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On k-Step Fibonacci Functions and k-Step Fibonacci Numbers

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Abstract

In this manuscript, we study k-step Fibonacci functions defined as $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+k) = f(x+k-1)+f(x+k-2)+\cdots+f(x)$, for all $x \in \mathbb{R}$. Moreover, we develop some properties of these functions including notions of f-even and f-odd functions. Furthermore, we show that $\lim_{x\to\infty} \frac{f(x+i)}{f(x)} = \alpha^i$ for all $i \in \mathbb{N}$, where α is the k-nacci constant.

1 Introduction

The definition of Fibonacci functions was inspired from the well-known Fibonacci numbers. As the recurrence relation of Fibonacci numbers is $F_n = F_{n-1} + F_{n-2}$, where $F_0 = 0$ and $F_1 = 1$, the Fibonacci function is defined in the same fashion to be $f(x + 2) = f(x + 1) + f(x)$ for any $x \in \mathbb{R}$. Meanwhile, many researchers found plenty of useful results from this Fibonacci function. For instance, Parker [5] studied properties of derivation of this function. Also, the general form of the recurrence relation of the Fibonacci function was presented by Gandhi [2] as $f(x + n) = F_n f(x + 1) + F_{n-1} f(x)$, where F_n is the full Fibonacci sequence. Later, Neggers et al. [3] discovered

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interesting properties of the Fibonacci function using the concept of f-even and f-odd functions. Recently, Parizi and Gordji [2] and Sharma and Panwar [6] obtained similar results in terms of Tribonacci functions and Tetranacci functions.

In this paper, we provide analogous properties of the k-step Fibonacci function with the idea of f-even and f-odd functions. Also, we obtain the limit of the quotient $\frac{f(x+1)}{f(x)}$ to be the k-nacci constant α. Moreover, we have $\lim_{x\to\infty} \frac{f(x+i)}{f(x)} = \alpha^i$ for all $i \in \mathbb{N}$.

2 k-Step Fibonacci Numbers and k-Step Fibonacci Functions

Let $k \in \mathbb{N}$ with $k \geq 2$. The k-step Fibonacci sequence is defined as

$$F_n = \begin{cases} 0, & \text{if } 0 \leq n < k-1 \\ 1, & \text{if } n = k-1 \\ \sum_{i=1}^{k} F_{n-i}, & \text{if } n \geq k \end{cases}$$

A function f defined on the real numbers $\mathbb{R}$ is said to be a *k-step Fibonacci function* if it satisfies the following formula

$$f(x+k) = f(x+k-1) + f(x+k-2) + \cdots + f(x)$$

for all $x \in \mathbb{R}$.

Example 2.1. *Let $f(x) = a^x$ be a k-step Fibonacci function, where $a > 0$. Then $a^x a^k = f(x+k) = f(x+k-1) + f(x+k-2) + \cdots + f(x) = a^x \left(a^{k-1} + a^{k-2} + \cdots + 1\right)$. Since $a > 0$, we get $a^k - a^{k-1} - a^{k-2} - \cdots - 1 = 0$. It follows from Lemma 3.6 of [7] that this characteristic equation has one positive real root $\alpha \in (1, 2)$. Note that α is called k-nacci constant. Hence $f(x) = \alpha^x$ is a k-step Fibonacci function.*

Let $F_0 = 0, F_1 = 0, \ldots, F_{k-1} = 1$. Define the full k-step Fibonacci sequence as follows:

$$F_{-n} = -F_{-n+1} - \cdots - F_{-n+k-1} + F_{-n+k},$$

for negative indices and define F_n as a normal k-step Fibonacci number, for $n \in \mathbb{N}$.

Example 2.2. *Let* $\{u_n^{(1)}\}_{n=-\infty}^{\infty}, \{u_n^{(2)}\}_{n=-\infty}^{\infty}, \ldots, \{u_n^{(k)}\}_{n=-\infty}^{\infty}$ *be the full* k*-step Fibonacci sequences. Define a function* $f : \mathbb{R} \to \mathbb{R}$ *by*

$$f(x) = u_{\lfloor x \rfloor}^{(1)} + u_{\lfloor x \rfloor}^{(2)} t + \cdots + u_{\lfloor x \rfloor}^{(k)} t^{k-1}$$

where $t = x - \lfloor x \rfloor \in (0, 1)$*, for any* $x \in \mathbb{R}$*. Then*

$$\begin{aligned} f(x+k) &= u_{\lfloor x+k \rfloor}^{(1)} + u_{\lfloor x+k \rfloor}^{(2)} t + \cdots + u_{\lfloor x+k \rfloor}^{(k)} t^{k-1} \\ &= u_{\lfloor x \rfloor+k-1}^{(1)} + u_{\lfloor x \rfloor+k-2}^{(1)} + \cdots + u_{\lfloor x \rfloor}^{(1)} \\ &\quad + \left(u_{\lfloor x \rfloor+k-1}^{(2)} + u_{\lfloor x \rfloor+k-2}^{(2)} + \cdots + u_{\lfloor x \rfloor}^{(2)} \right) t \\ &\quad + \cdots + \left(u_{\lfloor x \rfloor+k-1}^{(k)} + u_{\lfloor x \rfloor+k-2}^{(k)} + \cdots + u_{\lfloor x \rfloor}^{(k)} \right) t^{k-1} \\ &= \left(u_{\lfloor x \rfloor+k-1}^{(1)} + u_{\lfloor x \rfloor+k-1}^{(2)} t + \cdots + u_{\lfloor x \rfloor+k-1}^{(k)} t^{k-1} \right) \\ &\quad + \left(u_{\lfloor x \rfloor+k-2}^{(1)} + u_{\lfloor x \rfloor+k-2}^{(2)} t + \cdots + u_{\lfloor x \rfloor+k-2}^{(k)} t^{k-1} \right) \\ &\quad + \cdots + \left(u_{\lfloor x \rfloor}^{(1)} + u_{\lfloor x \rfloor}^{(2)} t + \cdots + u_{\lfloor x \rfloor}^{(k)} t^{k-1} \right) \\ &= f(x+k-1) + f(x+k-2) + \cdots + f(x) \end{aligned}$$

This implies that f *is a* k*-step Fibonacci function.*

Note that if a k-step Fibonacci function is differentiable on $\mathbb{R}$, then its derivative is also a k-step Fibonacci function.

Proposition 2.3. *Let* f *be a* k*-step Fibonacci function,* $t \in \mathbb{R}$*. Define a function* $g : \mathbb{R} \to \mathbb{R}$ *by*

$$g(x) = f\left(x + t + t^2 + \cdots + t^{k-1}\right)$$

for any $x \in \mathbb{R}$*, then* g *is also a* k*-step Fibonacci function.*

Proof. Given $x \in \mathbb{R}$, we have

$$\begin{aligned} g(x+k) &= f\left(x + t + t^2 + \cdots + t^{k-1} + k\right) \\ &= f\left(x + t + t^2 + \cdots + t^{k-1} + k - 1\right) + \cdots + f\left(x + t + t^2 + \cdots + t^{k-1}\right) \\ &= g(x+k-1) + \cdots + g(x). \end{aligned}$$

Hence g is a k-step Fibonacci function. □

As an example, knowing that $f(x) = \alpha^x$ is a k-step Fibonacci function, $g(x) = \alpha^{x+t+t^2+\cdots+t^{k-1}} = \alpha^{t+t^2+\cdots+t^{k-1}} f(x)$ is also a k-step Fibonacci function.

Theorem 2.4. *Let $f(x)$ be a k-step Fibonacci function and and let $\{F_n\}$ be a sequence of k-step Fibonacci numbers with $F_0 = F_1 = \cdots = F_{k-2} = 0$ and $F_{k-1} = 1$. Then*

$$f(x+n) = F_n f(x+k-1) + (F_{n-1} + F_{n-2} + \cdots + F_{n-k+1})\, f(x+k-2) + \cdots + (F_{n-1} + F_{n-2})\, f(x+1) + F_{n-1} f(x)$$

for any $x \in \mathbb{R}$ and $n \geq k$.

Proof. First, we verify the base case; i.e., $n = k$. Observe that

$$\begin{aligned} f(x+k) &= f(x+k-1) + f(x+k-2) + \cdots + f(x+1) + f(x) \\ &= F_k f(x+k-1) + F_{k-1} f(x+k-2) + \cdots + F_{k-1} f(x+1) + F_{k-1} f(x) \\ &= F_k f(x+k-1) + (F_{k-1} + F_{k-2} + \cdots + F_1)\, f(x+k-2) \\ &\quad + \cdots + (F_{k-1} + F_{k-2})\, f(x+1) + F_{k-1} f(x) \end{aligned}$$

Let $j \in \mathbb{N}$ with $j \geq k$. We now assume that the formula holds for all cases of $n = k, k+1, \ldots, j$. Then

$$\begin{aligned} & f(x+j+1) = f(x+j) + f(x+j-1) + \cdots + f(x+j-(k-1)) \\ &= [F_j f(x+k-1) + (F_{j-1} + F_{j-2} + \cdots + F_{j-k+1})\, f(x+k-2) \\ &\quad + \cdots + (F_{j-1} + F_{j-2})\, f(x+1) + F_{j-1} f(x)] + [F_{j-1} f(x+k-1) + (F_{j-2} + F_{j-3} \\ &\quad + \cdots + F_{j-k})\, f(x+k-2) + \cdots + (F_{j-2} + F_{j-3})\, f(x+1) + F_{j-2} f(x)] \\ &\quad + \cdots + \left[F_{j-k+1} f(x+k-1) + \left(F_{j-k} + F_{j-k-1} + \cdots + F_{j-2(k-1)}\right) f(x+k-2) \right. \\ &\quad \left. + \cdots + (F_{j-k} + F_{j-k-1})\, f(x+1) + F_{j-k} f(x)\right] \\ &= (F_j + F_{j-1} + \cdots + F_{j-k+1})\, f(x+k-1) + [(F_{j-1} + F_{j-2} + \cdots + F_{j-k}) + \\ &\quad (F_{j-2} + F_{j-3} + \cdots + F_{j-k-1}) + \cdots + \left(F_{j-k+1} + F_{j-k} + \cdots + F_{j-2(k-1)}\right)]\, f(x+k-2) \\ &\quad + \cdots + [(F_{j-1} + F_{j-2} + \cdots + F_{j-k}) + (F_{j-2} + F_{j-3} + \cdots + F_{j-k-1})]\, f(x+1) \\ &\quad + [F_{j-1} + F_{j-2} + \cdots + F_{j-k}]\, f(x) \\ &= F_{j+1} f(x+k-1) + (F_j + F_{j-1} + \cdots + F_{j-k+2})\, f(x+k-2) + \ldots \\ &\quad + (F_j + F_{j-1})\, f(x+1) + F_j f(x). \end{aligned}$$

Using the principle of Mathematical Induction, the result holds for all natural number $n \geq k$. □

Definition 2.5. *[3] Let a function $a : \mathbb{R} \to \mathbb{R}$ be such that if $a(x)h(x) \equiv 0$ and h is continuous. Then $h(x) \equiv 0$. The map a is said to be an f-even function (resp., f-odd function) if $a(x+1) = a(x)$ (resp., $a(x+1) = -a(x)$) for any $x \in \mathbb{R}$.*

Theorem 2.6. *Let $f(x) = a(x)g(x)$ be a real-valued function, where a is an f-even function and g is continuous. Then f is a k-step Fibonacci function if and only if g is a k-step Fibonacci function.*

Proof. Assume that f is a k-step Fibonacci function. Then $a(x)g(x+k) = a(x+k)g(x+k) = f(x+k) = f(x+k-1) + f(x+k-2) + \cdots + f(x) = a(x+k-1)g(x+k-1) + \cdots + a(x)g(x) = a(x)\left(g(x+k-1) + \cdots + g(x)\right)$. Thus $a(x)\left(g(x+k-1) + \cdots + g(x) - g(x+k)\right) \equiv 0$. Hence $g(x+k-1) + \cdots + g(x) - g(x+k) \equiv 0$; i.e., $g(x+k) = g(x+k-1) + \cdots + g(x)$. Therefore g is a k-step Fibonacci function. On the other hand, suppose that g is a k-step Fibonacci function. Then $f(x+k) = a(x+k)g(x+k) = a(x)\left(g(x+k-1) + \cdots + g(x)\right) = a(x+k-1)g(x+k-1) + \cdots + a(x)g(x) + f(x+k-1) + \cdots + f(x)$. Hence f is a k-step Fibonacci function. □

Additionally, a real-valued function f satisfying

$$f(x+k) = (-1)^{2k-1}f(x+k-1) + (-1)^{2k-2}f(x+k-2) + \cdots + (-1)^k f(x)$$

is said to be an *odd k-step Fibonacci function*. The following result can be obtained by a similar method to the proof of Theorem 2.6.

Corollary 2.7. *Let $f(x) = a(x)g(x)$ be a real-valued function, where a is an f-odd function and g is continuous. Then f is a k-step Fibonacci function if and only if g is a odd k-step Fibonacci function.*

Finally, we consider the limit of the quotient of a k-step Fibonacci function.

Theorem 2.8. *If f is a k-step Fibonacci function, then the limit of $\frac{f(x+i)}{f(x)}$ exists for all $i \in \mathbb{N}$.*

Proof. Since we will be taking the limit as x tends to infinity, we can assume that $x \geq k$. Let $n = \lfloor x \rfloor$ and $y = x - \lfloor x \rfloor$. Then

$$\begin{aligned}
\frac{f(x+1)}{f(x)} &= \frac{f(y+n+1)}{f(y+n)} \\
&= \frac{F_{n+1}f(y+k-1) + \left(F_n + F_{n-1} + \cdots + F_{n-k+2}\right)f(y+k-2) + \cdots + F_n f(y)}{F_n f(y+k-1) + \left(F_{n-1} + F_{n-2} + \cdots + F_{n-k+1}\right)f(y+k-2) + \cdots + F_{n-1}f(y)} \\
&= \frac{F_{n+1}}{F_n}\left(\frac{f(y+k-1) + \left(\frac{F_n + F_{n-1} + \cdots + F_{n-k+2}}{F_{n+1}}\right)f(y+k-2) + \cdots + \frac{F_n}{F_{n+1}}f(y)}{f(y+k-1) + \left(\frac{F_{n-1} + F_{n-2} + \cdots + F_{n-k+1}}{F_n}\right)f(y+k-2) + \cdots + \frac{F_{n-1}}{F_n}f(y)}\right).
\end{aligned}$$

By Theorem 2.6 of [1], $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha$. Thus $\lim_{n\to\infty} \frac{F_{n+i}}{F_n} = \alpha^i$ for any $i \in \mathbb{N}$. Hence

$$\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = \lim_{n\to\infty} \frac{f(y+n+1)}{f(y+n)}$$
$$= \alpha \left(\frac{f(y+k-1) + \left(\frac{1}{\alpha} + \frac{1}{\alpha^2} + \cdots + \frac{1}{\alpha^{k-1}}\right) f(y+k-2) + \cdots + \frac{1}{\alpha} f(y)}{f(y+k-1) + \left(\frac{1}{\alpha} + \frac{1}{\alpha^2} + \cdots + \frac{1}{\alpha^{k-1}}\right) f(y+k-2) + \cdots + \frac{1}{\alpha} f(y)} \right)$$
$$= \alpha.$$

Similarly, one obtains $\lim_{x\to\infty} \frac{f(x+i)}{f(x)} = \alpha^i$ for all $i \in \mathbb{N}$. □

References

[1] J. B. Bacani, J. T. Rabago, On generalized Fibonacci Number, Appl. Math. Sci., **9**, no. 73, (2015), 3611-3622.

[2] K. R. Gandhi, Exploration of Fibonacci Function, International Journal of Advanced and Applied Sciences, **5**, no. 7, (2012), 57-62..

[3] J. S. Han, H. K. Kim, J. Neggers, On Fibonacci Functions with Fibonacci Numbers, Advances in Difference Equations, (2012).

[4] M. N. Parizi, M. E. Gordji, On Tribonacci Functions and Tribonacci Numbers, Int. J. Math. Comput. Sci., **11**, no. 1, (2016), 23-32.

[5] F. D. Parker, A Fibonacci Function, Fibonacci Quart., **6**, (1968), 1-2.

[6] K. K. Sharma, V. Panwar, On Tetranacci Functions and Tetranacci Numbers, Int. J. Math. Comput. Sci., **15**, no. 3, (2020), 923-932.

[7] D. A. Wolfram, Solving Generalized Fibonacci Recurrences, Fibonacci Quart., **36**, (1998), 129-145.