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Dual basis for a class of cell modules

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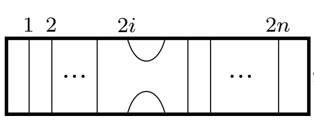
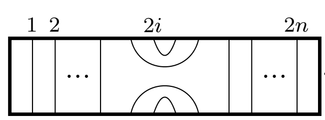
Abstract

In this paper, we construct a dual basis for a certain Fuss-Catalan algebra cell modules. We use a technique from the linear algebra to determine the new basis. This dual basis will give the opportunity to find the characters and the central primitive idempotents for the Fuss-Catalan algebras as a future work.

1 Introduction

The Fuss-Catalan algebras $FC_n(a,b)$, where $a,b \in \mathbb{C} - \{0\}$, were introduced by Bisch and Jones as a generalization of the Temperley-Lieb algebras [1]. The generators of these algebras were defined in [1, Proposition 4.1.3] by the diagrams, 1, ${}_1U_i$ and ${}_2U_i$, where $1 \leq i \leq n-1$, and

${}_1U_i =$ 

, ${}_2U_i =$ 

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Moreover, we have, [1, Corollary 2.1.7], for $n \geq 1$,

$$\dim FC_n(a,b) = \frac{1}{n+1}\binom{2n}{n}.$$

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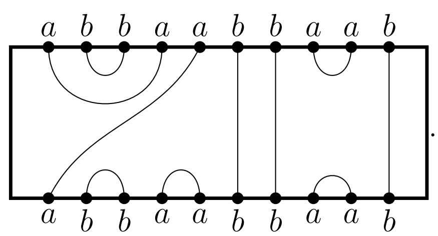
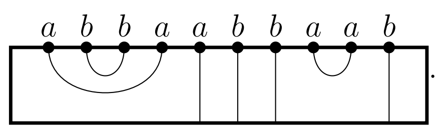
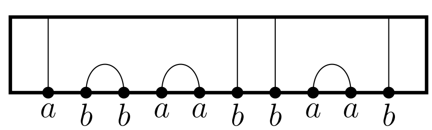
Let $\mathcal{J}_n$ be the set of basis diagrams for $FC_n(a,b)$, where a diagram $D \in \mathcal{J}_n$ is represented by $2n$ points and two parallel lines such that each line contains n points with assignments of colors $abbaabbaabb....a(orb)$. Two points with the same color either connected by a string at the same line or by a string at opposite lines provided that there are no crossings. For $D_1, D_2 \in \mathcal{J}_n$, we define the multiplication $D_1D_2 = a^{r_1}b^{r_2}D_3$ by stacking D_1 at the top of D_2 and connecting the points at the bottom line of D_1 with the corresponding points at the top line of D_2. Moreover, we remove r_1, and r_2 circles with a and b colors from D_3 respectively. Strings connecting two opposite points called *through strings* and the number of through strings in a diagram D is called *length* of D, the color associated to through strings in D is called *label* of D. The *initial part* of D is represented by the points at the top line of D with their strings, while the *final part* of D is represented by the points at the bottom line of D with their strings. For example, the following diagram $D \in FC_5$



its label is $abbb = ab^3$, and $\ell(D) = 4$. The initial part of D is the diagram



and the final part of D is the diagram



Section 2 contains the definition of cellular algebras and their cell modules as defined in [3]. Motivated by the work of Y. Li and D. Zhao [6] investigating the center of the symmetric cellular algebras, in section 3 we introduce the dual basis for a family of $FC_n(a,b)$-cell modules with certain labels by using the duality condition and the Gram matrices.

2 Cellular algebras and cell modules

Graham and Lehrer [3] defined a cellular algebra A and they introduced the theory that determine the irreducible representations of A. There are interesting classes of cellular algebras arising from mathematics and physics, for instance, Hecke algebras of finite type, Ariki–Koike Hecke algebras, q-Schur algebras, Brauer algebras, partition algebras, Birman–Wenzl algebras (see [2, 3, 8, 9] for details).

Definition 2.1 ([1, Definition 3.1.5]). *Two labels has the property $\lambda \leq \lambda'$ if λ is deduced from λ' by removing some (or no) letters of λ'.*

For example, we have $abbaaa \leq abbbbaaaabba$. Let Λ be the set of labels for the diagrams in $\mathcal{J}_n$ and let $\mathcal{W}_n(\lambda)$ be set of initial parts of diagrams in $\mathcal{J}_n$ with distinct labels. So we may define $\mathcal{J}_n = \{C^{\lambda}_{S,T} \mid \lambda \in \Lambda \text{ and } S, T \in \mathcal{W}_n(\lambda)\}$, where the diagram $C^{\lambda}_{S,T}$ constructed by placing S over T^* ($*$ is the function on $FC_n(a,b)$ that reflects a diagram upside down). In [4, Theorem 4.5], the author proved that the algebras $FC_n(a,b)$ with cell datum $(\Lambda, \mathcal{W}_n(\lambda), \mathcal{J}_n, *)$ are cellular in the sense of [3].

Definition 2.2 ([3, Definition 2.1]). *Let A be a cellular algebra with cell datum $(\Lambda, \mathcal{W}, \mathcal{B}, *)$. For each $\lambda \in \Lambda$ define the left A-module $\Delta(\lambda)$ as a free R-module with basis $\{C_S \mid S \in \mathcal{W}(\lambda)\}$ and A-action defined by*

$$aC_S = \sum_{S' \in \mathcal{W}(\lambda)} r_a(S', S) C_{S'} \qquad a \in A,\ S \in \mathcal{W}(\lambda),$$

where $r_a(S', S) \in R$. This module is called the cell module *of A labelled λ.*

Defining the action of $D \in FC_n(a,b)$ on the initial part M of $D' \in FC_n(a,b)$ to be the initial part of DD', we have

Definition 2.3 ([4, Definition 4.6]). *For each $\lambda \in \Lambda$, there is a cell module $\Delta_n(\lambda)$ with basis $\mathcal{W}_n(\lambda)$ such that for all $D \in FC_n(a,b)$ and $M \in \mathcal{W}_n(\lambda)$, the action is defined by*

$$D \cdot M = \begin{cases} DM, & \text{if } \ell(DM) = \ell(M), \\ 0, & \text{otherwise.} \end{cases}$$

3 Dual basis

In [6], the authors used the dual basis method to find the center of the graded symmetric cellular algebra A by studying the Higman ideal of A. In this section, we construct the dual basis for a large class of cell modules for the Fuss-Catalan algebras $FC_n(a,b)$, namely, $\Delta_n(\mu_n)$, where for $n \geq 4$,

$$\mu_n = \begin{cases} a^2(abba)^{m-2} & \text{if } n = 2m \\ a^2(abba)^{m-2}ab & \text{if } n = 2m+1. \end{cases} \tag{3.1}$$

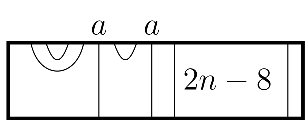
In Table 1, we give the label μ_n for the initial values of n.

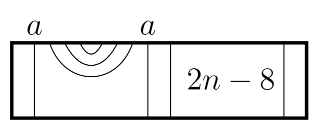
n	4	5	6	7	8
μ_n	a^2	$a^2(ab)$	$a^2(ab^2a)$	$a^2(ab^2a^2b)$	$a^2(ab^2a^2b^2a)$

Table 1: The label μ_n

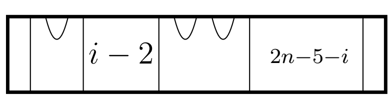
From [5, Proposition 5.1 and Proposition 5.4], we have

Proposition 3.1. *For $n \geq 4$, let μ_n be as defined in (3.1), then the dimension of $\Delta_n(\mu_n)$ is $(2n-4)$ and it is spanned by $\{M_i | i = 0, 1, \cdots, 2n-5\}$, where*

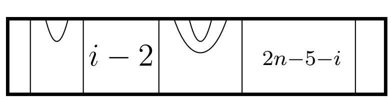
$M_0 =$ 

, $M_1 =$ 

,

$M_i =$ 

if $i = 2, 4, \ldots, 2n-6$,

$M_i =$ 

if $i = 3, 5, \ldots, 2n-5$.

The cell modules of a cellular algebra are irreducible if the bilinear form associated to each cell module is non-degenerate (see [3, Theorem 3.8]).

Definition 3.2 ([5, Definition 5.5]). *A bilinear form $\langle -, - \rangle \colon \Delta_n(\lambda) \times \Delta_n(\lambda) \to \mathbb{C}$ defined on basis diagrams M_1 and M_2 by rotating M_2 upside down and putting it over M_1. Now, if the number of through strings does not change in this connection, then $\langle M_1, M_2 \rangle = a^{k_1} b^{k_2}$, where k_1*

and k_2 are the number of closed a and b-circles that be constructed, but if the number of through strings go down, then $\langle M_1, M_2\rangle = 0$. We extend bilinearly to all of $\Delta_n(\lambda)$.

Lemma 3.3 ([5, Lemma 5.6]). *Let $D \in FC_n$ and M_1, $M_2 \in \Delta_n(\lambda)$ are basis diagrams. Then*

(i) $\langle M_1, M_2\rangle = \langle M_2, M_1\rangle$

(ii) $\langle M_1, DM_2\rangle = \langle D^* M_1, M_2\rangle$,

where D^* is the reflection of D about a horizontal line.

Definition 3.4. *The Gram matrix corresponding to the cell module $\Delta_n(\lambda)$ that has ordered basis $(M_1, M_2, \ldots, M_r)$ is defined by, for i, $j = 1, 2, \ldots, r$,*

$$G_n(\lambda)_{i,j} = \langle M_i, M_j\rangle.$$

Proposition 3.5 ([5, Proposition 5.8]). *Let μ_n be as defined in (3.1). The Gram matrix $G(\mu_n)$ for the cell module $\Delta_n(\mu_n)$ subject to the ordered basis $B = (M_0, M_1, \ldots, M_{2n-5})$ has the following form*

(i) $$G(\mu_n) = \begin{pmatrix} ab^2 & b^2 & b & 0 \\ b^2 & ab^2 & ab & b^2 \\ b & ab & ab^2 & b \\ 0 & b^2 & b & ab^2 \end{pmatrix} \quad \text{if } n = 4.$$

(ii) $$G(\mu_n) = \begin{pmatrix} G(\mu_{n-1}) & Q_s \\ Q_s^T & D_t \end{pmatrix} \text{ if } n > 4,$$

where $Q_s^T = \begin{pmatrix} 0 & \ldots & 0 & bs \\ 0 & \ldots & 0 & b \end{pmatrix}_{2\times(2n-6)}$, $D_t = \begin{pmatrix} ab^2 & bt \\ bt & ab^2 \end{pmatrix}$, *and $(s, t) = (a, b)$ if n even while $(s, t) = (b, a)$ if n odd.*

Now, our approach to construct a dual basis for $\Delta_n(\mu_n)$ depends on calculating the inverse of $G_n(\mu_n)$. However, this is a non-trivial task. Thus we will write the Gram matrix $G_n(\mu_n)$ in a block diagonal form by using a change of basis so that we can determine its inverse recursively.

Proposition 3.6. *Consider the ordered basis $B = \{M_0, M_1, \cdots, M_{2n-5}\}$ for the cell module $\Delta_n(\mu_n)$, where $n > 4$ and μ_n is as defined in (3.1). Then*

1) the set $B' = \{M_0, M_1, \cdots, M_{2n-7}, w_1, w_2\}$ is basis for $\Delta_n(\mu_n)$, where $w_1 = \frac{1}{1-t^2} M_{2n-8} - \frac{t}{1-t^2} M_{2n-7} - M_{2n-6}$ and $w_2 = \frac{1}{s(1-t^2)} M_{2n-8} - \frac{t}{s(1-t^2)} M_{2n-7} - M_{2n-5}$, *2) the Gram matrix corresponding to B' is given by*

$$G'(\mu_n) = \begin{pmatrix} G(\mu_{n-1}) & 0 \\ 0 & D' \end{pmatrix},$$

where $D' = \begin{pmatrix} ab^2\left(\frac{t^2-2}{t^2-1}\right) & bt\left(\frac{t^2-2}{t^2-1}\right) \\ bt\left(\frac{t^2-2}{t^2-1}\right) & bt\left(s - \frac{1}{s(t^2-1)}\right) \end{pmatrix}$ *and* $(s,t) = (a,b)$ *if* n *even while* $(s,t) = (b,a)$ *if* n *odd.*

Proof:
1) Since B and B' have same number of elements and the change of basis matrix from B to B' has determinant one, B' is linearly independent and hence B' is basis for $\Delta_n(\mu_n)$.

2) From Proposition 3.5, we have

$$G(\mu_n) = \begin{pmatrix} G(\mu_{n-1}) & Q_s \\ Q_s^T & D_t \end{pmatrix} = \left(\begin{array}{cc|c} G(\mu_{n-2}) & Q_t & \multirow{2}{*}{Q_s} \\ Q_t^T & D_s & \\ \hline \multicolumn{2}{c|}{Q_s^T} & D_t \end{array}\right).$$

Substituting the block matrices Q_s, Q_t^T, D_s, Q_s^T and D_t, we get

$G(\mu_n) =$

$\langle -,- \rangle$	$M_0 \quad M_1 \quad \cdots \quad M_{2n-9}$	$M_{2n-8} \quad M_{2n-7}$	$M_{2n-6} \quad M_{2n-5}$
$M_0, M_1, \vdots, M_{2n-9}$	$G(\mu_{n-2})$	Q_t	0
M_{2n-8}	$0 \quad 0 \quad \cdots \quad 0 \quad tb$	$ab^2 \quad sb$	$0 \quad 0$
M_{2n-7}	$0 \quad 0 \quad \cdots \quad 0 \quad b$	$sb \quad ab^2$	$sb \quad b$
M_{2n-6}	0	$0 \quad sb$	$ab^2 \quad tb$
M_{2n-5}		$0 \quad b$	$tb \quad ab^2$

To find the entries of $G'(\mu_n)$ we only need to compute $\langle w_i, M_j \rangle$ for $i = 1, 2$ and $j = 0, 1, \cdots, 2n-5$, while the remaining entries of $G'(\mu_n)$ are equal to the corresponding entries of $G(\mu_n)$; that is,

$$G'(\mu_n) = \begin{pmatrix} G(\mu_{n-1}) & X^T \\ X & Y \end{pmatrix},$$

where for $i = 1, 2$ and $j = 0, 1, \cdots, 2n-7$,

$$X_{ij} = \langle w_i, M_j \rangle,$$

and for $i, j = 1, 2$,

$$Y_{ij} = \langle w_i, w_j \rangle$$

.

One can check that
$\langle w_1, M_i \rangle = \langle M_{2n-6}, M_i \rangle = 0$ for $i = 0, 1, \cdots, 2n-7$,
$\langle w_2, M_i \rangle = \langle M_{2n-6}, M_i \rangle = 0$ for $i = 0, 1, \cdots, 2n-7$,
$\langle w_1, w_1 \rangle = ab^2 \left(\frac{t^2-2}{t^2-1} \right)$,
$\langle w_2, w_2 \rangle = bt \left(s - \frac{1}{s(t^2-1)} \right)$, and
$\langle w_1, w_2 \rangle = bt \left(\frac{t^2-2}{t^2-1} \right)$.
Then X is the zero matrix and $Y = D'$ as required.

For convenience, we rewrite the basis of $\Delta_n(\mu_n)$ that defined in Proposition 3.6 by setting $m_i = M_i$ for $i = 0, 1, \cdots, 2n-7$, $m_{2n-6} = w_1$, and $m_{2n-5} = w_2$. Then the basis of $\Delta_n(\mu_n)$ is $B' = \{m_0, m_1, \cdots, m_{2n-5}\}$.

Let $\{v_0, v_1, \cdots, v_r\}$ be a basis for a vector space V. We say that the set $\{v'_0, v'_1, \cdots, v'_r\}$ is a dual basis for V if it spans V and $\langle v_i, v'_j \rangle = \delta_{ij}$, where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$ (see [7, sec. 1]). Now, we are ready to introduce the dual basis for $\Delta_n(\mu_n)$. To construct the dual basis $\{v_0, v_1, \cdots, v_{2n-5}\}$ to the basis $B' = \{m_0, m_1, \cdots, m_{2n-5}\}$, we follow the method that described in [7, sec. 2]. We may express the dual basis v_i as a linear combination of the basis diagrams in B'. Then

$$v_i = \sum_{k=0}^{2n-5} c_{ik} m_k, \qquad 0 \leq i \leq 2n-5.$$

So, we have to find the coefficients c_{ik} such that the duality condition is satisfied, that is, $\langle m_j, v_i \rangle = \delta_{ij}$ for $0 \leq i, j \leq 2n-5$. The duality condition will give us a system of linear equations. For fix $0 \leq i \leq 2n-5$, we have

$$\delta_{ij} = \langle m_j, v_i \rangle = \sum_{k=0}^{2n-5} c_{ik} \langle m_j, m_k \rangle,$$

so the coefficients c_{ik} can be determined by solving the linear system

$$\begin{bmatrix} \langle m_0, m_0\rangle & \langle m_0, m_1\rangle & \cdots & \langle m_0, m_{2n-5}\rangle \\ \langle m_1, m_0\rangle & \langle m_1, m_1\rangle & \cdots & \langle m_1, m_{2n-5}\rangle \\ \vdots & \vdots & & \vdots \\ \langle m_i, m_0\rangle & \langle m_i, m_1\rangle & \cdots & \langle m_i, m_{2n-5}\rangle \\ \vdots & \vdots & & \vdots \\ \langle m_{2n-5}, m_0\rangle & \langle m_{2n-5}, m_1\rangle & \cdots & \langle m_{2n-5}, m_{2n-5}\rangle \end{bmatrix} \begin{bmatrix} c_{i0} \\ c_{i1} \\ \vdots \\ c_{ii} \\ \vdots \\ c_{i,2n-5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}.$$

This system can be written as $G'(\mu_n)C = E$, and hence, $C = (G'(\mu_n))^{-1}E$. Suppose that P is the change of basis matrix from B to B'. Then $G'(\mu_n) = P^T G(\mu_n)P$ and we get $G(\mu_n)^{-1} = PG'(\mu_n)^{-1}P^T$.

Proposition 3.7. *With the notations as above, for $n > 4$, the cell module $\Delta_n(\mu_n)$ with basis $B' = \{m_0, m_1, \cdots, m_{2n-5}\}$ has dual basis defined by*

$$v_i = \sum_{k=0}^{2n-5} c_{ik} m_k, \qquad (0 \le i \le 2n-5),$$

where, for fix i, the coefficients $c_{i,k}$ is given by the matrix

$$C = \begin{pmatrix} P\Big(G'(\mu_{n-1})\Big)^{-1} P^T & 0 \\ 0 & (D')^{-1} \end{pmatrix} E.$$

Proof:
From Proposition 3.6, we deduce

$$\Big(G'(\mu_n)\Big)^{-1} = \begin{pmatrix} \Big(G(\mu_n)\Big)^{-1} & 0 \\ 0 & (D')^{-1} \end{pmatrix}.$$

Let P be the change of basis matrix from B to B', then $G'(\mu_{n-1}) = P^T G(\mu_{n-1})P$, and hence, $G^{-1}(\mu_{n-1}) = P\Big(G'(\mu_{n-1})\Big)^{-1} P^T$. Hence the result follows.

Notice that, P can be found easily. So we can find the dual basis for $\Delta_n(\mu_n)$ recursively as explained in the following example.

Example 3.8. *For the case $n = 5$, using Proposition 3.5,*

$$G(\mu_4)^{-1} = \begin{pmatrix} \frac{a^2-1}{ab^2(a^2-2)} & \frac{-1}{b^2(a^2-2)} & 0 & \frac{1}{ab^2(a^2-2)} \\ \frac{-1}{b^2(a^2-2)} & \frac{a^2b^2-2}{ab^2(a^2-2)(b^2-1)} & \frac{-1}{ab(b^2-1)} & \frac{-1}{b^2(a^2-2)} \\ 0 & \frac{-1}{ab(b^2-1)} & \frac{1}{a(b^2-1)} & 0 \\ \frac{1}{ab^2(a^2-2)} & \frac{-1}{b^2(a^2-2)} & 0 & \frac{a^2-1}{ab^2(a^2-2)} \end{pmatrix}$$

$$\text{and } (D')^{-1} = \begin{pmatrix} \frac{1}{a(b^2-1)} & \frac{1}{ab(b^2-1)} \\ \frac{1}{ab(b^2-1)} & \frac{a^2b^2-b^2-1}{ab^2(a^2-2)(b^2-1)} \end{pmatrix}.$$

The dual basis $v_i = \sum_{k=0}^{5} c_{ik} m_k$ *are*
$v_0 = \frac{a^2-1}{ab^2(a^2-2)} m_0 + \frac{-1}{b^2(a^2-2)} m_1 + \frac{1}{ab^2(a^2-2)} m_3,$
$v_1 = \frac{-1}{b^2(a^2-2)} m_0 + \frac{a^2b^2-2}{ab^2(a^2-2)(b^2-1)} m_1 + \frac{-1}{ab(b^2-1)} m_2 + \frac{-1}{b^2(a^2-2)} m_3,$
$v_2 = \frac{-1}{ab(b^2-1)} m_1 + \frac{1}{a(b^2-1)} m_2,$
$v_3 = \frac{1}{ab^2(a^2-2)} m_0 + \frac{-1}{b^2(a^2-2)} m_1 + \frac{a^2-1}{ab^2(a^2-2)} m_3,$
$v_4 = \frac{1}{a(b^2-1)} m_4 + \frac{1}{ab(b^2-1)} m_5$
$v_5 = \frac{1}{ab(b^2-1)} m_4 + \frac{a^2b^2-b^2-1}{ab^2(a^2-2)(b^2-1)} m_5.$

We can find the dual basis for $n = 6$ *by using Proposition 3.7 since* $G'(\mu_{n-1})$ *is known for the case* $n = 5$.

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