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Natural Difference Labeling on Certain Graphs

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Abstract

A (p, q) graph G is said to admit natural difference labeling if its vertices can be labeled by non-negative integers such that the induced edge labels obtained by the absolute value of the difference of the labels of the end vertices are the first n natural numbers. A graph which admits natural difference labeling is called a natural difference graph. Merge graph $(C_3 * K_{1,n})$, Subdivision of the edges of the star graph $K_{1,n}$, Lilly graph I_n $(n \geq 2)$ admit natural difference labeling are proved in this paper.

1 Introduction

In this paper graphs that are considered are simple, finite and undirected. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$

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respectively. For various graph theoretic notations and terminology Harary [4] is followed.

Graph labeling traces its origin to labeling presented by Rosa in 1967. A dynamic survey of Graph Labeling was updated by Gallian [1]. Triangular sum labeling was introduced by Hegde and Shankaran[2]. Motivated by the concept of triangular sum labeling and centered triangular sum labeling [3], a concept called natural difference labeling of graphs is introduced.

Natural Difference Labeling has its applications in temporal networks; i.e., networks defined by a labeling assigning to each edge of an underlying graph G, a set of discrete time-labels.The labels of an edge which are natural numbers, indicate the discrete time moments at which the edge is available. Temporal networks, also known as time varying networks, has its applications in communication networks.

We give a brief summary of the definitions which are useful for the present investigations.

Definition 1.1. *If the vertices of the graph are assigned values subject to certain conditions, it is known as graph labeling.*

Definition 1.2. *A natural difference labeling of a graph G is a one to one function $f : V(G) \to I$(where I is the set of all non-negative integers) that induces a bijection $f^* : E(G) \to N$(where N is the set of all natural numbers) of the edges of G defined by $f^*(uv) = |f(u) - f(v)|\ \forall e = uv \in E(G)$. The edge labels are the first n natural numbers. The graph which admits such labeling is called natural difference graph.*

Definition 1.3. *[6] A merge graph $G_1 * G_2$ can be formed from two graphs G_1 and G_2 by merging a vertex of G_1 with a vertex of G_2.*

Definition 1.4. *[6] A subdivision of a graph G is a graph that can be obtained from G by a sequence of edge subdivisions.*

Definition 1.5. *[5] The Lilly graph $I_n, n \geq 2$ is constructed by using two star graphs $2K_{1,n}, n \geq 2$ and joining two paths $2P_n$, $n \geq 2$ sharing a common vertex; i.e.,*

$$I_n = 2K_{1,n} + 2P_n.$$

2 Main results

Here we prove that Merge graph, Subdivision of the edges of the star graph and Lilly graph admit natural difference labeling.

Theorem 2.1. *The Merge graph $C_3 * K_{1,n}$ admits natural difference labeling.*

Proof.
Let G be the merge graph $C_3 * K_{1,n}$. Let u_0, u_1, u_2 be the vertices of C_3 and $u_3 \dots u_{n+2}$ be the vertices of $K_{1,n}$ and the function $f : V(G) \rightarrow \{0, 1, 2 \dots n+2\}$ is defined by

$$f(u_i) = \frac{i(i+1)}{2}, \ (0 \leq i < 3)$$

$$f(u_i) = i + 1, \ (3 \leq i \leq n + 2)$$

and the induced edge labels obtained by the difference of the labels of the end vertices;
i.e.,

$$f^*(u_0u_1) = 1,$$

$$f^*(u_1u_2) = 2,$$

$$f^*(u_2u_0) = 3$$

and

$$f^*(u_0u_i) = i + 1, \ (3 \leq i \leq n + 2)$$

are the first $n + 3$ natural numbers. Thus the Merge graph admits natural difference labeling (See Figure 1).

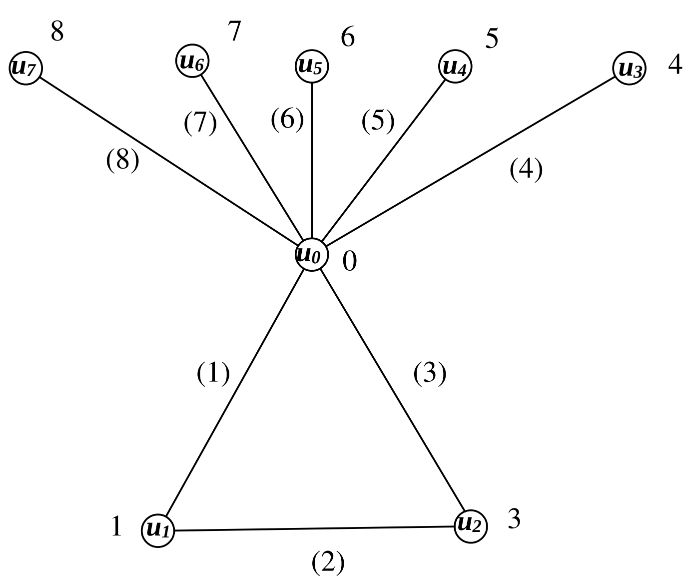

Figure 1: Natural Difference Labeling of Merge graph $(C_3 * K_{1,5})$

Theorem 2.2. *The Subdivision of the edges of the star graph $K_{1,n}$ admits natural difference labeling.*

Proof.
Let G be a graph obtained by the subdivision of the edges of the star graph $K_{1,n}$. Let $V(G) = \{u_0, u_1, \ldots u_{2n}\}$ be the vertex set. Define $f : V(G) \to \{0, 1, 2 \ldots 2n\}$ by

$$f(u_i) = i, \ (0 \le i \le n)$$

$$f(u_i) = i + j, \ (n + 1 \le i \le 2n) \ (j = 1, 2 \ldots n)$$

and the edge labels of the star graph obtained by the difference of the labels of the end vertices; i.e.,

$$f^*(u_0 u_i) = i, (1 \le i \le n)$$

and the edge labels of the subdivision of the star graph have the following induced labeling

$$f^*(u_j u_j + 5) = j + 5, \ (j = 1, 2 \ldots n)$$

are the first $2n$ natural numbers. Thus the subdivision of the edges of the star graph $K_{1,n}$ admits natural difference labeling (See Figure 2).

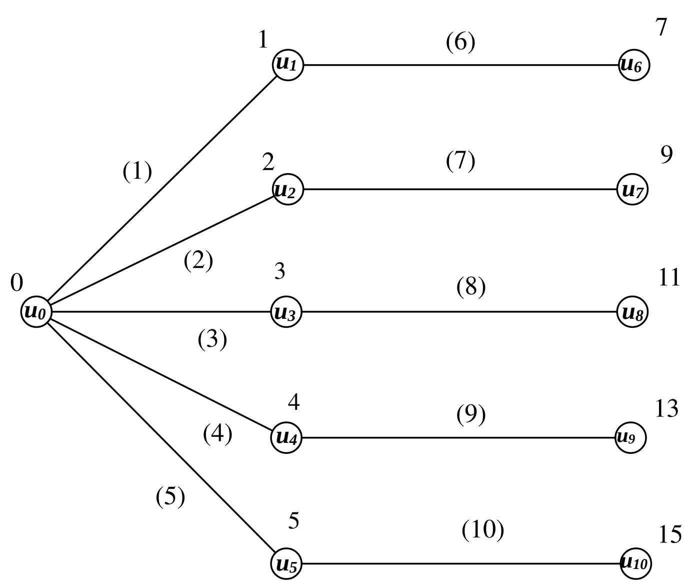

Figure 2: Natural Difference Labeling of the Subdivision of the edges of $K_{1,5}$

Theorem 2.3. *The Lilly graph I_n $(n \geq 2)$ admits natural difference labeling.*

Proof.
Let G be the Lilly graph I_n $(n \geq 2)$. Let $u_0, u_1, u_2 \ldots u_{4n-1}$ be the vertices of I_n.
Define a labeling $f : V(G) \rightarrow \{0, 1, 2 \ldots 4n-2\}$ as follows:
For n odd

$$f(u_i) = \frac{i(i+1)}{2} \; (0 \leq i \leq n+t)$$
$$f(v_i) = (nj-1) + k \; (i = 1, 2 \ldots 2n)$$

t takes odd values, $j \in N$ and j takes the values from 3 in succession and k takes the values from $0, 1, 2 \ldots$; i.e.,

$$\text{for } n = 3, t = 1, j = 3, k = 0, 1, 2 \ldots$$

$$\text{for } n = 5, t = 3, j = 4, k = 0, 1, 2 \ldots$$

and so on.
and the induced function $f^* : E(G) \rightarrow N$ is defined by

$$f^*(u_{i-1}u_i) = i(\; 0 \leq i \leq n+t)$$

$$f^*(u_{n-1}v_m) = n+t+m \; (m = 1, 2 \ldots 2n)$$

and the induced edge labels obtained by the difference of the labels of the end vertices are the first $3n+i$ natural numbers where i is odd. For n even,

$$f(u_i) = \frac{i(i+1)}{2} \; (0 \leq i \leq n+t)$$

$$f(v_i) = nj + k \; (i = 1, 2 \ldots 2n)$$

t takes even values, $j \in N$ and j takes the values from 2 in succession and k takes the values from $0, 1, 2 \ldots$ in succession;

$$i.e., \text{ for } n = 2, t = 0, j = 2, k = 0, 1, 2 \ldots$$

$$\text{for } n = 4, t = 2, j = 3, k = 1, 2 \ldots$$

$$for \; n = 6, t = 4, j = 4, k = 2, 3 \ldots$$

and so on.
and the induced function $f^* : E(G) \rightarrow N$ is defined by

$$f^*(u_{i-1}u_i) = i(\; 0 \leq i \leq n+t)$$

$$f^*(u_{n-1}v_m) = n + t + m (m = 1, 2 \dots 2n)$$

and the induced edge labels obtained by the difference of the labels of the end vertices are the first $4n - 2$ natural numbers. Thus the Lilly graph admits natural difference labeling (See Figure 3).

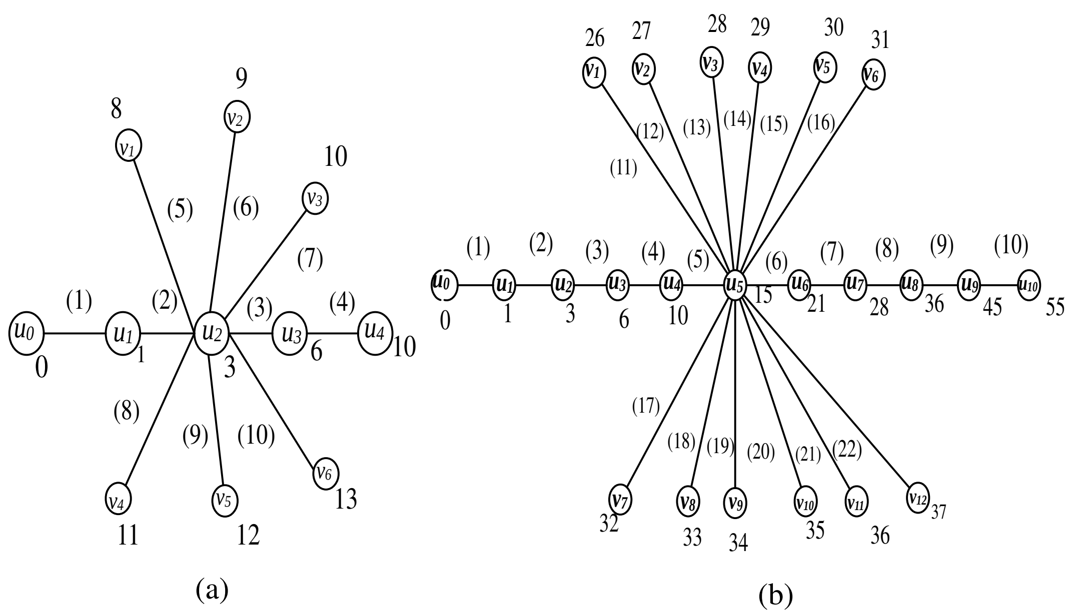

Figure 3: Lilly graphs (a)$I_3 = 2K_{1,3} + 2P_3$, (b) $I_6 = 2K_{1,6} + 2P_6$

3 Conclusion

Merge graph, Subdivision of the edges of the star graph and Lilly graph admit natural difference labeling are proved in this paper. Natural Difference Labeling for interconnection networks and other graphs are under investigation.

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