{n-1} > -\beta^n$. Similarly, if n is even, then it is immediate that $\{b\alpha\} + \beta^n > 0$ and by using Theorem 3.1, we obtain $\{b\alpha\} < 1 + \sqrt{5}\beta^{n-1} < 1 - \beta^n$. This completes the proof. □

It is possible to extend the range of b in Theorem 3.1 and Corollary 3.2 but the results are not nice and we do not need them in our application. Therefore, we only give some special cases in an example and leave the general case to the interested readers.

Example 3.3. *Let $n \ge 5$, $k \ge n+2$, $b = F_k$, and $B = \{b\alpha\} + \beta^n$. Then the following statements hold.*

(i) *If k and n are odd, then $-1 < B < 0$.*

(ii) *If $k \not\equiv n \pmod 2$, then $0 < B < 1$.*

(iii) *If k and n are even, then $1 < B < 2$.*

Proof. By Lemma 2.5, $B = \beta^n - \beta^k + [k \equiv 0 \pmod 2]$.
Case 1 k is odd. Then $B = \beta^n - \beta^k$. If n is odd, then $-1 < \beta^n < \beta^{n+2} \le \beta^k < 0$, and so $-1 < B < 0$. If n is even, then $k \ge n+3$, and $0 < B \le \beta^n - \beta^{n+3} = -2\beta^{n+1} < 1$.
Case 2 k is even. Then $B = \beta^n - \beta^k + 1$. If n is odd, then $B < 1$, $k \ge n+3$, and $B \ge 1 + \beta^n - \beta^{n+3} = 1 - 2\beta^{n+1} > 0$. If n is even, then $0 < \beta^k \le \beta^{n+2} < \beta^n < 1$, and so $1 < B < 2$. This completes the proof. □

References

[1] A. S. Fraenkel, *How to beat your Wythoff games' opponent on three fronts*, Amer. Math. Monthly, **89,** (1982), 353–361.

[2] A. S. Fraenkel, *Heap games, numeration systems and sequences*, Ann. Comb., **2,** (1998), 197–210.

[3] R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics : A Foundation for Computer Science*, Second Edition, Addison–Wesley, 1994

[4] S. Kawsumarng, T. Khemaratchatakumthorn, P. Noppakeaw, P. Pongsriiam, *Sumsets Associated with Wythoff Sequences and Fibonacci Numbers*, Period. Math. Hungar, online first version available at https://doi.org/10.1007/s10998-020-00343-0

[5] C. Kimberling, *Beatty sequence and Wythoff sequences, generalized*, Fibonacci Quart., **49,** (2011), 195–200.

[6] D. A. Klarner, *Representation if N as a sum of distinct elements from special sequence*, Fibonacci Quart., **4,** (1966), 289–305.

[7] K. Onphaeng, P. Pongsriiam, *Jacobsthal and Jacobsthal-Lucas numbers and sums introduced by Jacobsthal and Tverberg*, J. Integer Seq., **20,** (2017), Article 17.3.6

[8] J. Pitman, *Sumsets of finite Beatty sequences*, Electron. J. Combin,, **8,** (2001), Article R15, 1–23.

[9] P. Pongsriiam, R. C. Vaughan, *The divisor function on residue classes I*, Acta Arith., **168,** (2015), 369–381.

[10] W. A. Wythoff, *A modification of the game of nim*, Nieuw Arch. Wiskd., **2,** (1905–1907), 199–202.

[11] E. Zeckendorf, *Représentation des nombres par une somme des nombres de Fibonacci ou de nombres de Lucas*, Bull. Soc. e Roy, Sci. Liége, **41,** (1972), 179–182.