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Generalized Opial-Type Inequalities on Time Scales

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Abstract

Some dynamic inequalities of Opial-type inequalities are generalized through Jensen's and Hölder's inequalities for convex functions. In particular, Hua's and Rozanova's inequalities on time scales are refined.

1 Introduction

Inequalities involving integrals of a function and its derivative were established by Opial in [9]. They turned out to be one of the most useful inequalities in analysis and have been receiving continuous attention. The results are as follows:

Theorem 1.1. *If $f(x)$ is absolutely continuous on $[0,h]$ such that $f(0) = f(h) = 0$ and $f(x) > 0$ on $(0,h)$, then*

$$\int_0^h |f(x)f'(x)|\,dx \leq \frac{h}{4}\int_0^h (f'(x))^2 dx. \tag{1.1}$$

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$\frac{h}{4}$ is the best possible constant.
Olech [8] provided a modified version of the result in the following theorem:

Theorem 1.2. *If $f(x)$ is absolutely continuous on $[0,h]$ with $f(0)=0$, then*

$$\int_0^h f(x)f'(x)dx \leq \frac{h}{2}\int_0^h (f'(x))^2dx. \tag{1.2}$$

He observed that the absolute value in (1.1) is not necessary. A non-trivial generalization of (1.2) was established by Hua in [6] as:

Theorem 1.3. *Let $x(t)$ be absolutely continuous on $[0,a]$ and $x(0)=0$. If $l>0$, then*

$$\int_0^a |x^l(t)x'(t)|dt \leq \frac{a^l}{l+1}\int_0^a |x'(t)|^{l+1}dt. \tag{1.3}$$

Rozanova [11] established the following result:

Theorem 1.4. *Let $f(t)$ and $g(t)$ be convex and increasing functions on $[0,\infty]$ with $f(0)=0$ and let $p(t)\geq 0$, $p'(t)>0$, $t\in[\alpha,r]$ with $p(t)=0$. If $x(t)$ is absolutely continuous on $[\alpha,r)$ and $x(\alpha)=0$, then*

$$f\left(\int_\alpha^r p'(t)g\left(\frac{|x'(t)|}{p'(t)}\right)dt\right) \geq \int_\alpha^r p'(t)g\left(\frac{|x'(t)|}{p'(t)}\right)\left[f'\left(p(t)g\left(\frac{|x(t)|}{p(t)}\right)\right)\right]dt. \tag{1.4}$$

Time scale ($\mathbb{T}$) calculus was initiated by Stefan Hilger [5] in order to create a theory that can unify discrete and continuous analysis. A *time scale* is an arbitrary non-empty closed subset of the real numbers. The three most popular examples of time scale calculus are differential calculus, difference calculus and quantum calculus; that is, $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{N}$, $\mathbb{T}=q^{\mathbb{N}_0}=\{q^t : t\in\mathbb{N}_0\}$, where $q>1$, Kac and Cheung [7]. The Delta derivative f^Δ of a function f is defined on $\mathbb{T}$ as:
(i)$f^\Delta=f'$ is the usual derivative if $\mathbb{T}=\mathbb{R}$; and
(ii)$f^\Delta=\Delta f$ is the usual forward difference operator if $\mathbb{T}=\mathbb{Z}$.
The summary of time scale calculus and its applications could be sourced from ([1], [4], [10], [13]) and the references therein.
Saker [12] established the following result on time scale:

Theorem 1.5. *Let $\mathbb{T}$ be a time scale with $a,b\in\mathbb{T}$ and m,n be positive numbers such that $m>1$. Let r,s be non–negative rd–continuous functions on $(b,c)_\mathbb{T}$ such that $\int_a^b r^{-1/(m+n-1)}(t)\Delta t<\infty$. If $y:[a,b]\cap\mathbb{T}\to\mathbb{R}^+$ is delta* *differentiable with* $y(a) = 0$, *(and* y^{Δ} *does not change sign in* $(a,b)_{\mathbb{T}}$*), then one has*

$$\int_a^b s(x)|y(x) + y^{\sigma}(x)|^m \left|y^{\Delta}(x)\right|^n \Delta x \leq K(a,b,m,n) \int_a^b r(x) \left|y^{\Delta}(x)\right|^{m+n} \Delta x, \tag{1.5}$$

where

$$\begin{aligned} &K(a,b,m,n) \\ &= 2^{2m-1}\left(\frac{n}{m+n}\right)^{\frac{n}{m+n}} \\ &\times \left(\int_a^b (s(x))^{(m+n)/m}(r(x))^{-n/m}\left(\int_a^x r^{-1/(m+n-1)}(t)\Delta t\right)^{m+n-1}\Delta x\right)^{\frac{m}{m+n}} \\ &+ 2^{m-1}\sup_{a\leq x\leq b}\left(\mu^m(x)\frac{s(x)}{r(x)}\right). \end{aligned} \tag{1.6}$$

Throughout this article, we shall assume that a time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. Moreover, we denote $f^{\sigma}: = fo\sigma$, where the forward jump operator σ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. Also, if $t < sup\mathbb{T}$ and $\sigma(t) = t$, then t is called right–dense and if $t > inf\mathbb{T}$ and $\rho(t) = t$, then t is called left–dense. Points that are right–dense and left–dense at the same time are called dense. The graininess function $\mu : \mathbb{T} \to \mathbb{R}^+$ is defined by $\mu := \sigma(t) - t$. If $\mathbb{T}^k := \mathbb{T} - \{m\}$ if $\mathbb{T}$ has a left–scattered maximum m; otherwise, $\mathbb{T}^k := \mathbb{T}$. We will assume that $\sup\mathbb{T} = \infty$ and define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}} := [a, b\cap\mathbb{T}]$. The aim of this work is to generalize some inequalities of Opial-type by using Jensen′s and Hölder's inequalities for convex functions.

2 Main Result

2.1 Some adaptations of H$\ddot{o}$lder′s inequality

Hölder's inequality on time scale as expressed in [2] states that:

$$\int_{\alpha}^{\beta} |f(t)g(t)|\,\Delta t \leq \left[\int_{\alpha}^{\beta} |f(t)|^k \,\Delta t\right]^{\frac{1}{k}} \left[\int_{\alpha}^{\beta} |g(t)|^l \,\Delta t\right]^{\frac{1}{l}}, \tag{2.7}$$

where $\alpha, \beta \in \mathbb{T}$ and $f, g \in C_{rd}(\mathbb{I}, \mathbb{R})$, $k > 1$ and $1/k + 1/l = 1$.

$$|c+d|^m \leq 2^{m-1}(|c|^m + |d|^m), m \geq 1, \tag{2.8}$$

where c, d are positive real numbers.

Theorem 2.1. *Let $\mathbb{T}$ be a time scale, $f(t)$ be absolutely continuous and non-decreasing function on $[a,b]$, and $0 \leq a \leq b < \infty$ with $f(t) > 0$ for $t > 0$. Suppose that $p \geq l \geq 1, q > 0,\ 0 < l+q \leq p$ and $\delta > 0$. Then,*

$$\begin{aligned}&\int_a^b t^{\delta l-1} f^q(t) \left[\int_t^b f(s)\Delta s\right]^l \Delta t \\ &\leq [\delta^{-1}]^{(lp-(l+q)+p)/p} \left[\frac{p}{l+q}\right] \left[\int_a^b (f^p(s)) s^{\frac{lp(1+\delta)}{l+q}-1} \Delta s\right]^{\frac{(l+q)}{p}} \\ &+ [\delta^{-1}]^{(lp-(l+q)+p)/p} \left[\frac{p}{l+q}\right] a^{\delta \frac{(l+q)}{p}} \left[\int_a^b f^p(s) s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)} \Delta s\right]^{\frac{(l+q)}{p}}\end{aligned} \tag{2.9}$$

Proof: Suppose $q = 1$ and $p = l+1$ in (2.9). Then,

$$\begin{aligned}\left|\int_a^b t^{\delta l-1} f(t) \left[\int_t^b f(s)\Delta s\right]^l \Delta t\right| &\leq [\delta^{-1}]^l a^\delta \left[\left|\int_a^b f^{l+1}(s) s^{(l-1)(1+\delta)} \Delta s\right|\right] \\ &+ [\delta^{-1}]^l \left[\left|\int_a^b f^{l+1}(s) s^{l(1+\delta)-1} \Delta s\right|\right]\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}\left|\int_a^b t^{\delta l-1} f(t) \left[\int_t^b f(s)\Delta s\right]^l \Delta t\right| &\leq [\delta^{-1}]^l a^\delta \left[\int_a^b \left|f^{l+1}(s)\right| s^{(l-1)(1+\delta)} \Delta s\right] \\ &+ [\delta^{-1}]^l \left[\int_a^b \left|f^{l+1}(s)\right| s^{l(1+\delta)-1} \Delta s\right]\end{aligned} \tag{2.11}$$

$$= [\delta^{-1}]^l \left[\int_a^b \left|f^{l+1}(t)\right| t^{l+\delta l-1} \Delta t\right] + [\delta^{-1}]^l a^\delta \left[\int_a^b \left|f^{l+1}(t)\right| t^{l-\delta+\delta l-1} \Delta t\right].$$

Rearranging and factoring out $t^{\delta l-1}$ implies

$$0 \leq \int_a^b t^{\delta l-1} \left[[\delta^{-1}]^l \left|f^{l+1}(t)\right| t^l + [\delta^{-1}]^l a^\delta \left|f^{l+1}(t)\right| t^{l-\delta} - \left|f(t) \left[\int_t^b f(s)\Delta s\right]^l\right|\right] \Delta t. \tag{2.12}$$

If $t \geq 0$ on $[a,b]$, then

$$0 \leq \int_a^b \left[[\delta^{-1}]^l \left| f^{l+1}(t) \right| t^l + [\delta^{-1}]^l a^\delta \left| f^{l+1}(t) \right| t^{l-\delta} - \left| f(t) \left[\int_t^b f(s)\Delta s \right]^l \right| \right] \Delta t. \tag{2.13}$$

Hence

$$\begin{aligned} \int_a^b \left| f(t) \left[\int_t^b f(s)\Delta s \right]^l \right| \Delta t \leq \int_a^b [\delta^{-1}]^l \left| f^{l+1}(t) \right| t^l \Delta t \\ + [\delta^{-1}]^l a^\delta \int_a^b \left| f^{l+1}(t) \right| t^{l-\delta} \Delta t. \end{aligned} \tag{2.14}$$

Setting $t^l = b^l$ for $t \in [a,b]$ and $l > 0$, we have

$$\begin{aligned} \int_a^b \left| f(t) \left[\int_t^b f(s)\Delta s \right]^l \right| \Delta t \leq [\delta^{-1}]^l b^l \int_a^b \left| f^{l+1}(t) \right| \Delta t \\ + [\delta^{-1}]^l a^\delta \int_a^b \left| f^{l+1}(t) \right| t^{-\delta} \Delta t. \end{aligned} \tag{2.15}$$

Since $|f(t)| = |-f(t)|$, (2.15) becomes

$$\begin{aligned} \int_a^b \left| -f(t) \left[\int_t^b f(s)\Delta s \right]^l \right| \Delta t \leq [\delta^{-1}]^l b^l \int_a^b \left| -f^{l+1}(t) \right| \Delta t \\ + [\delta^{-1}]^l a^\delta b^l \int_a^b \left| -f^{l+1}(t) \right| t^{-\delta} \Delta t. \end{aligned} \tag{2.16}$$

Suppose $\delta = (1+l)^{\frac{1}{l}}$ in (2.16). Then

$$\begin{aligned} \int_a^b \left| -f(t) \left[\int_t^b f(s)\Delta s \right]^l \right| \Delta t \leq \frac{b^l}{l+1} \int_a^b \left| -f^{l+1}(t) \right| \Delta t \\ + \frac{b^l}{l+1} a^{(l+1)^{\frac{1}{l}}} \int_a^b \left| -f^{l+1}(t) \right| t^{-\delta} \Delta t. \end{aligned} \tag{2.17}$$

Replacing $\int_t^b f(s)\Delta s = x(t)$ and $|-f(t)| = \left| x^\Delta(t) \right|$ in (2.17) lead to

$$\begin{aligned} \int_a^b |x(t)|^l \left| x^\Delta(t) \right| \Delta t \leq \frac{b^l}{l+1} \int_a^b \left| x^\Delta(t) \right|^{l+1} \Delta t \\ + \frac{b^l}{l+1} a^{(l+1)^{\frac{1}{l}}} \int_a^b \left| x^\Delta(t) \right|^{l+1} \Delta t. \end{aligned} \tag{2.18}$$

Let $\mathbb{T} = \mathbb{R}$ and $a \to 0$ in (2.18). Setting
(i) $b = h$, $l = 1$, $x' = f' = x^{\Delta}$, we have (1.2);
while setting
(ii) $x' = x^{\Delta}$, we have (1.3). Hence the proof is complete.

2.2 Generalization of Rozanova′s Inequality on Time-scale

Theorem 2.2. *Let $f(t)$ and $g(t)$ be convex and increasing functions on $[0, \infty]$ with $f(0) = 0$ and let $p(t) \geq 0$, $p^{\Delta}(t) > 0$, $t \in [\alpha, \tau]$ with $p(t) = 0$. If $x(t)$ is absolutely continuous on $[\alpha, \tau)$ and $x(\alpha) = 0$, then*

$$f\left(\int_{\alpha}^{\tau} p^{\Delta}(t) g\left(\frac{\left|x^{\Delta}(t)\right|}{p^{\Delta}(t)}\right) \Delta t\right) \geq \int_{\alpha}^{\tau} p^{\Delta}(t) g\left(\frac{\left|x^{\Delta}(t)\right|}{p^{\Delta}(t)}\right) \left[f^{\Delta}\left(p(t) g\left(\frac{|x(t)|}{p(t)}\right)\right)\right] \Delta t.$$

Proof: Let $y(t) = \int_a^t \left|x^{\Delta}(s)\right| \Delta s$, $t \in [\alpha, \tau]$. Then $y^{\Delta}(t) = \left|x^{\Delta}(t)\right|$ and $y(t) \geq |x(t)|$. By using Jensen's inequality and by setting $\int_{\alpha}^{\tau} \Delta\lambda(t) = 1$, we have

$$f\left[\int_{\alpha}^{\tau} \psi(t) \Delta\lambda(t)\right] \geq \left[\int_{\alpha}^{\tau} f(\psi(t)) \Delta\lambda(t)\right], \tag{2.19}$$

with $\psi(t) = g\left(\frac{\left|x^{\Delta}(t)\right|}{p^{\Delta}(t)}\right)$ and $\Delta\lambda(t) = p^{\Delta}(t)dt$. For any convex functions $f(u) = u^l$

$$\left(\int_{\alpha}^{\tau} \psi(t) \Delta\lambda(t)\right)^l = \left(\int_{\alpha}^{\tau} p^{\Delta}(t) g\left(\frac{\left|x^{\Delta}(t)\right|}{p^{\Delta}(t)}\right) \Delta t\right)^l. \tag{2.20}$$

Hence

$$\left(\int_{\alpha}^{\tau} p^{\Delta}(t) g\left(\frac{\left|x^{\Delta}(t)\right|}{p^{\Delta}(t)}\right) \Delta t\right)^l = f\left(\int_{\alpha}^{\tau} p^{\Delta}(t) g\left(\frac{\left|x^{\Delta}(t)\right|}{p^{\Delta}(t)}\right) \Delta t\right), \tag{2.21}$$

which implies

$$\left(\int_{\alpha}^{\tau} p^{\Delta}(t) g\left(\frac{\left|x^{\Delta}(t)\right|}{p^{\Delta}(t)}\right) \Delta t\right)^l = f\left(\int_{\alpha}^{\tau} p^{\Delta}(t) g\left(\frac{\left|y^{\Delta}(t)\right|}{p^{\Delta}(t)}\right) \Delta t\right). \tag{2.22}$$

The right hand side of (2.22) can be written as greater than or equal to:

$$\int_{\alpha}^{\tau} \left[f\left(\int_{\alpha}^{t} p^{\Delta}(s) g\left(\frac{\left|y^{\Delta}(s)\right|}{p^{\Delta}(s)}\right) \Delta s\right) \Delta t\right]^{\Delta}. \tag{2.23}$$

Hence

$$f\left(\int_\alpha^\tau p^\Delta(t)g\left(\frac{|x^\Delta(t)|}{p^\Delta(t)}\right)\Delta t\right) \geq \int_\alpha^\tau p^\Delta(t)g\left(\frac{|x^\Delta(t)|}{p^\Delta(t)}\right)\left[f^\Delta\left(p(t)g\left(\frac{|x(t)|}{p(t)}\right)\right)\right]\Delta t.$$

The proof is complete.

Theorem 2.3. *Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$ and m, n be positive real numbers such that $m > 1$. If $y : [a, b] \cap \mathbb{T} \to \mathbb{R}^+$ is delta differentiable with $y(a) = 0$, (and y^Δ does not change sign in $(a, b)_\mathbb{T}$), then*

$$\int_a^b |y(x) + y^\sigma(x)|^m \left|y^\Delta(x)\right|^n \Delta x \leq K(a, b, m, n) \int_a^b \left|y^\Delta(x)\right|^{m+n} \Delta x, \quad (2.24)$$

where

$$\begin{aligned} K(a, b, m, n) = {} & 2^{2m-1}\left(\frac{n}{m+n}\right)^{\frac{n}{m+n}} \\ & \times \left(\int_a^b (r(x))^{-n/m}\left(\int_a^x r^{-1/(m+n-1)}(t)\Delta t\right)^{m+n-1}\Delta x\right)^{\frac{m}{m+n}} \\ & + 2^{m-1}\sup_{a\leq x\leq b}\left(\mu^m(x)\frac{1}{r(x)}\right). \end{aligned} \quad (2.25)$$

Proof: Since $x \in [a, b]_\mathbb{T}$,

$$|f(x)| = \int_a^x \left|f^\Delta(t)\right| \Delta t. \quad (2.26)$$

Then

$$|f(x)| = \int_a^x \frac{1}{(r(t))^{1/(m+n)}}(r(t))^{1/(m+n)}\left|f^\Delta(t)\right|\Delta t. \quad (2.27)$$

By inequality (2.7),

$$\left\{\begin{matrix} f(t) = \dfrac{1}{(r(t))^{1/(m+n)}}, & g(t) = (r(t))^{1/(m+n)}\left|f^\Delta(t)\right|\Delta t, \\ k = \dfrac{m+n}{m+n-1}, & l = m+n \end{matrix}\right\} \quad (2.28)$$

$$\begin{aligned} \int_a^x \left|f^\Delta(t)\right|\Delta t \leq {} & \left(\int_a^x \frac{1}{(r(t))^{1/(m+n-1)}}\Delta t\right)^{\frac{m+n-1}{m+n}} \\ & \times \left(\int_a^x r(t)\left|f^\Delta(t)\right|^{m+n}\Delta t\right)^{\frac{1}{m+n}}. \end{aligned} \quad (2.29)$$

For $a \leq x \leq b$,

$$\begin{aligned}\int_a^x |f(x)|^m \leq \left(\int_a^x \frac{1}{(r(t))^{1/(m+n-1)}} \Delta t\right)^{\frac{m(m+n-1)}{m+n}} \\ \times \left(\int_a^x r(t) \left|f^\Delta(t)\right|^{m+n} \Delta t\right)^{\frac{m}{m+n}}.\end{aligned} \tag{2.30}$$

Since

$$f^\sigma(t) = f(t) + \mu(t) f^\Delta(t), \tag{2.31}$$

adding $f(t)$ to both sides of (2.31), we have

$$f(t) + f^\sigma(t) = 2f(t) + \mu(t) f^\Delta(t). \tag{2.32}$$

Applying (2.8) on (2.32),

$$\begin{aligned}|f(t) + f^\sigma(t)|^m &\leq 2^{m-1}\left(2^m |f(t)|^m + \mu^m(t)\left|f^\Delta(t)\right|^m\right) \\ &= 2^{2m-1}|f(t)|^m + 2^{m-1}\mu^m(t)\left|f^\Delta(t)\right|^m.\end{aligned} \tag{2.33}$$

Set

$$q(x) = \int_a^x r(t) \left|f^\Delta(t)\right|^{m+n} \Delta t, \tag{2.34}$$

$$q^\Delta(x) = r(x)\left|f^\Delta(x)\right|^{m+n}. \tag{2.35}$$

From (2.35), we have

$$\left|f^\Delta(x)\right|^{m+n} = \frac{q^\Delta(x)}{r(x)} \quad \Rightarrow \left|f^\Delta(x)\right|^n = \left(\frac{q^\Delta(x)}{r(x)}\right)^{\frac{n}{m+n}}. \tag{2.36}$$

Multiplying $\left|f^\Delta(x)\right|^n$ by (2.33), we have

$$\begin{aligned}&|f(x) + f^\sigma(x)|^m \left|f^\Delta(x)\right|^n \leq 2^{2m-1}|f(x)|^m \left|f^\Delta(x)\right|^n + 2^{m-1}\mu^m(x) \\ &\times \left|f^\Delta(x)\right|^{m+n} = 2^{2m-1}|f(x)|^m \left(\frac{q^\Delta(x)}{r(x)}\right)^{\frac{n}{m+n}} + 2^{m-1}\mu^m(x)\left(\frac{q^\Delta(x)}{r(x)}\right) \\ &= 2^{2m-1}|f(x)|^m \left(\frac{1}{r(x)}\right)^{\frac{n}{m+n}} \left(q^\Delta(x)\right)^{\frac{n}{m+n}} + 2^{m-1}\mu^m(x)\left(\frac{q^\Delta(x)}{r(x)}\right).\end{aligned} \tag{2.37}$$

Substituting (2.30) into (2.37), we have

$$\begin{aligned} &|f(x)+f^{\sigma}(x)|^{m}\left|f^{\Delta}(x)\right|^{n} \leq 2^{2m-1}|f(x)|^{m}\left(\frac{1}{r(x)}\right)^{\frac{n}{m+n}}\left(q^{\Delta}(x)\right)^{\frac{n}{m+n}} \\ &+2^{m-1}\mu^{m}(x)\left(\frac{q^{\Delta}(x)}{r(x)}\right) \leq 2^{2m-1}\left(\frac{1}{r(x)}\right)^{\frac{n}{m+n}}\left(q^{\Delta}(x)\right)^{\frac{n}{m+n}} \\ &\times\left(\int_{a}^{x}\frac{1}{(r(t))^{1/(m+n-1)}}\Delta t\right)^{\frac{m(m+n-1)}{m+n}}\left(\int_{a}^{x} r(t)\left|f^{\Delta}(t)\right|^{m+n}\Delta t\right)^{\frac{m}{m+n}} \\ &+2^{m-1}\mu^{m}(x)\left(\frac{q^{\Delta}(x)}{r(x)}\right). \end{aligned} \tag{2.38}$$

Also, substituting (2.35) into (2.38), we have

$$\begin{aligned} &\int_{a}^{b}|f(x)+f^{\sigma}(x)|^{m}\left|f^{\Delta}(x)\right|^{n}\Delta x \leq 2^{2m-1}\int_{a}^{b}\left(\frac{1}{r(x)}\right)^{\frac{n}{m+n}}\left(q^{\Delta}(x)\right)^{\frac{n}{m+n}}\Delta x \\ &\times\left(\int_{a}^{x}\frac{1}{(r(t))^{1/(m+n-1)}}\Delta t\right)^{\frac{m(m+n-1)}{m+n}}\left(q^{\Delta}(x)\right)^{\frac{m}{m+n}} \\ &+2^{m-1}\mu^{m}(x)\int_{a}^{b}\left(\frac{q^{\Delta}(x)}{r(x)}\right)\Delta x. \end{aligned} \tag{2.39}$$

Rearranging the right hand side of (2.39), we have

$$\begin{aligned} &\int_{a}^{b}|f(x)+f^{\sigma}(x)|^{m}\left|f^{\Delta}(x)\right|^{n}\Delta x \leq 2^{2m-1}\int_{a}^{b}\left(\frac{1}{r(x)}\right)^{\frac{n}{m+n}} \\ &\times\left(\int_{a}^{x}\frac{1}{(r(t))^{1/(m+n-1)}}\Delta t\right)^{\frac{m(m+n-1)}{m+n}}\int_{a}^{b}\left(q^{\Delta}(x)\right)^{\frac{n}{m+n}}\left(q^{\Delta}(x)\right)^{\frac{m}{m+n}}\Delta x \\ &+2^{m-1}\sup_{a\leq x\leq b}\left(\frac{\mu^{m}(x)}{r(x)}\right)\int_{a}^{b} q^{\Delta}(x)\Delta x. \end{aligned} \tag{2.40}$$

Using Hölder's inequality with indices $\frac{m+n}{m}$ and $\frac{m+n}{n}$, we have

$$\begin{aligned} &\int_{a}^{b}|f(x)+f^{\sigma}(x)|^{m}\left|f^{\Delta}(x)\right|^{n}\Delta x \\ &\leq 2^{2m-1}\left[\int_{a}^{b}\left(\frac{1}{r(x)}\right)^{\frac{n}{m}}\left(\int_{a}^{x}\frac{\Delta t}{(r(t))^{1/(m+n-1)}}\right)^{(m+n-1)}\Delta x\right]^{\frac{m}{m+n}} \\ &\times\int_{a}^{b}\left(\left(q^{\Delta}(x)\right)\left(q^{\Delta}(x)\right)^{\frac{m}{n}}\right)^{\frac{n}{m+n}}+2^{m-1}\sup_{a\leq x\leq b}\left(\frac{\mu^{m}(x)}{r(x)}\right)\int_{a}^{b} q^{\Delta}(x)\Delta x. \end{aligned} \tag{2.41}$$

Bohner and Peterson [3] stated that:

$$\left(q^{\Delta}(x)\right)\left(q^{\Delta}(x)\right)^{\frac{m}{n}} \leq \frac{n}{m+n}\left(q^{\frac{m+n}{n}}(x)\right)^{\Delta}. \tag{2.42}$$

Substituting (2.42) into (2.41) and simplifying, we have

$$\begin{aligned}
&\int_a^b |f(x)+f^{\sigma}(x)|^m \left|f^{\Delta}(x)\right|^n \Delta x \\
&\leq 2^{2m-1}\left[\int_a^b \left(\frac{1}{r(x)}\right)^{\frac{n}{m}} \left(\int_a^x \frac{\Delta t}{(r(t))^{1/(m+n-1)}}\right)^{(m+n-1)} \Delta x\right]^{\frac{m}{m+n}} \\
&\times \left(\frac{n}{m+n}\right)^{\frac{n}{m+n}} \int_a^b q^{\Delta}(x)\Delta x \\
&+ 2^{m-1} \sup_{a\leq x\leq b}\left(\frac{\mu^m(x)}{r(x)}\right)\int_a^b q^{\Delta}(x)\Delta x.
\end{aligned} \tag{2.43}$$

Set $\int_a^b q^{\Delta}(x)\Delta x = q(x)$. Then (2.43) becomes

$$\begin{aligned}
&\int_a^b |f(x)+f^{\sigma}(x)|^m \left|f^{\Delta}(x)\right|^n \Delta x \\
&\leq 2^{2m-1}\left[\int_a^b \left(\frac{1}{r(x)}\right)^{\frac{n}{m}} \left(\int_a^x \frac{\Delta t}{(r(t))^{1/(m+n-1)}}\right)^{(m+n-1)} \Delta x\right]^{\frac{m}{m+n}} \\
&\times \left(\frac{n}{m+n}\right)^{\frac{n}{m+n}} q(x) + 2^{m-1} \sup_{a\leq x\leq b}\left(\frac{\mu^m(x)}{r(x)}\right) q(x).
\end{aligned} \tag{2.44}$$

Factoring out $q(x)$ in (2.44) and setting $r(x)=1$, we get

$$\int_a^b |f(x)+f^{\sigma}(x)|^m \left|f^{\Delta}(x)\right|^n \Delta x \leq K(a,b,m,n)\int_a^b \left|f^{\Delta}(x)\right|^{m+n} \Delta x. \tag{2.45}$$

From the chain rule in time scale,

$$\begin{aligned}
\int_a^b (x-a)^{m+n-1}\Delta x \; &\leq \frac{1}{m+n}\int_a^b ((x-a)^{m+n})^{\Delta} \\
= \frac{(b-a)^{m+n}}{m+n}.
\end{aligned} \tag{2.46}$$

Substituting (2.46) into (2.25) with $r(t) = 1$,

$$\begin{aligned} K(a,b,m,n) &= 2^{2m-1}\left(\frac{n}{m+n}\right)^{\frac{n}{m+n}}\left(\int_a^b (x-a)^{(m+n-1)}\Delta x\right)^{\frac{m}{m+n}} \\ &+ 2^{m-1}\sup_{a\le x\le b}\mu^m(x) \\ &= 2^{2m-1}\left(\frac{n}{m+n}\right)^{\frac{n}{m+n}}\left(\frac{(b-a)^{m+n}}{m+n}\right)^{\frac{m}{m+n}} + 2^{m-1}\sup_{a\le x\le b}\mu^m(x) \\ &\le 2^{2m-1}\frac{n^{n/(m+n)}}{m+n}(b-a)^m + 2^{m-1}\sup_{a\le x\le b}\mu^m(x). \end{aligned} \tag{2.47}$$

Let $f^\sigma(x) = f(x)$, $\mu(x) = 0$ and substitute (2.47) into (2.45). This implies,

$$\int_a^b |f(x)|^m \left|f^\Delta(x)\right|^n \Delta x \le 2^{2m-1}\frac{n^{n/(m+n)}}{m+n}(b-a)^m \int_a^b \left|f^\Delta(x)\right|^{m+n}\Delta x. \tag{2.48}$$

Remark: Each of the following two cases yields an Opial-type inequality.
(i) If $\mathbb{T} = \mathbb{R}$ and set $m = n = 1$ and $b = h, a = 0$; and
(ii) If $\mathbb{T} = \mathbb{R}$ and set $m = \frac{1}{2}$, $n = 1$ and $b = h, a = 0$.

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