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Study of Generalized $k-$hypergeometric Functions

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Abstract

In this paper, a unified approach to generalized $k-$hypergeometric function ${}_pF_{q,k}$, is given. As a result, generalized $k-$hypergeometric series and solution of its ordinary differential equation are obtained. In addition, we introduce integral representations and regions of convergence of different generalized series and deduce the m^{th} derivative of $k-$functions.

1 Introduction

The study of hypergeometric functions have prompted renewed interest in function theory in the last decades. This is evidenced in almost more than 2000 papers listed just in the last ten years in mathematics reviews under the topic of Hypergeometric functions which is an important class of special functions. Special functions are particular mathematical functions which have more or less established names and notations due to their importance in mathematical analysis, functional analysis, physics, or other applications. Numerous solutions of differential equations relate to many special functions

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in mathematics, physics, and engineering. For instance, the solution could be presented by a power series [3], continued fraction [7] and hypergeometric series [2, 3, 4]. For more details about the recent works the readers may refer to [1, 8, 9] and references cited there in. Differential equations have a huge impact in society as they occur significantly in every branch of science. Linear homogeneous differential equations with rational function coefficients are very common in mathematics, combinatorics, physics and engineering. Finding closed form solutions (solutions expressible in terms of well studied special functions, for example: Bessel, Kummer, Liouville, Hypergeometric) of such differential equations is a fascinating area of research in computer algebra [1, 3, 4]. It is well known that many phenomena in physical and technical applications are governed by a variety of differential equations. We note that these differential equations have appeared in many different research fields, for instance in the theory of automorphic function, in conformal mapping theory, in the theory of representations of Lie algebras, and in the theory of difference equations. Analytical and numerical methods to solve ordinary differential equations form an interesting research direction in differentiable dynamical systems and their applications.

2 Preliminaries

In this section, we briefly review some basic definitions and facts concerning the $k-$hypergeometric series and ordinary differential equations. Solutions to the hypergeometric differential equation are built out of the hypergeometric series. The equation

$$kx(1-kx)D^{''}y+(c-(a+b+k)kx)D^{'}y-aby=0 \tag{2.1}$$

is called Gauss $k-$hypergeometric differential equation discussed in [10, 7]. The point $x=0$ is a regular singular point for the equation and a power series technique is a method to solve such an equation. Some literature on $k-$hypergeometric series and the $k-$hypergeometric differential equation can be found in [8, 9] and [6]. Diaz and Pariguan [8] have introduced and proved some identities of $k-$gamma function, $k-$beta function and $k-$pochhammer symbol. They deduced integral representations of $k-$gamma and $k-$beta function respectively given by

$$\Gamma_k(\mu)=\int_o^{\infty} t^{\mu-1}e^{-\frac{t^k}{k}}dt, \qquad Re(\mu)>0, \quad k>0, \tag{2.2}$$

$$\beta_k(\mu,\nu) = \frac{1}{k}\int_o^1 t^{\frac{\mu}{k}-1}(1-t)^{\frac{\nu}{k}-1}dt, \quad \mu > 0, \quad \nu > 0. \tag{2.3}$$

In addition, they provided

$$\beta_k(\mu,\nu) = \frac{\Gamma_k(\mu)\Gamma_k(\nu)}{\Gamma_k(\mu+\nu)}. \tag{2.4}$$

The $k-$Pochhammer symbol is defined as

$$(\sigma)_{n,k} = \sigma(\sigma+k)(\sigma+2k)...(\sigma+(n-1)k) = \prod_{j=1}^{n}(\sigma+jk-k), \tag{2.5}$$

where $\sigma \neq 0$, $(\sigma)_{0,k} = 1$, $(\sigma)_n = (\sigma)_{n,1}$, $k > 0$. Also,

$$(\sigma)_{m+n,k} = (\sigma+mk)_{n,k}(\sigma)_{m,k}, (\sigma)_{mn,k} = m^{mn}\prod_{i=1}^{m}\left(\frac{\sigma+ki-k}{m}\right)_{n,k}. \tag{2.6}$$

The relation between the $k-$Pochhammer symbol and the $k-$gamma function is

$$(\sigma)_{n,k} = \frac{\Gamma_k(\sigma+nk)}{\Gamma_k(\sigma)}. \tag{2.7}$$

The hypergeometric function [3, 4, 2] with three parameters μ, ν, c, two parameters μ, ν in the numerator and one parameter c in the denominator, is defined as

$$_2F_1\left(\mu,\nu;c;u\right) = \sum_{n=0}^{\infty}\frac{(\mu)_n(\nu)_n u^n}{(c)_n n!}. \tag{2.8}$$

Definition 1. Mubeen [10] defined $k-$hypergeometric function with three parameters μ,ν,c, two parameters μ,ν in the numerator and one parameter c in the denominator, by

$$_2F_{1,k}\left((\mu,k),(\nu,k);(c,k);u\right) = \sum_{n=0}^{\infty}\frac{(\mu)_{n,k}(\nu)_{n,k}u^n}{(c)_{n,k}n!}. \tag{2.9}$$

As a natural generalization of (2.9) we have the following definition:
Definition 2. The generalized $k-$hypergeometric function is defined by

$$_pF_{q,k}\left((a_1,k),(a_2,k),...,(a_p,k);(b_1,k),(b_2,k),...(b_q,k);u\right) = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n,k}}{\prod_{i=1}^{q}(b_i)_{n,k}}\frac{u^n}{n!} \tag{2.10}$$

in which $k > 0$ and no denominator parameter b_i is allowed to be zero or a negative integer.

3 Solution of the differential equation

Recall that [7, 10] the ordinary $k-$hypergeometric function ${}_2F_{1,k}((a,k),(b,k);(c,k);u)$ satisfies the differential equation

$$ku(1-ku)D^{''}y + (c-(a+b+k)ku)D^{'}y - aby = 0, \tag{3.1}$$

or, in terms of the differential operator $\theta = u\frac{d}{du}$, the differential equation

$$[\theta(k\theta + c - k) - u(k\theta + a)(k\theta + b)]y = 0. \tag{3.2}$$

With equation (2.10) before us, we can proceed as follows

$$\phi = {}_pF_{q,k} = \sum_{n=o}^{\infty} \frac{\prod_{i=1}^{p}(a_i)_{n,k}}{\prod_{i=1}^{q}(b_i)_{n,k}} \frac{u^n}{n!}.$$

Since $k\theta u^n = knu^n$, it follows that

$$k\theta \prod_{i=0}^{p}(k\theta + b_i - k)\phi = \sum_{n=o}^{\infty} \frac{\prod_{i=0}^{p}(a_i)_{n,k}}{\prod_{i=0}^{q}(b_i)_{n,k}} \frac{(kn + b_i - k)knu^n}{n!}.$$

Since the last factor in $(b_i)_{n,k}$ is $(kn + b_i - k)$, we have

$$k\theta \prod_{i=0}^{p}(k\theta + b_i - k)\phi = k\sum_{n=1}^{\infty} \frac{\prod_{i=0}^{p}(a_i)_{n,k}}{\prod_{i=0}^{q}(b_i)_{n-1,k}} \frac{u^n}{(n-1)!}.$$

Now we replace n by $n+1$ and get

$$k\theta \prod_{i=0}^{p}(k\theta + b_i - k)\phi = k\sum_{n=o}^{\infty} \frac{\prod_{i=0}^{p}(a_i)_{n+1,k}}{\prod_{i=0}^{q}(b_i)_{n,k}} \frac{u^{n+1}}{n!}$$

$$= ku\sum_{n=o}^{\infty}\frac{\prod\limits_{i=0}^{p}(a_i)_{n,k}\prod\limits_{i=0}^{p}(a_i+kn)}{\prod\limits_{i=0}^{q}(b_i)_{n,k}}\frac{u^n}{n!},$$

$$= ku\prod_{i=0}^{p}(a_i+k\theta)\phi.$$

Thus we have shown that $\phi = {}_pF_{q,k}$ is a solution of the differential equation

$$[\theta\prod_{i=0}^{p}(k\theta+b_i-k)-u\prod_{i=0}^{p}(a_i+k\theta)]\phi=0, \tag{3.3}$$

when no b_i is a non positive integer. The solution is valid for all finite u when $p \leq q$. If $p = q + 1$, then the solution [11] is valid in $|u| < 1$.

4 Integral representation

Theorem 4.1. *If $p \leq q+1$, $Re(b_1) > Re(a_1) > 0$, and if none of $b_1, b_2, ..., b_q$ is zero or a negative integer and if $|u| < 1$, then*

$${}_pF_{q,k}\left[\begin{array}{ll}(a_1,k),(a_2,k),...,(a_p,k); & \\ (b_1,k),(b_2,k),...,(b_q,k); & u\end{array}\right]=\frac{\Gamma_k(a_1)}{k\Gamma_k(b_1)\Gamma_k(b_1-a_1)}$$

$$\times\int_o^1 t^{\frac{a_1}{k}-1}(1-t)^{\frac{b_1-a_1}{k}-1}{}_{p-1}F_{q-1,k}\left[\begin{array}{ll}(a_2,k),(a_3,k),...,(a_p,k); & \\ (b_2,k),(b_3,k),...,(b_q,k); & tu\end{array}\right]dt. \tag{4.1}$$

Proof: From (2.4)

$$\beta_k(a_1+nk,b_1-a_1)=\frac{\Gamma_k(a_1+nk)\Gamma_k(b_1-a_1)}{\Gamma_k(b_1+nk)}.$$

Using (2.3) and (2.5), we have

$$\beta_k(a_1+nk,b_1-a_1)=\frac{(a_1)_{n,k}\Gamma_k(a_1)\Gamma_k(b_1-a_1)}{(b_1)_{n,k}\Gamma_k(b_1)}$$

$$=\frac{1}{k}\int_o^1 t^{\frac{a_1+nk}{k}-1}(1-t)^{\frac{b_1-a_1}{k}-1}dt.$$

By (2.10), we have

$$\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n,k}}{\prod_{i=1}^{q}(b_i)_{n,k}}\frac{u^n}{n!}=\frac{\Gamma_k(b_1)}{k\Gamma_k(a_1)\Gamma_k(b_1-a_1)}\int_o^1 t^{\frac{a_1}{k}-1}(1-t)^{\frac{b_1-a_1}{k}-1}\sum_{n=0}^{\infty}\frac{\prod_{i=2}^{p}(a_i)_{n,k}}{\prod_{i=2}^{q}(b_i)_{n,k}}\frac{(tu)^n}{n!}dt.$$

which completes the proof.
Note. If $p \leq q$, then the condition $|u| < 1$ may be omitted.

Theorem 4.2. *If $Re(\alpha) > 0, Re(\beta) > 0$, $k > 0$, if no b_i is zero or a negative integer, m and s are to be positive integers, then inside the region of convergence of the resultant series we have*

$$\frac{t^{1-\frac{\alpha+\beta}{k}}}{kB_k(\alpha,\beta)}\int_o^1 x^{\frac{\alpha}{k}-1}(t-x)^{\frac{\beta}{k}-1}{}_pF_q\left[\begin{array}{l}(a_1,k),(a_2,k),...,(a_p,k);\\(b_1,k),(b_2,k),...,(b_q,k);cx^m(t-x)^s\end{array}\right]dx$$

$$={}_{p+m+s}F_{q+m+s,k}\left[\begin{array}{l}(a_1,k),...,(a_p,k);\left(\frac{\alpha}{m},k\right),...,\left(\frac{\alpha+(m-1)k}{m},k\right);\left(\frac{\beta}{s},k\right),...,\left(\frac{\beta+(s-1)k}{s},k\right);\\(b_1,k),...,(b_q,k);\left(\frac{\alpha+\beta}{m+s},k\right),\left(\frac{\alpha+\beta+k}{m+s},k\right),...,\left(\frac{\alpha+\beta+(m+s-1)k}{m+s},k\right);\frac{c\ m^m s^s t^{m+s}}{(m+s)^{m+s}}\end{array}\right]$$

Proof: Consider

$$A(t)=\frac{1}{k}\int_o^1 x^{\frac{\alpha}{k}-1}(t-x)^{\frac{\beta}{k}-1}\times{}_pF_q\left[\begin{array}{l}(a_1,k),(a_2,k),...,(a_p,k);\\(b_1,k),(b_2,k),...\quad,(b_q,k);cx^m(t-x)^s\end{array}\right]dx.$$

We evaluate $A(t)$ by term by term integration. Let $x = tv$. Then

$$A(t)=\frac{t^{\frac{\alpha+\beta}{k}-1}}{k}\int_o^1 v^{\frac{\alpha}{k}-1}(1-v)^{\frac{\beta}{k}-1}{}_pF_q\left[\begin{array}{l}(a_1,k),(a_2,k),...,(a_p,k);\\(b_1,k),(b_2,k),...,(b_q,k);ct^{m+s}v^m(1-v)^s\end{array}\right]dv,$$

$$=\frac{t^{\frac{\alpha+\beta}{k}-1}}{k}\int_o^1\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n,k}c^n t^{\frac{(m+s)nk}{k}}v^{\frac{\alpha+mnk}{k}-1}(1-v)^{\frac{\beta+snk}{k}-1}}{\prod_{i=1}^{q}(b_i)_{n,k}n!}dv.$$

Using (2.3), we get

$$A(t)=t^{\frac{\alpha+\beta}{k}-1}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n,k}c^n t^{\frac{(m+s)nk}{k}}B_k(\alpha+mnk,\beta+snk)}{\prod_{i=1}^{q}(b_i)_{n,k}n!}. \tag{4.3}$$

Now, applying (2.4) and (2.7), we get

$$B_k(\alpha+mnk,\beta+snk)=\frac{\Gamma_k(\alpha+mnk)\Gamma_k(\beta+snk)}{\Gamma_k(\alpha+\beta+(m+s)nk)},$$

$$=\frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)}\frac{(\alpha)_{mn,k}(\beta)_{sn,k}}{(\alpha+\beta)_{(m+s)n,k}}.$$

Since

$$(\alpha)_{mn,k}=m^{mn}\prod_{j=1}^{m}\left(\frac{\alpha+kj-k}{m}\right)_{n,k},$$

$$B_k(\alpha+mn,\beta+sn)=\frac{B_k(\alpha,\beta)m^{mn}s^{sn}\prod_{j=1}^{m}\left(\frac{\alpha+kj-k}{m}\right)_{n,k}\prod_{w=1}^{s}\left(\frac{\beta+kw-k}{s}\right)_{n,k}}{(m+s)^{(m+s)n}\prod_{v=1}^{m+s}\left(\frac{\alpha+\beta+kv-k}{m+s}\right)_{n,k}}. \quad (4.4)$$

The use of (4.4) on the right side of (4.3) yields

$$A(t)=t^{\frac{\alpha+\beta}{k}-1}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n,k}c^n t^{\frac{(m+s)nk}{k}}}{\prod_{i=1}^{q}(b_i)_{n,k}n!}\frac{B_k(\alpha,\beta)m^{mn}s^{sn}\prod_{j=1}^{m}\left(\frac{\alpha+kj-k}{m}\right)_{n,k}\prod_{w=1}^{s}\left(\frac{\beta+kw-k}{s}\right)_{n,k}}{(m+s)^{(m+s)n}\prod_{v=1}^{m+s}\left(\frac{\alpha+\beta+kv-k}{m+s}\right)_{n,k}}$$

which after simplification gives (4.2).

Corollary 4.3. *If $Re(\alpha)>0, Re(\beta)>0,\ k>0$ and if m is positive integer, then inside the region of convergence of the resultant series we have*

$$\frac{t^{1-\frac{\alpha+\beta}{k}}}{kB_k(\alpha,\beta)}\int_o^1 x^{\frac{\alpha}{k}-1}(t-x)^{\frac{\beta}{k}-1}{}_pF_q\left[\begin{array}{l}(a_1,k),(a_2,k),...,(a_p,k);\\(b_1,k),(b_2,k),...\quad,(b_q,k);cx^m\end{array}\right]dx$$

$$={}_{p+m}F_{q+m,k}\left[\begin{array}{l}(a_1,k),...,(a_p,k);\left(\frac{\alpha}{m},k\right),...,\left(\frac{\alpha+(m-1)k}{m},k\right);\\(b_1,k),...,(b_q,k);\left(\frac{\alpha+\beta}{m},k\right),\left(\frac{\alpha+\beta+k}{m},k\right),...,\left(\frac{\alpha+\beta+(m-1)k}{m},k\right);ct^m\end{array}\right] \quad (4.5)$$

Corollary 4.4. *If $Re(\alpha)>0, Re(\beta)>0,\ k>0$ and if s is positive integer, then inside the region of convergence of the resultant series we have*

$$\frac{t^{1-\frac{\alpha+\beta}{k}}}{kB_k(\alpha,\beta)}\int_o^1 x^{\frac{\alpha}{k}-1}(t-x)^{\frac{\beta}{k}-1}{}_pF_q\left[\begin{array}{l}(a_1,k),(a_2,k),...,(a_p,k);\\(b_1,k),(b_2,k),...,(b_q,k);c(t-x)^s\end{array}\right]dx$$

$$={}_{p+s}F_{q+s,k}\left[\begin{array}{l}(a_1,k),...,(a_p,k);\left(\frac{\beta}{s},k\right),...,\left(\frac{\beta+(s-1)k}{s},k\right);\\(b_1,k),...\quad,(b_q,k);\left(\frac{\alpha+\beta}{s},k\right),\left(\frac{\alpha+\beta+k}{s},k\right),...,\left(\frac{\alpha+\beta+(s-1)k}{s},k\right);ct^s\end{array}\right]. \tag{4.6}$$

Proof: The proof of (4.5) and (4.6) are direct consequences of Theorem (4.2), following the same steps in the derivation of (4.2).

Theorem 4.5. *If $p \le q+1$, $Re(b_1) > Re(a_1) > 0$, and if no one of $b_1, b_2, ..., b_q$ is zero or a negative integer, then*

$$\frac{d^m}{du^m}{}_pF_{q,k}\left[\begin{array}{r}(a_1,k),...,(a_p,k);\\(b_1,k),...,(b_q,k);\ u\end{array}\right]=\frac{\prod_{i=1}^{p}(a_i)_{m,k}}{\prod_{i=1}^{q}(b_i)_{m,k}}{}_pF_{q,k}\left[\begin{array}{l}(a_1+mk,k),...,(a_p+mk,k);\\(b_1+mk,k),...,(b_q+mk,k);u\end{array}\right]. \tag{4.7}$$

Proof: Differentiate ${}_pF_{q,k}$ with respect to u

$$\frac{d}{du}{}_pF_{q,k}=\frac{d}{du}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n,k}}{\prod_{i=1}^{q}(b_i)_{n,k}}\frac{u^n}{n!}=\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n,k}}{\prod_{i=1}^{q}(b_i)_{n,k}}\frac{u^{n-1}}{(n-1)!},$$

with a shift of index from n to $n+1$

$$=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n+1,k}}{\prod_{i=1}^{q}(b_i)_{n+1,k}}\frac{u^n}{n!}=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i+k)_{n,k}\prod_{i=1}^{p}(a_i)}{\prod_{i=1}^{q}(b_i+k)_{n,k}\prod_{i=1}^{q}(b_i)}\frac{u^n}{n!},$$

$$\frac{d}{du}{}_pF_{q,k}=\frac{\prod_{i=1}^{p}(a_i)}{\prod_{i=1}^{q}(b_i)}{}_pF_{q,k}\left[\begin{array}{l}(a_1+k,k),...,(a_p+k,k);\\(b_1+k,k),...\quad,(b_q+k,k);u\end{array}\right].$$

Now, m repeated applications of the formula for the derivative of a ${}_pF_{q,k}$ yield

$$\frac{d^m}{du^m}{}_pF_{q,k} = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i+mk)_{n,k}\prod_{i=1}^{p}(a_i)_{m,k}}{\prod_{i=1}^{q}(b_i+mk)_{n,k}\prod_{i=1}^{q}(b_i)_{m,k}}\frac{u^n}{n!}$$

$$= \frac{\prod_{i=1}^{p}(a_i)_{m,k}}{\prod_{i=1}^{q}(b_i)_{m,k}}{}_pF_{q,k}\left[\begin{array}{c}(a_1+mk,k),...,(a_p+mk,k);\\(b_1+mk,k),...,(b_q+mk,k)u\end{array}\right]$$

which completes the proof.

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