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Estimates for Constrained Approximation in $\mathbb{L}_{p,r}^{\alpha,\beta}$ space: Piecewise Polynomials

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Abstract

This paper deals with weighted constrained approximation, on the interval $[-1,1]$, by piecewise polynomials. These results are applied to obtain Jackson-type estimates for the approximation of non-continuous functions. Our results improve, generalize and extend some weighted approximation results in the literature.

1 Introduction

K. A. Kopotun has a significant impact on the weighted approximation of f, for the past five years (see [11, 12, 13, 16, 17, 20]), and we are affected by his contribution in the kth symmetric difference (see, [10, proof of Lemma 4.1]). Let $\Delta_h^k(f,x)$ be the kth symmetric difference of f given [7] by

$$\Delta_h^k(f,x) = \left\{ \begin{array}{ll} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x + (\frac{2i-k}{2})h) \,; & x \pm \frac{kh}{2} \in [-1,1] \\ 0\,; & \text{otherwise.} \end{array} \right\}.$$

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Let $\|.\|_p = \|.\|_{L_p[-1,1]}, 0 < p \leq \infty$ and $\phi(x) = \sqrt{1-x^2}$. Then the Ditzian-Totik modulus of smoothness (DTMS) of a function $f \in L_p[-1,1]$ is defined [6] by

$$\omega_{k,r}^{\phi}(f,t)_p = \sup_{0<h\leq t} ||\phi^r \Delta_{h\phi}^k(f,x)||_p, \quad k,r \in \mathbb{N}_\circ.$$

Also, the kth usually modulus of smoothness of $f \in L_p[-1,1]$ is defined [7] by

$$\omega_k(f,\delta,[-1,1])_p = \sup_{0<h\leq \delta} ||\Delta_h^k(f,x)||_p, \quad \delta > 0, p \leq \infty.$$

Denote by $AC_{loc}(-1,1)$ and $AC[-1,1]$ the set of functions that are locally absolutely continuous on $(-1,1)$ and absolutely continuous on $[-1,1]$ respectively.

Definition 1.1. *[22] Let $w_{\alpha,\beta}(x) = (1+x)^\alpha(1-x)^\beta$ be the (classical) Jacobi weight, and let*

$$\alpha, \beta \in J_p = \begin{cases} (-1/p, \infty), & \textit{if}\, p < \infty, \\ [0,\infty), & \textit{if}\, p = \infty. \end{cases}$$

Define

$$\mathbb{L}_p^{\alpha,\beta} = \{f : [-1,1] \longrightarrow \mathbb{R} : \|w_{\alpha,\beta} f\|_p < \infty, \ \ \textit{and} \ \ 0 < p < \infty\},$$

and

$$\mathbb{L}_{p,r}^{\alpha,\beta} = \{f : [-1,1] \longrightarrow \mathbb{R} : f^{(r-1)} \in AC_{loc}(-1,1), 1 \leq p \leq \infty, ||w_{\alpha,\beta} f^{(r)}||_p < \infty\},$$

and for convenience let $\mathbb{L}_{p,0}^{\alpha,\beta} = \mathbb{L}_p^{\alpha,\beta}$.

Definition 1.2. *[27] Let $f \in \mathbb{C}$ be a continuous function on $[-1,1]$, $r \geq 1$ and $\phi(x) = \sqrt{1-x^2}$. Denote by $\mathbb{C}^{(r)}$ the space of continuous functions which possess an absolutely continuous $(r-1)$st derivative in $[-1,1]$ such that $f^{(r)}$ is almost everywhere bounded; that is,*

$$\mathbb{C}^{(r)} = \{f \in \mathbb{C} : f^{(r-1)} \in AC[-1,1] \ \ \textit{and} \ \ \|f^{(r)}\| < \infty\}.$$

Denote by $\overline{\mathbb{C}}^{(r)}$ the space of continuous functions which possess a locally absolutely continuous $(r-1)$st derivative in $(-1,1)$ such that

$$\overline{\mathbb{C}}^{(r)} = \{f \in \mathbb{C} : f^{(r-1)} \in AC_{loc}(-1,1) \ \ \textit{and} \ \ \|\phi^r f^{(r)}\|_p < \infty\}.$$

Definition 1.3. *[18] The polynomial p_n is a constrained polynomial, if p_n is shape preserving of the function f, except that being an unconstrained polynomial.*

Definition 1.4. *[10] For $k \in \mathbb{N}$, $r \in \mathbb{N}_\circ$ and $1 \leq p \leq \infty$, we define*

$$K^{\phi}_{k,r}(f^{(r)}, t^k)_p = \inf_{g \in \mathbb{C}_2^{(r+k)}} (\|(f^{(r)} - g^{(r)})\phi^r\|_p + t^k \|g^{(k+r)}\phi^{k+r}\|_p).$$

Definition 1.5. *[28] A subset X of $\mathbb{R}^n$ is a convex set if $[x,y] \subseteq X$, whenever $x, y \in X$. Equivalently, X is convex if*

$$(1-\lambda)x + \lambda y \in X, \ \text{for all } x, y \in X \text{ and } \lambda \in [0,1].$$

Theorem 1.6. *[29] Let f be a function from $\mathbb{R}^n$ to $[-\infty, \infty]$. Then f is convex if and only if*

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)\zeta + \lambda\xi, \ 0 < \lambda < 1,$$

whenever $f(x) < \zeta$ and $f(y) < \xi$.

Definition 1.7. *[23] Let π_n be the space of all algebraic polynomials of degree $\leq n-1$, and $\Delta^{(2)}$ be the set of all convex functions on $[-1,1]$. For $f \in \mathbb{C}([-1,1]) \cap \Delta^{(2)}$, the degree of best convex polynomial approximation of f is*

$$E_n^{(2)}(f) = \inf \{\|f - p_n\|, \ p_n \in \pi_n \cap \Delta^{(2)}\}.$$

Definition 1.8. *[8] Let $Y_s = \{y_i\}_{i=1}^s$, $s \in \mathbb{N}$ be a partition of $[-1,1]$; that is, a collection of s fixed points y_i such that*

$$y_{s+1} = -1 < y_s < \cdots < y_1 < 1 = y_\circ.$$

Let $\Delta^{(2)}(Y_s)$ be the set of continuous functions on $[-1,1]$ that are convex downwards on the segment $[y_{i+1}, y_i]$ if i is even and convex upwards on the same segment if i is odd. The functions from $\Delta^{(2)}(Y_s)$ are called coconvex.

Definition 1.9. *[14] Let $\Delta^{(2)}(Y_s)$ be the collection of all functions f in $\mathbb{C}([-1,1])$ that change convexity at the points of the set Y_s, and are convex in $[y_s, 1]$. The degree of best coconvex polynomial approximation of f is defined by $E_n^{(2)}(f, Y_s) = \inf \{\|f - p_n\|, \ p_n \in \pi_n \cap \Delta^{(2)}(Y_s)\}$.*

Definition 1.10. *[21] A function f is said to be k-monotone, $k \geq 1$ on $[a,b]$, if and only if for all choices of $k+1$ distinct points $x_\circ, \cdots, x_k$ in $[a,b]$ the inequality $f[x_\circ, \cdots, x_k] \geq 0$, holds, where*

$$f[x_\circ, \cdots, x_k] = \sum_{j=0}^{k} \frac{f(x_j)}{\theta'(x_j)}\,, \quad \theta(x_j) = \prod_{j=0}^{k} (x - x_j)$$

denote the kth divided difference of f at $x_\circ, \cdots, x_k$.

Definition 1.11. *[24] Denote by $\Delta^{(1)}$ the set of monotone, (say nondecreasing) functions $f \in \mathbb{C}([-1,1])$, and, as usual, for $f \in \Delta^{(1)}$, let*

$$E_n^{(1)}(f) = \inf \{ \|f - p_n\|,\ p_n \in \pi_n \cap \Delta^{(1)} \}$$

be the degree of best monotone polynomial approximation of f.

The degree of best comonotone polynomial approximation, $1 \leq p \leq \infty$, is defined as follows [4, 26]:

Definition 1.12. *For a collection $\Delta^{(1)}(Y_s)$, if $f \in \mathbb{C}([-1,1])$ and $(-1)^i f$ is nondecreasing on $[y_{i+1}, y_i]$, $0 \leq i \leq s$, the degree of best comonotone polynomial approximation of f is defined by $E_n^{(1)}(f, Y_s) = \inf \{ \|f - p_n\|,\ p_n \in \pi_n \cap \Delta^{(1)}(Y_s)\}$. If $Y_s = \emptyset$, then $E_n^{(1)}(f, \emptyset)_p = E_n^{(1)}(f)_p$ which is usually referred to as the degree of best monotone polynomial approximation.*

Remark 1.13. *From Definitions 1.8 and 1.12, we use the following notations*

$$sgn(f(x)) = \begin{cases} 1, & \text{if } x \in I, \\ -1, & \text{if } x \notin I. \end{cases}$$

and

$$conv(f(x)) = \begin{cases} 1, & \text{if } i \text{ even in } [y_{i+1}, y_i], \\ -1, & \text{if } i \text{ odd in } [y_{i+1}, y_i]. \end{cases}$$

The first authors to deal with such cases (constrained and unconstrained) were DeVore, Leviatan and Xiangming [5] followed by Kopotun in [18] and [19]. In 2003, Leviatan and Shevshuk [25] were interested in estimating the degree of approximation of f by piecewise polynomials, where f is coconvex function. For estimates on constrained approximation with piecewise polynomials by Leviatan, Shevchuk and Vlasiuk, see [26, 30, 31]. The following results were proven in [26].

Theorem 1.14. *Given $Y_s \in \mathbb{Y}_s$, $s \geq 2$, $\mathcal{N} \geq 1$ and $1 < \sigma \leq 2$, there exist constants $c(\sigma, s)$ and $\mathcal{N}^*(\sigma, Y_s)$, such that for all functions $\Delta^{(1)}(Y_s)$ satisfying*

$$n^\sigma E_n(f) \leq 1, \quad n \geq s+2,$$

$$n^\sigma E_n^{(1)}(f, Y_s) \leq c(1, \sigma, s, \mathcal{N}), \quad n \geq \mathcal{N}^*.$$

Theorem 1.15. *Given $Y_s \in \mathbb{Y}_s$, $s \geq 3$, $\mathcal{N} \geq 1$ and $2 < \sigma \leq 4$, there exist constants $c(\sigma, s)$ and $\mathcal{N}^*(\sigma, Y_s)$, such that for all functions $\Delta^{(2)}(Y_s)$ satisfying*

$$n^\sigma E_n(f) \leq 1, \quad n \geq s+3,$$

$$n^\sigma E_n^{(2)}(f, Y_s) \leq c(2, \sigma, s, \mathcal{N}), \quad n \geq \mathcal{N}^*.$$

In 2019, Kopotun et al. ([9, Theorem 2.1]) proposed a convex piecewise polynomial approximation $S \in \mathbb{S}(\hat{T}_n, r+2) \cap \Delta^{(2)})$, if there is a number $\mathcal{N}$ depending on a natural number r and the convex function f in $\mathbb{C}^{(r)}([-1, 1])$; with the knots of CH. PR. $\hat{T}_n$ (Chebyshev partition). Then

$$|f(x) - S(x)| \leq c(r)(\min \phi(x)^2, n^{-1}\phi(x))^r \omega_2(f^{(r)}, n^{-1}\phi(x)). \tag{1.1}$$

A main contribution to this paper is our question about the function

$$f(x) = \begin{cases} e^x & ; \text{if } x \in [-1, 0)/Q^c, \\ 6x & ; \text{if } x \in [0, 1]/Q^c, \end{cases} \tag{1.2}$$

such that Q^c is an irrational number.

What is happening for constrained approximation of piecewise polynomials for the function f, if it is discontinuous?

In other words, if the function is discontinuous, we see that (1.1) is invalid. The proofs of our main results will answer the above question.

We will adopt the constants c throughout the paper, which may depend on r, α, β and, may depend on $\omega_{1,r}^{\phi}$ or f, p, k, x_*, $x^{\#}$.
Note that the constants r, p, k are nonnegative integers while α, β, x_*, $x^{\#}$ are real numbers.

2 Definitions and Facts

In this section, we will present the linear space of Lebesgue Stieltjes integrablei functions. First, let us recall the definition of the Lebesgue Stieltjes integrablei, given in [1].

Definition 2.1. *Let $\mathbb{D}$ be a measurable set. Let $f : \mathbb{D} \longrightarrow \mathbb{R}$ be a bounded function. Let $\mathcal{L}_i : \mathbb{D} \longrightarrow \mathbb{R}$ be nondecreasing function for $i \in \Lambda$. For a Lebesgue partition $\mathbf{P}$ of $\mathbb{D}$, put $\underline{\mathrm{LS}}(f,\ \mathbf{P},\ \underline{\mathcal{L}}) = \sum_{j=1}^{n} \prod_{i\in\Lambda}\ m_j\ \mathcal{L}_i(\mu(\mathbb{D}_j))$ and $\overline{\mathrm{LS}}(f,\ \mathbf{P},\ \underline{\mathcal{L}}) = \sum_{j=1}^{n} \prod_{i\in\Lambda}\ M_j\ \mathcal{L}_i(\mu(\mathbb{D}_j))$ where μ is a measure function of $\mathbb{D}$, $m_j = \inf\{f(x) :\ \ x \in \mathbb{D}_j\}$, $M_j = \sup\{f(x) :\ \ x \in \mathbb{D}_j\}$, and $\underline{\mathcal{L}} = \mathcal{L}_1,\ \mathcal{L}_2,\ \cdots$. Also, $\mathcal{L}_i(x_j) - \mathcal{L}_i(x_{j-1}) > 0$, $\underline{\mathrm{LS}}(f,\ \mathbf{P},\ \underline{\mathcal{L}}) \leq \overline{\mathrm{LS}}(f,\ \mathbf{P},\ \underline{\mathcal{L}})$, $\prod_{i\in\Lambda} \underline{\int}_i^{\mathbb{D}} f\ \underline{\mathrm{d}\mathcal{L}} = \sup\{\underline{\mathrm{LS}}(f,\ \underline{\mathcal{L}})\}$ and $\prod_{i\in\Lambda} \overline{\int}_i^{\mathbb{D}} f\ \underline{\mathrm{d}\mathcal{L}} = \inf\{\overline{\mathrm{LS}}(f,\ \underline{\mathcal{L}})\}$ where $\underline{\mathrm{LS}}(f,\ \underline{\mathcal{L}}) = \{\underline{\mathrm{LS}}(f,\ \mathbf{P},\ \underline{\mathcal{L}}) :\ \mathbf{P}$ part of set $\mathbb{D}\}$ and $\overline{\mathrm{LS}}(f,\ \underline{\mathcal{L}}) = \{\overline{\mathrm{LS}}(f,\ \mathbf{P},\ \underline{\mathcal{L}}) :$ $\mathbf{P}$ part of set $\mathbb{D}\}$. If $\prod_{i\in\Lambda} \underline{\int}_i^{\mathbb{D}} f\ \underline{\mathrm{d}\mathcal{L}} = \prod_{i\in\Lambda} \overline{\int}_i^{\mathbb{D}} f\ \underline{\mathrm{d}\mathcal{L}}$ where $\underline{\mathrm{d}\mathcal{L}} = \mathrm{d}\mathcal{L}_1 \times\ \mathrm{d}\mathcal{L}_2 \times \cdots$, then f is integral $\int_i$ according to $\mathcal{L}_i$ for $i \in \Lambda$.*

Lemma 2.2. *If f is a function of Lebesgue Stieltjes integral-i, then υf is a function Lebesgue Stieltjes integral-i, where $\upsilon > 0$ is real number, and*

$$\prod_{i\in\Lambda} \int_i^{\mathbb{D}} \upsilon f\ \underline{\mathrm{d}\mathcal{L}} = \upsilon \prod_{i\in\Lambda} \int_i^{\mathbb{D}} f\ \underline{\mathrm{d}\mathcal{L}}\ , \tag{2.1}$$

holds.

Proof Let f be a function on the set $\mathbb{D}$ and satisfying Definition 2.1, $\mathbb{D}$ be a bounded set, and $\mathbf{P}$ be a Lebesgue partition of $\mathbb{D}$. From Definition 2.1, if $\upsilon > 0$ is real number, then $\underline{\mathrm{LS}}(\upsilon\ f,\ \mathbf{P},\ \underline{\mathcal{L}}) = \upsilon\ \underline{\mathrm{LS}}(f,\ \mathbf{P},\ \underline{\mathcal{L}})$, and $\overline{\mathrm{LS}}(\upsilon\ f,\ \mathbf{P},\ \underline{\mathcal{L}}) = \upsilon\ \overline{\mathrm{LS}}(f,\ \mathbf{P},\ \underline{\mathcal{L}})$. Thus $\prod_{i\in\Lambda} \underline{\int}_i^{\mathbb{D}} \upsilon\ f\ \underline{\mathrm{d}\mathcal{L}} = \upsilon\ \prod_{i\in\Lambda} \underline{\int}_i^{\mathbb{D}} f\ \underline{\mathrm{d}\mathcal{L}}$ and $\prod_{i\in\Lambda} \overline{\int}_i^{\mathbb{D}} \upsilon\ f\ \underline{\mathrm{d}\mathcal{L}} = \upsilon\ \prod_{i\in\Lambda} \overline{\int}_i^{\mathbb{D}} f\ \underline{\mathrm{d}\mathcal{L}}$.
Since f is function of integrable according to $\mathcal{L}_i$,

$$\prod_{i\in\Lambda} \underline{\int}_i^{\mathbb{D}} \upsilon\ f\ \underline{\mathrm{d}\mathcal{L}} = \prod_{i\in\Lambda} \overline{\int}_i^{\mathbb{D}} \upsilon\ f\ \underline{\mathrm{d}\mathcal{L}} = \upsilon\ \prod_{i\in\Lambda} \overline{\int}_i^{\mathbb{D}} f\ \underline{\mathrm{d}\mathcal{L}} = \upsilon\ \prod_{i\in\Lambda} \underline{\int}_i^{\mathbb{D}} f\ \underline{\mathrm{d}\mathcal{L}}.$$

Hence, (2.1) with $\upsilon > 0$ real number, holds.

Lemma 2.3. *If the functions f_1, f_2 are integrable on the set $\mathbb{D}$ according to $\mathcal{L}_i$, for $i \in \Lambda$, then $f_1 + f_2$ is the function of integrable according to $\mathcal{L}_i$, for $i \in \Lambda$, such that*

$$\prod_{i\in\Lambda} \underline{\int}_i^{\mathbb{D}} (f_1 + f_2)\ \underline{\mathrm{d}\mathcal{L}} = \prod_{i\in\Lambda} \underline{\int}_i^{\mathbb{D}} f_1\ \underline{\mathrm{d}\mathcal{L}} + \prod_{i\in\Lambda} \underline{\int}_i^{\mathbb{D}} f_2\ \underline{\mathrm{d}\mathcal{L}}.$$

Proof Let f_1 and f_2 be functions on the set $\mathbb{D}$ and satisfying Definition 2.1, $\mathbb{D}$ be a bounded set, and $\mathbf{P}$ be a Lebesgue partition of $\mathbb{D}$.
Then $\underline{\text{LS}}(f_1 + f_2,\ \mathbf{P},\ \underline{\mathcal{L}}) \geq \underline{\text{LS}}(f_1,\ \mathbf{P},\ \underline{\mathcal{L}}) + \underline{\text{LS}}(f_2,\ \mathbf{P},\ \underline{\mathcal{L}})$ and $\overline{\text{LS}}(f_1 + f_2,\ \mathbf{P},\ \underline{\mathcal{L}}) \geq \overline{\text{LS}}(f_1,\ \mathbf{P},\ \underline{\mathcal{L}}) + \overline{\text{LS}}(f_2,\ \mathbf{P},\ \underline{\mathcal{L}})$. For $\varepsilon > 0$, there exist two Lebesgue Partitions $\mathbf{P}_1$ and $\mathbf{P}_2$ of $\mathbb{D}$ whose union is $\mathbf{P}$ such that $\overline{\text{LS}}(f_1,\ \mathbf{P}_1,\ \underline{\mathcal{L}}) - \underline{\text{LS}}(f_1,\ \mathbf{P}_1,\ \underline{\mathcal{L}}) \leq \frac{\varepsilon}{2}$ and $\overline{\text{LS}}(f_2,\ \mathbf{P}_2,\ \underline{\mathcal{L}}) - \underline{\text{LS}}(f_2,\ \mathbf{P}_2,\ \underline{\mathcal{L}}) \leq \frac{\varepsilon}{2}$. Therefore,

$$\underline{\text{LS}}(f_1,\ \mathbf{P}_1,\ \underline{\mathcal{L}}) - \underline{\text{LS}}(f_2,\ \mathbf{P}_2,\ \underline{\mathcal{L}}) \leq \overline{\text{LS}}(f_1,\ \mathbf{P}_1,\ \underline{\mathcal{L}}) - \overline{\text{LS}}(f_2,\ \mathbf{P}_2,\ \underline{\mathcal{L}}) + \varepsilon.$$

Definition 2.4. *[2] A domain $\mathbb{D}$ of convex polynomial p_n of $\Delta^{(2)}$ is a subset of X and $X \subseteq \mathbb{R}$, satisfying the following properties:*

1. $\mathbb{D} \in \mathbb{K}^N$ *, where*
   $\mathbb{K}^N = \{\mathbb{D} : \mathbb{D}$ *is a compact subset of* $X\}$
   is the class of all domain of convex polynomial,
2. *there is the point $t \in X/\mathbb{D}$, such that*
   $|p_n(t)| > \sup\{|p_n(x)| : x \in \mathbb{D}\}$*, and*
3. *there is the function f of $\Delta^{(2)}$, such that*
   $\|f - p_n\| \leq \frac{c}{n^2}\, \omega^{\phi}_{2,2}(f'', \frac{1}{2})$.

From Definitions 2.1 and 2.4, if the function f is convex, then $\mathbb{D}$ is domain of convex function of f.

Remark 2.5. *Let I_f be the class of all functions of integrable f satisfying Definition 2.1; i.e.,*

$$\begin{aligned} I_f &= \{f : f \text{ is integrable function according to } \mathcal{L}_i,\ i \in \Lambda\} \\ &= \{f : \prod_{i\in\Lambda} \underline{\int}_i^{\mathbb{D}} f\ \underline{\mathrm{d}\mathcal{L}} = \prod_{i\in\Lambda} \overline{\int}_i^{\mathbb{D}} f\ \underline{\mathrm{d}\mathcal{L}}\}. \end{aligned}$$

Definition 2.6. *For $f \in I_f$ and $r \in \mathbb{N}_\circ$, we define*

$$\omega^{\phi}_{i,r}\ (f^{(r)},\ \|\theta_N\|, [-1,1])_{w_{\alpha,\beta},p} = \sup\ \{\|w_{\alpha,\beta}\ \phi^r \Delta^i_{h\phi}(f^{(r)},\ x)\|_p\ ,\ 0 < h \leq \|\theta_N\|\},$$

where

$$\Delta^i_{h\phi}(f^{(r)},\ x) = \prod_{i\in\Lambda} \int_i^{\mathbb{D}} f\ \underline{\mathrm{d}\mathcal{L}}$$

and $\|\theta_N\| < 2(i^{-1}),\ \mathcal{N} \geq 2$.

In this paper, for f a function of Lebesgue-Stieltjes integral-i, we will discuss the relationship between integral and estimate of Jackson type theorems in the $\mathbb{L}_{p,r}^{\alpha,\beta}$ space.

3 Main Results

An immediate application of Definition 2.1, together with Lemmas 2.2 and 2.3, are the following main results. Denote by I, I_α and I_β the intervals whose are $[-1,1]$, $[-1,-1+\alpha]$ and $[1,1-\beta]$ respectively.

Theorem 3.1. *For $r \in \mathbb{N}_\circ$, $\alpha,\ \beta \in J_p$, there is a constant $c = c(r,\alpha,\beta,p)$ such that if $f \in \Delta^{(2)} \cap \mathbb{L}_{p,r}^{\alpha,\beta}$, there, a number $\mathcal{N} = \mathcal{N}(f, \omega_{1,r}^{\phi}\ (f^{(r)}, \|\theta_{\mathcal{N}}\|, I)_{w_{\alpha,\beta},p})$ for $n \geq \mathcal{N}$ and $S \in \mathbb{S}(\hat{T}_n, r+2) \cap \Delta^{(2)} \cap \mathbb{L}_{p,r}^{\alpha,\beta}$, such that*

$$\|f^{(r)} - S^{(r)}\|_{w_{\alpha,\beta},p} \leq c_{r,\alpha,\beta,p,\omega_{1,r}^{\phi}} \min\{\omega_{i,r}^{\phi}\ (f^{(r)}, \|\theta_{\mathcal{N}}\|, I_\alpha)_{w_{\alpha,\beta},p}\ ,\ \omega_{i,r}^{\phi}\ (f^{(r)}, \|\theta_{\mathcal{N}}\|, I_\beta)_{w_{\alpha,\beta},p}\}, \tag{3.1}$$

where

$$\Delta_{h\phi,\alpha}^{i}(f^{(r)},\ x) = \int_1^{\mathbb{D}} \int_2^{\mathbb{D}} \cdots \int_i^{\mathbb{D}} \cdots f^{(r)}\ \mathrm{d}\mathcal{L}_{1\mathrm{t},\alpha}\ \mathrm{d}\mathcal{L}_{2\mathrm{t},\alpha} \cdots \mathrm{d}\mathcal{L}_{\mathrm{it},\alpha} \cdots = \prod_{i\in\Lambda} \int_i^{\mathbb{D}} f^{(r)}\ \underline{\mathrm{d}\mathcal{L}_{\mathrm{t}\phi,\alpha}}\ ,$$

$$\Delta_{h\phi,\beta}^{i}(f^{(r)},\ x) = \int_1^{\mathbb{D}} \int_2^{\mathbb{D}} \cdots \int_i^{\mathbb{D}} \cdots f^{(r)}\ \mathrm{d}\mathcal{L}_{1\mathrm{t},\beta}\ \mathrm{d}\mathcal{L}_{2\mathrm{t},\beta} \cdots \mathrm{d}\mathcal{L}_{\mathrm{it},\beta} \cdots = \prod_{i\in\Lambda} \int_i^{\mathbb{D}} f^{(r)}\ \underline{\mathrm{d}\mathcal{L}_{\mathrm{t}\phi,\beta}}\ .$$

Moreover, if $r, \alpha, \beta = 0$, then

$$\|f - S\|_p \leq c(\omega_i^{\phi}) \omega_i^{\phi}\ (f, \|\theta_{\mathcal{N}}\|, I)_p\ . \tag{3.2}$$

In particular,

$$\|f^{(r)} - S^{(r)}\|_{w_{\alpha,\beta},p} \leq c_r \omega_{1,r}^{\phi}\ (f^{(r)}, \|\theta_{\mathcal{N}}\|, I)_{w_{\alpha,\beta},p}\ . \tag{3.3}$$

Proof It suffices to prove the result on $I = [a,b]$, since we can then get the general result for (3.1) by applying a piecewise convex polynomial. Additionally, by subtracting a linear polynomial interpolating f at endpoints

$f(-1) = f(1) = 0$, it is also clear that we can assume that f is not a constant function, and so, because of its convexity, $f(x) < 0$, for all $x \in (-1, 1)$.
If $z_{r,\alpha}$ and $z_{r,\beta}$ are defined by

$$z_{r,\alpha}(x) = \frac{x - x_i}{x_* - x^\#}\left(\frac{x - x^\#}{x_* - x_i}\; z_{r,\alpha}(x_*) - \frac{x - x_*}{x_i - x^\#}\; z_{r,\alpha}(x^\#)\right)$$

and

$$z_{r,\beta}(x) = \frac{x - x_i}{x_* - x^\#}\left(\frac{x - x^\#}{x_* - x_i}\; z_{r,\beta}(x_*) - \frac{x - x_*}{x_i - x^\#}\; z_{r,\beta}(x^\#)\right)$$

where $x_i \in [\frac{x_i + x^\#}{2}, \frac{x_i + x_*}{2}] \subseteq \theta_\mathcal{N},\ x^\# = x_{j(i)+1},\ x_* = x_{j(i)-2},\ \theta_\mathcal{N} = \theta_\mathcal{N}[-1,1] = \{x_i\}_{i=0}^{\mathcal{N}} = \{-1 = x_\circ \leq \cdots \leq x_{\mathcal{N}-1} \leq x_\mathcal{N} = 1\}$ and $\|\theta_\mathcal{N}\| = \max_{0 \leq i \leq \mathcal{N}-1}\{x_{i+1} - x_i\}$ the length of the largest interval in that partition. Also,

$$z_{r,\alpha}(x^\#) = \begin{cases} \frac{C(conv(f(x^\#)))^{-1}}{\omega_1^\phi\, (f, \|\theta_\mathcal{N}\|, I_\alpha)_{w_{\alpha,\beta},p}}, & \text{if } \|f(x^\#)\|_{w_{\alpha,\beta},p} \leq c\, \omega_1^\phi\, (f, \|\theta_\mathcal{N}\|, I_\alpha)_{w_{\alpha,\beta},p}, \\ f(x^\#), & \text{otherwise} \end{cases}$$

and

$$z_{r,\alpha}(x_*) = \begin{cases} \frac{C(conv(f(x_*)))^{-1}}{\omega_1^\phi\, (f, \|\theta_\mathcal{N}\|, I_\alpha)_{w_{\alpha,\beta},p}}, & \text{if } \|f(x_*)\|_{w_{\alpha,\beta},p} \leq c\, \omega_1^\phi\, (f, \|\theta_\mathcal{N}\|, I_\alpha)_{w_{\alpha,\beta},p}, \\ f(x_*), & \text{otherwise.} \end{cases}$$

Choose $z_{r,\beta}$ as defined

$$z_{r,\beta}(x^\#) = \begin{cases} \frac{C(conv(f(x^\#)))^{-1}}{\omega_1^\phi\, (f, \|\theta_\mathcal{N}\|, I_\beta)_{w_{\alpha,\beta},p}}, & if \|f(x^\#)\|_{w_{\alpha,\beta},p} \leq c\, \omega_1^\phi\, (f, \|\theta_\mathcal{N}\|, I_\beta)_{w_{\alpha,\beta},p}, \\ f(x^\#), & \text{otherwise.} \end{cases}$$

and

$$z_{r,\beta}(x_*) = \begin{cases} \frac{C(conv(f(x_*)))^{-1}}{\omega_1^\phi\, (f, \|\theta_\mathcal{N}\|, I_\beta)_{w_{\alpha,\beta},p}}, & \text{if } \|f(x_*)\|_{w_{\alpha,\beta},p} \leq c\, \omega_1^\phi\, (f, \|\theta_\mathcal{N}\|, I_\beta)_{w_{\alpha,\beta},p}, \\ f(x_*), & \text{otherwise.} \end{cases}$$

Let $z_{r,\alpha}$ and $z_{r,\beta}$ be polynomials of degree $\leq r+1$ having Lebesgue-Stieltjes integral-i properties, and measurable on $\mathbb{D}$ and $t \in \mathbb{D}$. Suppose the following:
(i) t is real number,

(ii) $\alpha, \beta \in J_p\ ,\ p < \infty\ ,$
(iii) $\mathcal{L}_{t,\alpha}(x) = \frac{\alpha+1-\alpha^2}{\alpha(x-ty_1)^{\frac{1}{2}}}$ and $\mathcal{L}_{t,\beta}(x) = \frac{\beta+1-\beta^2}{\beta(x-ty_1)^{\frac{1}{2}}}\ ,$
where $x \in \mathbb{D}_t \subseteq \mathbb{D},\ t \in \mathbb{D}_t$ and y_1 is an inflection point that change (piecewise) convexity; like zero point of the convex function in (1.2).
Let $\mathcal{L}_{b,\alpha}(x)$ be a function such that $\mathcal{L}_{b,\alpha}(x) = 1,\ x \in \mathbb{D}$. Put $g_i(x) = z_{r,\beta}\mathcal{L}_{b,\alpha}(x)$ such that $x < b \in \mathbb{D}$.
Then $0 \leq \mathcal{L}_{t,\beta}(x) \leq \mathcal{L}_{b,\alpha}(x) = 1$. So $0 \leq z_{r,\alpha}\mathcal{L}_{t,\beta}(x) \leq z_{r,\beta}\mathcal{L}_{b,\alpha}(x) = g_i(x)$.
Let f be bounded function for $x \in \mathbb{D}$, and $u_j = -B + \frac{2B}{n}j,\ 0 \leq j \leq n$ and $B > |f(x)|$ for $x \in \mathbb{D}$. For j put $\mathbb{D}_j = f^{-1}[u_{j-1}, u_j)$, such that the family of sets $\mathbf{P}_n = \{\mathbb{D}_j\}$ is Lebesgue partition of $\mathbb{D}$.

$$\overline{\mathrm{LS}}(f,\ \mathbf{P}_n,\ \underline{\mathrm{g}}) - \underline{\mathrm{LS}}(f,\ \mathbf{P}_n,\ \underline{\mathrm{g}}) = \sum_{j=1}^{n}\prod_{i\in\Lambda}(M_j - m_j)\ g_i(\mu(\mathbb{D}_j))$$

$$\leq \sum_{j=1}^{n}\prod_{i\in\Lambda}(\frac{2B}{n})\ g_i(\mu(\mathbb{D}_j)) = \prod_{i\in\Lambda}(\frac{2B}{n})\ g_i(\mu(\mathbb{D})).$$

Now, for $\varepsilon > 0$, there is $\daleth \in \Lambda$ such that $\prod_{i\in\Lambda}(\frac{2B}{n})\ g_i(\mu(\mathbb{D})) \leq \varepsilon$.

Therefore, $\overline{\mathrm{LS}}(f,\ \mathbf{P}_\daleth,\ \underline{\mathrm{g}}) - \underline{\mathrm{LS}}(f,\ \mathbf{P}_\daleth,\ \underline{\mathrm{g}}) \leq \varepsilon$.

Hence, f satisfies the Lebesgue-Stieltjes integral-i.

Let $\mathbb{D}$ be a partition $\mathbf{P}'' = \{\mathbb{D}_j\}_{j=1}^{n}$ of some points in $\mathbb{D}$. Then, $\mathbb{D}$ is refinement of $\mathbf{P}_n$ such that $\sup\ \{\mu(\mathbb{D}_j) :\ j = 1, \cdots, n\} < \frac{1}{2}\delta(\varepsilon)$, and $\frac{2}{m} < \inf\ \{\mu(\mathbb{D}_j) : j = 1, \cdots, n\},\quad m \in \mathbb{N}$, where $\cap_{j=1}^{n}\mathbb{D}_j \neq \emptyset$.
Since $\alpha, \beta \in J_p\ ,\ p < \infty$, either $\alpha, \beta \neq 0$ or $\alpha, \beta = 0$.

Case I. If $\alpha, \beta \neq 0$, then we assume that $\alpha \leq \beta$, implies $1 - \beta \leq \alpha$. We now construct the piecewise polynomials $s \in \mathbb{S}(\hat{T}_n, r+2) \cap \Delta^{(2)} \cap \mathbb{L}_{p,r}^{\alpha,\beta}$. Let

$$s(x) = \begin{cases} z_{r,\alpha}(f, x), & \text{if } x \in [-1, \alpha), \\ z_{r,\beta}(f, x), & \text{if } x \in (\beta, 1]. \end{cases}$$

In this case, we will use Riesz's Representation Theorem (R.R.T).

$$\|f^{(r)} - s^{(r)}\|_{w_{\alpha,\beta},p} = \|w_{\alpha,\beta}(f^{(r)} - s^{(r)})\|_p = \|w_{\alpha,\beta}(f^{(r)} - (z_{r,\alpha}^{(r)}f + z_{r,\beta}^{(r)}f))\|_p$$

$$\leq (\int_{-1}^{1}|w_{\alpha,\beta}[f^{(r)} - ((\prod_{i\in\Lambda}\int_i^{\mathbb{D}} f^{(r)}\ \underline{\mathrm{d}\mathcal{L}_{\mathrm{t}\phi,\alpha}}) + (\prod_{i\in\Lambda}\int_i^{\mathbb{D}} f^{(r)}\ \underline{\mathrm{d}\mathcal{L}_{\mathrm{t}\phi,\beta}}))]|^p dx)^{\frac{1}{p}},\quad \textit{by (R.R.T.)}$$

$$\leq (\int_{-1}^{1} |w_{\alpha,\beta} f^{(r)} - (w_{\alpha,\beta}(\prod_{i\in\Lambda}\int_{i}^{\mathbb{D}} f^{(r)}\ \underline{\mathrm{d}\mathcal{L}_{\mathrm{t}\phi,\alpha}}) + w_{\alpha,\beta}(\prod_{i\in\Lambda}\int_{i}^{\mathbb{D}} f^{(r)}\ \underline{\mathrm{d}\mathcal{L}_{\mathrm{t}\phi,\beta}}))|^{p} dx)^{\frac{1}{p}}$$

$$\leq (\int_{-1}^{1} |w_{\alpha,\beta} f^{(r)}|^{p} dx)^{\frac{1}{p}} + (\int_{-1}^{1} |w_{\alpha,\beta}(\prod_{i\in\Lambda}\int_{i}^{\mathbb{D}} f^{(r)}\ \underline{\mathrm{d}\mathcal{L}_{\mathrm{t}\phi,\alpha}})|^{p} dx)^{\frac{1}{p}} +$$

$$(\int_{-1}^{1} |w_{\alpha,\beta}(\prod_{i\in\Lambda}\int_{i}^{\mathbb{D}} f^{(r)}\ \underline{\mathrm{d}\mathcal{L}_{\mathrm{t}\phi,\beta}}))|^{p} dx)^{\frac{1}{p}}$$

$$\leq (\int_{-1}^{1} |w_{\alpha,\beta} f^{(r)}|^{p} dx)^{\frac{1}{p}} + (\int_{-1}^{1} \prod_{i\in\Lambda}\int_{i}^{\mathbb{D}} |w_{\alpha,\beta} f^{(r)}|^{p} \underline{\mathrm{d}\mathcal{L}_{\mathrm{t}\phi,\alpha}}) dx)^{\frac{1}{p}} +$$

$$(\int_{-1}^{1} \prod_{i\in\Lambda}\int_{i}^{\mathbb{D}} |w_{\alpha,\beta} f^{(r)}|^{p} \underline{\mathrm{d}\mathcal{L}_{\mathrm{t}\phi,\beta}})) dx)^{\frac{1}{p}}$$

$$\leq (|\sup L(w_{\alpha,\beta} \times f^{(r)})|^{p})^{\frac{1}{p}} + (|\sup L(w_{\alpha,\beta} \times \sup LS(f^{(r)}, \underline{\mathcal{L}_{\mathrm{t}\phi,\alpha}}))|^{p})^{\frac{1}{p}} +$$

$$(|\sup L(w_{\alpha,\beta} \times \sup LS(f^{(r)}, \underline{\mathcal{L}_{\mathrm{t}\phi,\beta}}))|^{p})^{\frac{1}{p}}$$

$$\leq \sup |(m_{w_{\alpha,\beta}, f^{(r)}}) \times \mu(\mathbb{D}_k)| + \sup |(m_{w_{\alpha,\beta}} \times \sup(\sum_{j=1}^{n}\prod_{i\in\Lambda} m_j \mathcal{L}_{it,\alpha}(\mu(\mathbb{D}_j)))) \times \mu(\mathbb{D}_k)| +$$

$$\sup |(m_{w_{\alpha,\beta}} \times \sup(\sum_{j=1}^{n}\prod_{i\in\Lambda} m_j \mathcal{L}_{it,\beta}(\mu(\mathbb{D}_j)))) \times \mu(\mathbb{D}_k)|, \quad \textit{where}\ \mathbb{D}_k \in \{\mathbb{D}_j\}_{j=1}^{n},$$

$$\leq \sup |\inf(w_{\alpha,\beta} \times f^{(r)}(x)) \times \mu(\mathbb{D}_k)| + \sup |\inf(w_{\alpha,\beta} \times \sup(\sum_{j=1}^{n}\prod_{i\in\Lambda} \inf_{j}(f^{(r)}(x)) \mathcal{L}_{it,\alpha}(\mu(\mathbb{D}_j))) \times \mu(\mathbb{D}_k))| +$$

$$\sup|\inf(w_{\alpha,\beta}\times\sup(\sum_{j=1}^{n}\prod_{i\in\Lambda}\inf_{j}(f^{(r)}(x))\mathcal{L}_{it,\beta}(\mu(\mathbb{D}_j)))\times\mu(\mathbb{D}_k))|$$

$$\leq\inf|\sup((w_{\alpha,\beta}\times f^{(r)}(x))\times\mu(\mathbb{D}_k))|+$$

$$\inf|\sup(w_{\alpha,\beta}\times\sup(\sum_{j=1}^{n}\inf_{j}(f^{(r)}(x))[(\frac{\alpha+1-\alpha^2}{\alpha(\mu(\mathbb{D}_1)-ty_1)^{\frac{1}{2}}})\times(\frac{\alpha+1-\alpha^2}{\alpha(\mu(\mathbb{D}_2)-ty_1)^{\frac{1}{2}}})\cdots])\times\mu(\mathbb{D}_k))|+$$

$$\inf|\sup(w_{\alpha,\beta}\times\sup(\sum_{j=1}^{n}\inf_{j}(f^{(r)}(x))[(\frac{\beta+1-\beta^2}{\beta(\mu(\mathbb{D}_1)-ty_1)^{\frac{1}{2}}})\times(\frac{\beta+1-\beta^2}{\beta(\mu(\mathbb{D}_2)-ty_1)^{\frac{1}{2}}})\cdots])\times\mu(\mathbb{D}_k))|$$

$$\leq\inf(\sup\{\|w_{\alpha,\beta}\phi^r\Delta^1_{h\phi}(f^{(r)},\cdot)\|_p,\ 0<h\leq\|\theta_{\mathcal{N}}\|\})+$$

$$c_\alpha\inf(\sup\{\|w_{\alpha,\beta}\phi^r\Delta^i_{h\phi,\alpha}(f^{(r)},\cdot)\|_p,0<h\leq\|\theta_{\mathcal{N}}\|\})$$

$$+c_\beta\inf(\sup\{\|w_{\alpha,\beta}\phi^r\Delta^i_{h\phi,\beta}(f^{(r)},\cdot)\|_p,\ 0<h\leq\|\theta_{\mathcal{N}}\|\})$$
$$\leq\inf\omega^\phi_{1,r}(f^{(r)},\|\theta_{\mathcal{N}}\|,I)_{w_{\alpha,\beta},p}+c_\alpha\inf\omega^\phi_{i,r}(f^{(r)},\|\theta_{\mathcal{N}}\|,I_\alpha)_{w_{\alpha,\beta},p}+$$
$$c_\beta\inf\omega^\phi_{i,r}(f^{(r)},\|\theta_{\mathcal{N}}\|,I_\beta)_{w_{\alpha,\beta},p}. \tag{3.4}$$

Case II. If $\alpha,\beta=0$, $p\leq\infty$, then from (3.4), [4] and [15], we get (3.2). The case (3.3) is clear.

Lemma 3.2. *If $\Delta^i_{h\phi,\alpha}(f^{(r)},x)$ and $\Delta^i_{h\phi,\beta}(f^{(r)},x)$ are defined in Theorem 3.1, $r\in\mathbb{N}$ and $\alpha,\beta,\alpha-\delta,\beta-\delta\in J_p$, $\delta<1$, then, for any $I_{f^{(r)}}$ is Lebesgue-Stieltjes integral-i of f, such that $f\in\mathbb{L}^{\alpha-\delta,\beta-\delta}_{p,r}$, we have*

$$\|w_{\alpha-\delta,\beta-\delta}\ I_{f^{(r)}}\|_p\leq c_{\delta,r,p}\|w_{\alpha,\beta}\phi^r I_{f^{(r)}}\|_p\leq c_{\alpha,\beta}\omega^\phi_{i,r}(f^{(r)},\|\theta_{\mathcal{N}}\|,I)_{w_{\alpha,\beta},p}. \tag{3.5}$$

In particular, $f \in \mathbb{L}_{p,r}^{\alpha,\beta}$.

Proof Assume that $\|w_{\alpha-\delta,\beta-\delta}\|_p < \infty$, such that $\delta < 1$, and $f^{(r)}(0) = 0$. From ([22, pp. 126]), we get

$$w_{\alpha+\frac{r}{2},\beta+\frac{r}{2}}\ I_{f^{(r)}} = w_{\alpha,\beta}\phi^r I_{f^{(r)}}\ . \tag{3.6}$$

Now, either $-\delta \neq \frac{r}{2}$ or $-\delta = \frac{r}{2}$.

Case I. If $-\delta \neq \frac{r}{2}$, since $r \in \mathbb{N}$, implies $-\delta < \frac{r}{2}$ and by (3.6), then

$$\|w_{\alpha-\delta,\beta-\delta}\ I_{f^{(r)}}\|_p = \|w_{\alpha,\beta}\phi^{-\delta}\ I_{f^{(r)}}\|_p = \|w_{\alpha,\beta}\phi^{-\delta}\ I_{f^{(r)}} \times (\phi^{-r}\phi^r)\|_p =$$

$$\|w_{\alpha,\beta}\phi^r\ I_{f^{(r)}} \times (\phi^{-\delta-r})\|_p$$

$$< \|\phi^{-\delta-r}\|_p \times \|w_{\alpha,\beta}\phi^r\ I_{f^{(r)}}\|_p < c_{\delta,r,p}\|w_{\alpha,\beta}\phi^r\ I_{f^{(r)}}\|_p.$$

Case II. If $-\delta = \frac{r}{2}$, then from (3.6) we get $-\delta = \frac{r}{2} \geq 1$ and $w_{\alpha-\delta,\beta-\delta}\ I_{f^{(r)}} = w_{\alpha,\beta}\phi^r\ I_{f^{(r)}}$. Thus,

$$\|w_{\alpha-\delta,\beta-\delta}\ I_{f^{(r)}}\|_p = \|w_{\alpha,\beta}\phi^r\ I_{f^{(r)}}\|_p. \tag{3.7}$$

From cases (I) and (II), $\|w_{\alpha-\delta,\beta-\delta}\ I_{f^{(r)}}\|_p \leq c_{\delta,r,p}\|w_{\alpha,\beta}\phi^r\ I_{f^{(r)}}\|_p$ and $\mathbb{L}_{p,r}^{\alpha-\delta,\beta-\delta} \subset \mathbb{L}_{p,r}^{\alpha,\beta}$.

Finally, by using $\Delta^i_{h\phi,\alpha}(f^{(r)},x)$ and $\Delta^i_{h\phi,\beta}(f^{(r)},x)$, (3.5) follows from (3.7) and (3.4) through cases (I) and (II).

Theorem 3.3. *Let* Δ^k *be the space of all* k*-monotone functions. If* $f \in \Delta^k \cap \mathbb{L}_{p,r}^{\alpha,\beta}$ *is such that* $f^{(r)}(x) = p_n^{(r)}(x)$*, where* $p_n \in \pi_n \cap \Delta^k$*,* $N \geq k \geq 2$ *and* $s \in \mathbb{S}(\hat{T}_n, r+2) \cap \Delta^k \cap \mathbb{L}_{p,r}^{\alpha,\beta}$*, then*

$$\|f - s\|_{w_{\alpha,\beta},p} \leq c(f,p,k,\alpha,\beta,x_*,x^{\#})\omega_{i,r}^{\phi}(f,\|\theta_{\mathcal{N}}\|,I)_{w_{\alpha,\beta},p}\ . \tag{3.8}$$

In particular, if f *is a convex function and* p_n *is a convex polynomial or piecewise convex polynomial, then*

$$\|f-s\|_{w_{\alpha,\beta},p} \le c_k\ \omega^{\phi}_{i,r}(f,\|\theta_{\mathcal{N}}\|,I)_{w_{\alpha,\beta},p}\ . \tag{3.9}$$

Proof Suppose that $p_n \in \pi_n \cap \Delta^k$, $k \ge 2$ and $f^{(r)}(x) = p_n^{(r)}(x)$, ($f$ is k-monotone function and $\|w_{\alpha,\beta}\ f^{(r)}\|_p < \infty$, $1 \le p \le \infty$). Let z_r be defined in Theorem 3.1, on the interval I, where $\mathrm{sgn}(f)$ replaces $\mathrm{conv}(f)$. Let $s(x)$ be a piecewise polynomial of degree $\le n$, and

$$s(x) = \begin{cases} |p_n(f,\cdot|x_\circ,\cdots,x_{k-1}) - z_r|v(x) + z_r, & \text{if } x \in [x^{\#},x_*], \\ p_n(f,\cdot|x_\circ,\cdots,x_{k-1}), & \text{otherwise.} \end{cases}$$

such that

$$v(x) = \begin{cases} 1, & \text{if } x \notin I, \\ 0, & \text{if } x \in \cup_{i=1}^k[\frac{x_i+x^{\#}}{2},\frac{x_i+x_*}{2}], \\ C, & \text{if } x \in [x^{\#},\frac{x_i+x^{\#}}{2}],\ 1 \le i \le k. \end{cases}$$

From (3.7), we get

Case I. If $x \notin [x^{\#},x_*]$, then

$$\|f-p_n\|_{w_{\alpha,\beta},p} = \|w_{\alpha,\beta}(f-p_n(f,\cdot|x_\circ,\cdots,x_{k-1}))\|_p \tag{3.10}$$

$$= \|w_{\alpha,\beta}f - w_{\alpha,\beta}\times p_n(f,\cdot|x_\circ,\cdots,x_{k-1}) + w_{\alpha,\beta}z_r - w_{\alpha,\beta}z_r\|_p$$

$$= \|w_{\alpha,\beta}f - w_{\alpha,\beta}\sum_{j=1}^k f(t_j)\prod_{\substack{i=1\\ i\neq j}}^k \frac{x-x_i}{x_j-x_i} + w_{\alpha,\beta}(\frac{x-x_i}{x_*-x^{\#}}\times\frac{x-x^{\#}}{x_*-x_i}\ f(x_*) - \frac{x-x_i}{x_*-x^{\#}}\times\frac{x-x_*}{x_i-x^{\#}}f(x^{\#}))$$

$$-w_{\alpha,\beta}(\frac{x-x_i}{x_*-x^{\#}}\times\frac{x-x^{\#}}{x_*-x_i}\ f(x_*) - \frac{x-x_i}{x_*-x^{\#}}\times\frac{x-x_*}{x_i-x^{\#}}f(x^{\#}))\|_p.$$

If $t_\kappa,\ t_\iota \in \{t_i\}_{i=1}^k$ such that $t_\kappa = x_*$, $t_\iota = x^{\#}$, then we have

$$\|f-p_n\|_{w_{\alpha,\beta},p} \le \|w_{\alpha,\beta}f - w_{\alpha,\beta}\sum_{j=1}^k f(t_j)\prod_{\substack{i=1\\ i\neq j}}^k \frac{x-x_i}{x_j-x_i}\times F(x_*)F(x^{\#})\|_p.$$

Put

$q_{n,r}(f,\cdot|x_\circ,\cdots,x_{k-1},x_*,x^\#)=\sum_{j=1}^{k}f(t_j)F(x_*)F(x^\#)\prod_{\substack{i=1\\ i\neq j}}^{k}\frac{x-x_i}{x_j-x_i}$,

where $F(x_*)=f(x_*)\times(\frac{x-x_i}{x_*-x_i})$ and $F(x^\#)=f(x^\#)\times(\frac{x-x_i}{x_i-x^\#})$.

Then

$$\|f-p_n\|_{w_{\alpha,\beta},p}\leq c_{p,k,x_*,x^\#}\|w_{\alpha,\beta}f-w_{\alpha,\beta}\times q_{n,r}(f,\cdot|x_\circ,\cdots,x_{k-1},x_*,x^\#)\|_p$$

$$\leq c_{p,k,x_*,x^\#}\times[(\int_{-1}^{1}|w_{\alpha,\beta}f|^pdx)^{\frac{1}{p}}+(\int_{-1}^{1}|w_{\alpha,\beta}\times q_{n,r}(f,\cdot|x_\circ,\cdots,x_{k-1},x_*,x^\#)|^pdx)^{\frac{1}{p}}]$$

$$\leq c_{p,k,x_*,x^\#}\times[(\int_{-1}^{1}|w_{\alpha,\beta}f|^pdx)^{\frac{1}{p}}+(\int_{-1}^{1}|w_{\alpha,\beta}\prod_{i\in\Lambda}\int_{i}^{\mathbb{D}}f\ \mathrm{d}\underline{\mathcal{L}_\phi}|^pdx)^{\frac{1}{p}}],\quad \textit{by (R.R.T.)}$$

$$\leq c_{p,k,x_*,x^\#}\times[(|\sup L(w_{\alpha,\beta}f)|^p)^{\frac{1}{p}}+(|\sup L(w_{\alpha,\beta}\times\sup LS(f,\underline{\mathcal{L}_\phi}))|^p)^{\frac{1}{p}}]$$

$$\leq c(p,k,x_*,x^\#)\times\sup|\sum_{j=1}^{n}\prod_{i\in\Lambda}m_j\ \mathcal{L}_i(\mu(\mathbb{D}_j))|. \tag{3.11}$$

Case II. If $x\in[x^\#,x_*]$, then

$$\|f-s\|_{w_{\alpha,\beta},p}=\|w_{\alpha,\beta}(f-s)\|_p=\|w_{\alpha,\beta}(f-|p_n(f,\cdot|x_\circ,\cdots,x_{k-1})-z_r|v-z_r)||_p$$

$$\leq c\|w_{\alpha,\beta}(f-p_n(f,\cdot|x_\circ,\cdots,x_{k-1})+z_r-z_r)||_p.$$

Case II implies (3.10). So Case I and (3.10) immediately give (3.11). Consequently, Theorem 3.1 and (3.11) implies (3.8). In particular, if $k=2$, then we get (3.9).

Conclusion. If $f \in \mathbb{L}_{p,r}^{\alpha,\beta} \cap I_f$, then: First, the function $f \overset{\text{approximated}}{\longrightarrow}$ piecewise polynomial (constrained); that is, if f is convex function of $\Delta^{(2)}$, then p_n is convex polynomial of $\Delta^{(2)}$. If f is k-monotone function of Δ^k, then p_n is k-monotone polynomial of Δ^k. Secondly, we defined

$$\Delta_h^i(f,\ x) = \begin{cases} \prod_{i\in\Lambda} \int_i^{\mathbb{D}} f \ \underline{\mathrm{d}\mathcal{L}}, & \text{if } f \in I_f, \\ 0, & \text{otherwise.} \end{cases}$$

Also, $\mathbb{L}_{p,r}^{\alpha-\delta,\beta-\delta} \subset \mathbb{L}_{p,r}^{\alpha,\beta}$.

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