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Formulas for the Exponential Matrix of $so(1,3)$

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Abstract

In this paper, we give the formula of the exponential matrix e^B when $B \in so(1,3)$, the Lie algebra of Lorentz matrix Lie group. The formula is a generalization of the well known Rodrigues formula for skew-symmetric matrices of order 3.

1 Introduction

The exponential matrix plays a very important role in many fields of mathematics and physics. It can be computed by many methods. The matrix exponential problem has been studied after the introduction of Lie group methods to solve systems of ordinary differential equations numerically. According to these methods the differential system is solved in a Lie algebra (and not in a Lie group). Consider a linear first-order constant coefficient ordinary differential equation: $x'(t) = Ax(t), \qquad x(0) = x_0$, where $x(t)$ and x_0

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are n-vectors, and A is an $n \times n$ matrix of complex constants. It is well known that the solution to this equation is given by $x(t) = e^{At}x_0$. It is therefore important to have an accurate numerical method for computing the matrix exponential function. Many methods for computing e^A were widely studied in 2001. Politi [3] gave the formula of exponential matrix e^A when A is a skew-symmetric real matrix of order 4 and in 2005 Kula, Karancan and Yayh [4] gave formula of exponential matrix e^A when A is a semi skew-symmetric real matrix of order 4. Among the explicit formulas, only the Rodrigues formula allows the computation of e^A when A is a skew-symmetric real matrix. If $A = \begin{bmatrix} 0 & u_3 & u_2 \\ -u_3 & 0 & u_1 \\ -u_2 & -u_2 & 0 \end{bmatrix}$, then the Rodrigues formula is

$$e^A = I + \frac{sin\alpha}{\alpha}A + \frac{cos\alpha - 1}{\alpha^2}A^2,$$

where $\alpha = u_1^2 + u_2^2 + u_3^2$. In this paper, we derive a generalization of the Rodrigues formula for the Lie algebra matrices of Lorentz matrix Lie group.

2 Main Results

The Lie algebra of Lorentz matrix Lie group is

$$so(1,3) = \{B \in M_4(\mathbb{C}) | JBJ = -B^T\},$$

where $J = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Then any $B \in so(1,3)$ is in the form

$$B = \begin{bmatrix} 0 & u_1 & u_2 & u_3 \\ u_1 & 0 & u_4 & u_5 \\ u_2 & -u_4 & 0 & u_6 \\ u_3 & -u_5 & -u_6 & 0 \end{bmatrix}.$$

First we give some preliminary results:

Lemma 2.1. *Let $p(\lambda)$ be the be characteristic polynomial of a matrix $B \in so(1,3)$. Then $p(-\lambda) = p(\lambda)$.*

Proof.

$$\begin{aligned} p(\lambda) &= \det(\lambda I - B) \\ &= \det(J\lambda J - JBJ) \\ &= \det(J\lambda J + B^T) \\ &= \det(B^T + J\lambda J) \\ &= \det(B^T + \lambda I) \\ &= p(-\lambda) \end{aligned}$$

□

Lemma 2.2. *If* $B \in so(1,3)$*, then* $p(\lambda) = \lambda^4 + a_2\lambda^2 + a_0$*, where*

$$\begin{aligned} a_2 &= -a^2 - b^2 - c^2 + d^2 + e^2 + f^2 \\ a_0 &= (af - be + cd)^2. \end{aligned} \tag{2.1}$$

Proof. Let $p(\lambda) = \lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$ be the characteristic polynomial of matrix B. Then, by Lemma 2.1, we have $a_1 = 0 = a_3$. Next, applying Laplace rule to compute $\det(\lambda I - B)$, we get the coefficients of λ^2 and a_0. □

Corollary 2.3. *The eigenvalues of* $B \in so(1,3)$ *are*

$$\begin{aligned} \lambda_1 &= i\sqrt{\frac{a_2 - \sqrt{a_2^2 - 4a_0}}{2}} = i\alpha, & \lambda_2 &= -i\sqrt{\frac{a_2 - \sqrt{a_2^2 - 4a_0}}{2}} = -i\alpha, \\ \lambda_3 &= i\sqrt{\frac{a_2 + \sqrt{a_2^2 - 4a_0}}{2}} = i\mu, & \lambda_4 &= -i\sqrt{\frac{a_2 + \sqrt{a_2^2 - 4a_0}}{2}} = -i\mu. \end{aligned} \tag{2.2}$$

Theorem 2.4. *If the eigenvalues of the matrix* $B \in so(1,3)$ *are distinct, then*

$$e^B = aB^3 + bB^2 + cB + dI,$$

where

$$\begin{aligned} a &= \frac{\alpha \sin\mu - \mu \sin\alpha}{\alpha\mu(\alpha^2 - \mu^2)}, & b &= \frac{\cos\mu - \cos\alpha}{\alpha^2 - \mu^2}, \\ c &= \frac{\sin\alpha}{\alpha} + a\alpha^2, & d &= \cos\alpha + b\alpha^2, \end{aligned} \tag{2.3}$$

I *being the identity matrix of order 4.*

Proof. Let $\lambda_1, \lambda_2, \lambda_3$ and λ_4 be the distinct eigenvalues of a matrix B. Then $\lambda_1 = i\alpha, \lambda_2 = -i\alpha, \lambda_3 = i\mu$ and $\lambda_4 = -i\mu$. From Sylvester formula, we have $e^B = \sum_{i=1}^{4} e^{\lambda_i} B_i$ with $B_i = \prod_{j=1, j\neq i}^{k} \frac{1}{\lambda_i - \lambda_j}(B - \lambda_j I)$. Then e^B takes the form $e^B = aB^3 + bB^2 + cB + dI$ but the coefficient of B^3, B^2, B and I are in tricky form. We will use the eigenvectors to find the values of those coefficients as follows: If x_i is an eigenvector corresponding to an eigenvalue λ_i, then we have

$$e^B x_i = aB^3 x_i + bB^2 x_i + cBx_i + dIx_i, \qquad i = 1, \ldots, 4 \tag{2.4}$$

which is a linear system with unknowns a, b, c and d having a unique solution. By doing row operations with matrix $F = \begin{bmatrix} \lambda_1^3 & \lambda_1^2 & \lambda_1 & 1 & e^{\lambda_1} \\ \lambda_2^3 & \lambda_2^2 & \lambda_2 & 1 & e^{\lambda_2} \\ \lambda_3^3 & \lambda_3^2 & \lambda_3 & 1 & e^{\lambda_3} \\ \lambda_4^3 & \lambda_4^2 & \lambda_4 & 1 & e^{\lambda_4} \end{bmatrix}$ we get

$$\begin{aligned} a &= \frac{\alpha \sin(\mu) - \mu \sin(\alpha)}{\alpha\mu(\alpha^2 - \mu^2)} \\ b &= \frac{\cos(\mu) - \cos(\alpha)}{\alpha^2 - \mu^2} \\ c &= \frac{\alpha^3 \sin(\mu) - \mu^3 \sin(\alpha)}{\alpha\mu(\alpha^2 - \mu^2)} = \frac{\sin\alpha}{\alpha} + a\alpha^2 \\ d &= \frac{\alpha^2 \cos(\mu) - \mu^2 \cos(\alpha)}{\alpha^2 - \mu^2} = \cos\alpha + b\alpha^2. \end{aligned}$$

□

If the eigenvalues of B are not distinct which have two cases, then the following theorem holds:

Theorem 2.5. *Let $B \in so(1,3)$. Then the characteristic polynomial of B is in the form $p(\lambda) = a_4\lambda^4 + a_2\lambda + a_0$.*

1. *If $a_0 \neq 0$ and $a_2^2 - 4a_0 = 0$, then $e^B = \cos\beta I + \frac{\sin\beta}{\beta}B$, where $\beta = \sqrt{\frac{a_2}{2}}$.*

2. *If $a_0 = 0$, then $e^B = aB^2 + bB + I$, where $a = \frac{1-\cos\mu}{\mu^2}$, $b = \frac{\sin\mu}{\mu}$ and $\mu = \sqrt{a_2}$.*

Proof. 1. Let $\beta = \sqrt{\frac{a_2}{2}}$. If $a_0 \neq 0$ and $a_2^2 - 4a_0 = 0$, then B has two distinct eigenvalues which are $\lambda_1 = i\beta$ and $\lambda_2 = -i\beta$. Computing directly from Sylvester formula, we have $e^B = \cos\beta I + \frac{\sin\beta}{\beta} B$.
2. Let $\mu = \sqrt{a_2}$. If $a_0 = 0$, then B has three distinct eigenvalues which are $\lambda_1 = 0$, $\lambda_3 = i\mu$ and $\lambda_4 = -i\mu$. From Sylvester formula, we have

$$e^B = e^{\lambda_1} B_1 + e^{\lambda_3} B_3 + e^{\lambda_4} B_4, \text{ where } B_i = \prod_{j=1, j\neq i}^{k} \frac{1}{\lambda_i - \lambda_j}(B - \lambda_j I).$$

Then e^B is in the form of $e^B = aB^2 + bB + cI$. If x_i is an eigenvector corresponding to an eigenvalue λ_i, then we have

$$e^{\lambda_i} x_i = a\lambda_i^2 x_i + b\lambda_i x_i + c x_i, \qquad i = 1, 3, 4 \tag{2.5}$$

which is a linear system with unknowns a, b and c having a unique solution

$$a = \frac{1 - \cos\mu}{\mu^2}, \;\; b = \frac{\sin\mu}{\mu} \;\;\; c = 1.$$

□

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