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# A graph polynomial for independent sets of Fibonacci trees

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**Abstract**

We investigate the visualized representation of Fibonacci tree and its corresponding independence polynomial. This representation helps us to obtain a recursive formula for independence polynomial and to study some relevant properties of Fibonacci trees.

## 1 Introduction

A systematic study of the mathematical properties of the independence polynomial is worth studying because of its role in the kinetics of absorption of diatomic molecules on metal surfaces and its usage in the modeling of some other pysico-chemical phenomena.

**Definition 1.1.** *[1] An independent set in a graph $G$ is a vertex subset $S \subseteq V(G)$ that contains no edge of $G$. The independence number of a graph is the maximum size of an independent set of vertices; i. e., $\alpha(G)$.*

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The independence polynomial was introduced by Gutman and Harary [2].

**Definition 1.2.** *[1] Let $s_k$ denote the number of independent sets of size $k$, which are induced sub graphs of $G$. Then $I(G,z) = \sum_{k=0}^{\alpha(G)} s_k z^k$, where $\alpha(G)$ is the independence number of $G$.*

The independence polynomials are almost everywhere, but it is an NP complete problem [4] to determine the independence polynomial of a graph.

**Theorem 1.3.** *[1] Let $G$ be a simple graph . Let $v \in V(G)$ and $N[v]$ be the closed neighborhood of $v$. Then $\boldsymbol{I}(G;z) = \boldsymbol{I}(G-v;z) + z\boldsymbol{I}(G-N[v];z)$.*

The next result is the main tool in building the various relations between different orders of the Fibonacci tree.

**Theorem 1.4.** *[1] Let $G_1$ and $G_2$ be two vertex disjoint graphs. Then $I(G_1 \cup G_2; z) = I(G_1; z).I(G_2; z)$.*

**Definition 1.5.** *[5] Fibonacci tree of order $n$ has the Fibonacci trees of orders $n-1$ and $n-2$ as left and right subtrees. It is denoted by $r_n$, where $n = 0, 1, 2, 3....$*

Fibonacci trees represent the recursive call structure of the Fibonacci computation. Calculating formulas for $I(G,z)$ is an extremely difficult task, but we will find a visual representation method for families of Fibonacci trees.

# 2 Main results

## 2.1 Diagrammatic representation of independence polynomial of a Fibonacci tree

To compute the independence polynomial of a graph, we use a visual aid in the form of a rooted tree of subgraphs. At the root of the tree, we have a node which is the original graph whose independence polynomial we are trying to calculate. On the next level of the tree, we introduce two nodes. In the first node, place the subgraph of $G-v$ and in the second node place the subgraph of $G-N[v]$ (here $N[v]$ is the closed neighborhood of $v$.) Use theorem 1.4 for vertex disjoint graphs. Finally, by using the recurrence relation $\mathbf{I}(G;x) = \mathbf{I}(G-v;x) + x\mathbf{I}(G-N[v];x)$. We can decompose the independence polynomial of a graph vertex by vertex.

**Theorem 2.1.** *Independence polynomial of Fibonacci tree of order* $n$ *is given by the following formula:*
$I(r_n, z) = I(r_{n-2}, z)[I(r_{n-1}, z) + zI(r_{n-3}, z)I(r_{n-3}, z)I(r_{n-4}, z)]$ *where* $n \geq 4$.

**Proof.**
*By the structure of Fibonacci tree,* $r_n$ *has a single vertex in the middle ,* $r_{n-1}$ *and* $r_{n-2}$ *as left and right sub trees. By applying theorem 1.9([3]),*$I(r_n, z) = I(r_n - v) + zI(r_n - N[v])$. *Let* $G_1$ *denote the graph of* $r_{n-1}$, $G_2$ *denote the graph of* $r_{n-2}$,$G_3$ *denote the graph of* $r_{n-3}$ *and* $G_4$ *denote the graph of* $r_{n-4}$.$r_n - v$ *consists of two vertex disjoint graphs* $G_1$ *and* $G_2$. *Then, by Theorem 1.12([3]),* $I(r_n - v, z) = I(G_1, z).I(G_2, z)$.*Similarly* $r_n - N[v]$ *consists of four vertex disjoint graphs namely* $G_2, G_3, G_3, G_4$, *where* $G_3$ *repeats twice because of the definition of* $r_{n-1}$ *and* $r_{n-2}$. *Again, by theorem 1.12([3]),* $I(r_n - N[v], z) = I(G_2, z)I(G_3, z)I(G_3, z)I(G_4, z)$. *Substituting these results in* $I(r_n, z) = I(G_1, z).I(G_2, z) + zI(G_2, z)I(G_3, z)I(G_3, z)I(G_4, z)$; *i.e.,* $I(r_n, z) = I(r_{n-1}, z)I(r_{n-2}, z) + zI(r_{n-2}, z)I(r_{n-3}, z)I(r_{n-3}, z)I(r_{n-4}, z)$.

Much information is represented by the roots of a graph polynomial. By investigating the nature of the roots of independence polynomials, we can obtain the following results.

**Theorem 2.2.** $z = -1$ *is one of the roots of Fibonacci tree of any order* $n$ *except* $n = 1$.

**Proof.**
*The proof is obvious for n= 2 and 3. When* $n \geq 4$, *we have* $I(r_n, z) = I(r_{n-1}, z)I(r_{n-2}, z) + zI(r_{n-2}, z)I(r_{n-3}, z)I(r_{n-3}, z)I(r_{n-4}, z)$
$=I(r_{n-2}, z)[I(r_{n-1}, z) + zI(r_{n-3}, z)I(r_{n-3}, z)I(r_{n-4}, z)] = I(r_{n-2}, z)P_1$, *where* $P_1 = [I(r_{n-1}, z) + zI(r_{n-3}, z)I(r_{n-3}, z)I(r_{n-4}, z)]$.
*But* $I(r_{n-2}, z) = I(r_{n-3}, z)I(r_{n-4}, z) + zI(r_{n-4}, z)I(r_{n-5}, z)I(r_{n-5}, z)I(r_{n-6}, z). = I(r_{n-4}, z)P_2$, *where* $P_2 = I(r_{n-3}, z) + zI(r_{n-5}, z)I(r_{n-5}, z)I(r_{n-6}, z)$. *Therefore,* $I(r_n, z) = I(r_{n-4}, z)P_1P_2$. *Continuing like this, we have* $I(r_n, z) = I(r_2, z)P_1P_2....P_m$ $=(1 + 4Z + 3Z^2)P_1P_2....P_m$ $=(z + 1)(3z + 1)P_1P_2....P_m$, *where* $P_m = (I(r_3, z)) + zI(r_1, z)I(r_1, z)I(r_0, z))$. *So when* $n \geq 4$, *the independence polynomial of Fibonacci tree contains* $z = -1$ *as zero. Therefore,* $z = -1$ *is a root of* $I(r_n, z)$ *for all orders except* $n = 1$.

**Corollary 2.3.** *For every Fibonacci tree, the total number of independent sets with odd cardinality equals the total number of independent sets with even cardinality except* $n = 1$.

**Definition 2.4.** *[6] A polynomial is said to be stable if all its roots lie in the open left half-plane.*

**Theorem 2.5.** *Independence polynomials of Fibonacci trees with all real roots are stable.*

**Proof.**

*Since every real root of* $I(r_n, z)$ *is negative, we have real roots of* $I(r_n)$ *are stable. But there are stable independence polynomials of* $r_n$ *with complex roots. To illustrate, we calculate the independence polynomial of Fibonacci tree of order 6. We have* $I(r_6, z)$=$756z^{17} + 9882z^{16} + 60414z^{15} + 247851z^{14} + 734908z^{13} + 1587057z^{12} + 2522044z^{11} + 2996744z^{10} + 2700613z^9 + 1862269z^8 + 985332z^7 + 398605z^6 + 121986z^5 + 27687z^4 + 4506z^3 + 496z^2 + 33z + 1$. *To find the stability of complex roots, check the roots lie in the disk* $|z+1| < 1$ *or if every complex root of* $z$ *has real part in* $(-2, 0)$, *then we have the stability of* $I(r_n, z)$ *for all roots. By inspection,* $I(r_6, z)$ *have all complex roots with real parts in* $(-2, 0)$. *So roots are stable .*

**Corollary 2.6.** *The only rational roots of* $I(r_n, z)$ *is of the form* $r = -(\frac{1}{d})$, *where* $d$ =$1, 2, 3$.

**Proof.**

*By theorem 2.1,*
$I(r_n, z) = I(r_{n-1}, z)I(r_{n-2}, z)+z(I(r_{n-2}, z)I(r_{n-3}, z)I(r_{n-3}, z)I(r_{n-4}, z)$ *where* $n \geq 4$.*So* $I(r_{n-2})$ *is a factor of* $I(r_n)$. *Since* $n \geq 4$ *, we have* $r_2$ *is the reduced factor of this. By theorem the factors of* $r_2$ *are* $(z + 1)(3z + 1)$. *So* $r = -1\& - (\frac{1}{3})$ *are the roots of* $r_n$. *For* $n < 4$, *we can easily show that the rational roots are* $z = -1, -(\frac{1}{2}), -(\frac{1}{3})$. *Hence the result.*

Hoshino [6] focused on the Euler characteristic, an invariant that can be associated to any simplicial complex [7].

**Definition 2.7.** *The Euler characteristic of a simplicial complex* $\Delta$ *is the alternating sum* $EC(\Delta) = f_1 - f_2 + f_3 - ....$, *where* $f_k$ *represents the number of faces of dimension* $k$ *in* $\Delta$.

**Theorem 2.8.** *[6] The graphs whose independence complexes collapse to a point have independence polynomials with a root of* $-1$. *In other words, their reduced Euler characteristic is* $-1$.

**Corollary 2.9.** *The reduced Euler characteristic of Fibonacci tree of order* $n$ $(n \neq 1)$ *is* $-1$.

**Theorem 2.10.** *[8] The number of vertices of the* $n^{th}$ *Fibonacci tree is exactly* $F_{n+2} - 1$.

**Question 2.11.** *If $r_n$ denotes the $n^{th}$ Fibonacci tree of order $n$ and let $g_n = I(r_n, z)$, then does $\frac{g_n}{1+z}$ give any other independence polynomial of Fibonacci tree?*

*Counter example: Let* $g_3 = I(r_3, z) = 2z^4 + 11z^3 + 15z^2 + 7z + 1 = (z+1)(2z+1)(2z^2+8z+1)$ $\frac{g_3}{1+z} = (2z+1)(2z^2+8z+1) = 4z^3 + 18z^2 + 10z + 1$ *This is not the independence polynomial for any Fibonacci tree since there is no* 10 *vertices Fibonacci tree , by theorem 2.10 [8].*

**Remark 2.12.** *It is interesting to study the independence polynomial of a Fibonacci tree. It remains to see the characterization of roots of independence polynomial of Fibonacci tree and their detailed analysis.*

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