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Certain Property of the q-analogue of Sigmoid Functions

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Abstract

In this particular work, we define a modified q-sigmoid function and use the mean function to study the convex and concave properties. We further apply the modified q-sigmoid function to geometric function theory defined in a complex plane with examples.

Key words and phrases: $q-$derivative operator, q-sigmoid functions, mean function, $\Upsilon\Phi$-convex, $\Upsilon\Phi$-concave, q-starlike, q-convex.
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1 Introduction and Preliminaries

The sigmoid function, known as standard logistic function, defined by

$$G(x) = \frac{e^x}{1+e^x} = \frac{1}{1+e^{-x}};\ x \in \mathbb{R} \tag{1.1}$$

$$= \frac{1}{2} + \frac{1}{2}\tanh(\frac{x}{2});\ x \in \mathbb{R} \tag{1.2}$$

serves as an activation function at the output of each neuron in artificial neural network. In [6], [7], [8], [9], [10], and [11] authors have developed results connecting sigmoid functions and univalent functions theory in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Recently Nantomah [1] and Zhang et al. [2] used the mean function to study the convexity and monotonicity properties of sigmoid functions. Many researchers used the mean function to develop and study certain convex, concave and monotone increasing (decreasing) function. For some details about these contributions see [16], [17], [18], [19], [20],[21] and their references. All these gave us the impetus to define a modified q-sigmoid function and use the mean function to study the convex and concave properties.

We write some notations and basic concepts of q-calculus which will be useful in this study. Let $0 < q < 1$ and $x \in \mathbb{R}$. We state the q-derivative operator $\mathcal{D}_{x,q}$ as follows

Definition 1.1. *[1] Let $0 < q < 1$. Then the q-derivative $\mathcal{D}_{x,q}$ of a function $f(x)$ is defined by*

$$\begin{cases} \mathcal{D}_{x,q}f(x) = \frac{f(x)-f(qx)}{x(1-q)}, & x \neq 0, \\ \mathcal{D}_{x,q}f(x)|_{x=0} = f'(0) & . \end{cases}$$

Definition 1.2. *[4], [5] For any fixed real number $q > 0$, non-negative integer r, the q-integers of the number r is defined by*

$$[r]_q = \begin{cases} \frac{1-q^r}{1-q}, & q \neq 1 \\ r, & q = 1 \\ 0, & r = 0 \end{cases}$$

Definition 1.3. *[4],[5] The q-fractional is defined by*

$$[r]_q! = \begin{cases} [r]q[r-1]q \cdots [1]q, & r = 1, 2, 3, ..., \\ 1, & r = 0. \end{cases}$$

Definition 1.4. *[4],[5] A q-analogue of the ordinary exponential function* $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ *is defined by*

$$e_q^x := \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!}.$$

Definition 1.5. *[4],[5] The q-Bernoulli numbers* $\mathfrak{B}_{k,q}$ *is defined by the generating function*

$$\frac{x}{e_q^x - 1} = \sum_{k=0}^{\infty} \mathfrak{B}_{k,q} \frac{x^k}{[k]_q!}.$$

Motivated by the above definitions, we have

Definition 1.6. *[22]*

The q-Sigmoid function

$$G_q(x) = \frac{1}{1+e_q^{-x}} = \frac{1}{2} + \frac{1}{2}\tanh_q(\frac{x}{2}),\ x \in \mathbb{R};\ \ \tanh_q(x) = \frac{e_q^x - e_q^{-x}}{e_q^x + e_q^{-x}}. \quad (1.3)$$

The first and second derivatives of $G_q(x)$ *are given by*

$$\begin{aligned} \mathcal{D}_{x,q}(G_q(x)) &= \frac{e_q^x}{(1+e_q^x)(1+e_q^{xq})} = G_q(x)(1-G_q(xq)) \text{ and} \\ \mathcal{D}^2_{x,q}(G_q(x)) &= \frac{e_q^x(1-qe_q^x)}{(1+e_q^x)(1+e_q^{xq})(1+e_q^{xq^2})} = G_q(x)(1-G_q(xq))(1-[2]_qG_q(xq^2)). \end{aligned}$$

If we let $y = G_q(x),\ y_q = G_q(xq),\ \ldots, y_{q^2} = G_q(xq^2)$ *, then*

$$\mathcal{D}_{x,q} y = y(1-y_q) \quad (1.4)$$

$$y_q(0) = \frac{1}{2} \quad (1.5)$$

and the function $\frac{(q-1)}{\ln q}\log(1+e_q^x)$ *is defined as q-softplus function, that implies* $\frac{(q-1)}{\ln q}\int G_q(x)\partial_q = \log(1+e_q^x) + C$*, where* C *is an integration constant.*

Definition 1.7. *[22]*

The modified q-Sigmoid function is

$$\gamma_q(x) = \frac{2}{1+e_q^{-x}} = 1 + \left(\sum_{k=1}^{\infty} \frac{(-1)^k}{2^k}\left[\sum_{n=1} \frac{(-1)^n}{[n]_q!}x^n\right]^k\right). \quad (1.6)$$

We review definitions and results obtained by [18] to suit our research.

Definition 1.8. *[18] Let ℓ be a subinterval of $\mathbb{R}_+$. Then*

(i) *A function $\mathcal{M} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is called a mean function if the following hold:-*

- $\mathcal{M}(x, y) = \mathcal{M}(y, x)$
- $\mathcal{M}(x, x) = x$
- $x < \mathcal{M}(x, y) < y$; *whenever* $x < y$
- $\mathcal{M}(\alpha x, \alpha y) = \alpha \mathcal{M}(x, y)$ *for all* $\alpha \geq 0$.

We outline some examples of the mean functions.

a *Arithmetic mean:* $\mathcal{A}(x, y) = \frac{x+y}{2}$,

b *Geometric mean:* $\mathcal{G}(x, y) = \sqrt{xy}$,

c *Harmonic mean:* $\mathcal{H}(x, y) = \frac{1}{\mathcal{A}(\frac{1}{x}, \frac{1}{y})} = \frac{2xy}{x+y}$,

d *Logarithmic mean:* $\mathcal{L}(x, y) = \begin{cases} \frac{x-y}{\ln x - \ln y}, & x \neq y; \\ x, & x = y. \end{cases}$

e *Identity mean:* $\mathcal{I}(x, y) = \begin{cases} \frac{1}{e}\left(\frac{x^x}{y^y}\right)\frac{1}{x-y}, & x \neq y; \\ x, & x = y. \end{cases}$

(ii) *A continuous function $\Theta : \ell \to \mathbb{R}_+$ with two mean functions say Υ and Φ, is said to be $\Upsilon\Phi$-convex (concave) if $\Theta(\Upsilon(x, y)) \leq (\geq)\Phi(\Theta(x), \Theta(y))$ for all $x, y \in \ell$.*

We review the q-analogue of Theorem 2.4 of [18] and its consequence to suit our research.

Theorem 1.1. *Let $\Theta : \ell \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function. In parts d through i, let $\ell = (0, \beta)$, $\beta \in \mathbb{R}_+ \cup \{0\}$. Then,*

(a) *Θ is $AA - q$-convex (or $AA - q$-concave) if and only if Θ is q-convex (q-concave).*

(b) *Θ is $AG - q$-convex (or $AG - q$-concave) if and only if $\log \Theta$ is q-convex (q-concave).*

(c) Θ *is* $AH - q$*-convex (or* $AH - q$*-concave) if and only if* $\frac{1}{\Theta}$ *is q-convex (q-concave).*

(d) Θ *is* $GA - q$*-convex (or* $GA - q$*-concave) on* ℓ *if and only if* $\Theta(\beta \exp[-\xi])$ *is q-convex (q-concave) on* $\mathbb{R}_+$.

(e) Θ *is* $GG - q$*-convex (or* $GG - q$*-concave) on* ℓ *if and only if* $\log \Theta(\beta \exp[-\xi])$ *is q-convex (q-concave) on* $\mathbb{R}_+$.

(f) Θ *is* $GH - q$*-convex (or* $GH - q$*-concave) on* ℓ *if and only if* $\frac{1}{\Theta(\beta \exp[-\xi])}$ *is q-concave (q-convex) on* $\mathbb{R}_+$.

(g) Θ *is* $HA - q$*-convex (or* $HA - q$*-concave) on* ℓ *if and only if* $\Theta(\frac{1}{x})$ *is q-convex (q-concave) on* $(\frac{1}{\beta}, \infty)$.

(h) Θ *is* $HG - q$*-convex (or* $HG - q$*-concave) on* ℓ *if and only if* $\log \Theta(\frac{1}{x})$ *is q-convex (q-concave) on* $(\frac{1}{\beta}, \infty)$.

(i) Θ *is* $HH - q$*-convex (or* $HH - q$*-concave) on* ℓ *if and only if* $\frac{1}{\Theta(\frac{1}{x})}$ *is q-convex (q-concave) on* $(\frac{1}{\beta}, \infty)$.

Corollary 1.2. *Let* $\Theta : \ell \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ *be a differentiable function. Then,*

(a) Θ *is* $AA - q$*-convex (or* $AA - q$*-concave) on* ℓ *if and only if* $\mathcal{D}_{x,q} f(x)$ *is increasing (or decreasing) for all* $x \in \ell$.

(b) Θ *is* $AG - q$*-convex (or* $AG - q$*-concave) on* ℓ *if and only if* $\frac{\mathcal{D}_{x,q} f(x)}{f(x)}$ *is increasing (or decreasing) for all* $x \in \ell$.

(c) Θ *is* $AH - q$*-convex (or* $AH - q$*-concave) on* ℓ *if and only if* $\frac{\mathcal{D}_{x,q} f(x)}{f(x) f(xq)}$ *is increasing (or decreasing) for all* $x \in \ell$.

(d) Θ *is* $GA - q$*-convex (or* $GA - q$*-concave) on* ℓ *if and only if* $x \mathcal{D}_{x,q} f(x)$ *is increasing (or decreasing) for all* $x \in \ell$.

(e) Θ *is* $GG - q$*-convex (or* $GG - q$*-concave) on* ℓ *if and only if* $\frac{x \mathcal{D}_{x,q} f(x)}{f(x)}$ *is increasing (or decreasing) for all* $x \in \ell$.

(f) Θ *is* $GH - q$*-convex (or* $GH - q$*-concave) on* ℓ *if and only if* $\frac{x \mathcal{D}_{x,q} f(x)}{f(x) f(xq)}$ *is increasing (or decreasing) for all* $x \in \ell$.

(g) Θ *is* $HA - q$*-convex (or* $HA - q$*-concave) on* ℓ *if and only if* $x^2 \mathcal{D}_{x,q} f(x)$ *is increasing (or decreasing) for all* $x \in \ell$.

(h) Θ *is* $HG - q$*-convex (or* $HG - q$*-concave) on* ℓ *if and only if* $\frac{x^2\mathcal{D}_{x,q}f(x)}{f(x)}$ *is increasing (or decreasing) for all* $x \in \ell$.

(i) Θ *is* $HH - q$*-convex (or* $HH - q$*-concave) on* ℓ *if and only if* $\frac{x^2\mathcal{D}_{x,q}f(x)}{f(x)f(xq)}$ *is increasing (or decreasing) for all* $x \in \ell$.

2 Application of mean function to q-Sigmoid function

Theorem 2.1. *The q-sigmoid function* $G_q(x)$ *satisfies the inequality* $G_q(x+y) < G_q(x) + G_q(y)$*; that is,* $G_q(x)$ *is subadditive on* $\mathbb{R}$.

Proof 2.1.

$$\mathcal{D}_{x,q}\left(\frac{G_q(x)}{x}\right) = \frac{xe_q^x - e_q^x - e_q^{[2]_q x}}{x^2 q(1+e_q^x)(1+e_q^{xq})}.$$

Since $x < 1 + e_q^x$ *for all* $x \in \mathbb{R}$, $\mathcal{D}_{x,q}\left(\frac{G_q(x)}{x}\right) < 0$

Theorem 2.2. *Let* $x, y \in \mathbb{R}$, *with* $\frac{1}{e} + \frac{1}{f} = 1$, $e, f > 1$ *and* $\frac{fx+ye}{e+f} < x$. *Then,*

$$G_q\left(\frac{x}{e} + \frac{y}{f}\right) \geq [G_q(x)]^{\frac{1}{e}} [G_q(y)]^{\frac{1}{f}}; \tag{2.7}$$

that is, the function $G_q(x)$ *is q-logarithmical concave on* $\mathbb{R}$.

Proof 2.2. *Assume* $\frac{q-1}{\ln q}\Theta(x) = \ln G_q(x)$. *Then,*

$$\frac{q-1}{\ln q}\mathcal{D}^2_{x,q}(\Theta(x)) = \mathcal{D}_{x,q}\left(\frac{\mathcal{D}_{x,q}(G_q(x))}{G_q(x)}\right) = -\frac{qe_q^{xq}}{(1+e_q^{qx})(1+e_q^{q^2x})} \leq 0. \tag{2.8}$$

Corollary 2.3. *Under the hypothesis of Theorem 2.2, these inequalities hold:*

$$\mathcal{D}^2_{x,q}(G_q(x))G_q(x) - (G_q(x))^2 \leq 0; \ x \in \mathbb{R}, \tag{2.9}$$

and

$$G_q(1+\mu)G_q(1-\mu) \leq \left(\frac{e_q}{1+e_q}\right)^2, \ \mu \in \mathbb{R}_+. \tag{2.10}$$

Proof 2.3. *Assume* $e = f$, $x = 1+\mu$, $y = 1-\mu$. *Then Corollary 2.3 is a consequence of Theorem 2.2.*

Theorem 2.4. *We have*

$$G_{x,q}\left(\frac{x+y}{2}\right) \geq \frac{2G_{x,q}(x)G_{x,q}(y)}{G_{x,q}(x)+G_{x,q}(y)}; \tag{2.11}$$

that is, the function $G_q(x,q)$ is AH-q-concave on $\mathbb{R}_+$.

Proof 2.4.

$$\mathcal{D}_{x,q}\left(\frac{\mathcal{D}_{x,q}G_{x,q}(x)}{G_{x,q}(x)G_{x,q}(xq)}\right) = -\frac{q}{e_q^{xq^2}} < 0. \tag{2.12}$$

Theorem 2.5. *The function $G_q(x,q)$ is HH-q-convex on $(0,[2]_q)$ and HH-q-concave on $([2]_q,\infty)$; that is,*

$$G_{x,q}\left(\frac{2xy}{x+y}\right) \geq \frac{2G_{x,q}(x)G_{x,q}(y)}{G_{x,q}(x)+G_{x,q}(y)} \tag{2.13}$$

for all $0 < x, y < [2]_q$ and

$$G_{x,q}\left(\frac{2xy}{x+y}\right) \leq \frac{2G_{x,q}(x)G_{x,q}(y)}{G_{x,q}(x)+G_{x,q}(y)} \tag{2.14}$$

for all $[2]_q < x, y < \infty$. Equality holds for $x = y$. That is the function $G_q(x,q)$ is AH-q-concave on $\mathbb{R}_+$.

Proof 2.5.

$$\left(\frac{x^2\mathcal{D}_{x,q}G_{x,q}(x)}{G_{x,q}(x)G_{x,q}(xq)}\right) = \frac{x^2}{e_q^{xq}}. \tag{2.15}$$

$$\mathcal{D}_{x,q}\left(\frac{x^2\mathcal{D}_{x,q}G_{x,q}(x)}{G_{x,q}(x)G_{x,q}(xq)}\right) = \frac{x([2]_q - qx)}{e_q^{xq^2}}. \tag{2.16}$$

3 Application to Geometric Functions Theory

Let $\mathcal{PS}_q$ denote q-starlike functions in $\mathcal{U}$ such that

$$\left|\frac{z\mathcal{D}_{z,q}f(z)}{f(z)} - \frac{1}{1-q}\right| \leq \frac{1}{1-q}; (0 < q < 1; z \in \mathcal{U}). \tag{3.17}$$

Lemma 3.1. *([12], [15]) A function is in the class $\mathcal{PS}_q$ if and only if*

$$\left|\frac{f(qz)}{f(z)}\right| \leq 1,;\ \ z \in \mathcal{U}).$$

In 1989, Srivastava and Owa [12] proposed the study of the class, $\mathcal{PC}_q$ of q-convex functions in $\mathcal{U}$ such that

$$\left|\frac{z\mathcal{D}^2_{z,q}f(z)}{\mathcal{D}_{z,q}f(z)} - \frac{1}{1-q}\right| \leq \frac{1}{1-q}; (0 < q < 1; z \in \mathcal{U}). \tag{3.18}$$

Lemma 3.2. *([13], [14]) A function is in the class $\mathcal{PC}_q$ if and only if*

$$\left|\frac{f(q^2z) - f(qz)}{f(qz) - f(z)}\right| \leq q, (f(qz) \neq f(z);\ z \in \mathcal{U}).$$

Example 3.1. *Substituting the following*

$$\frac{z\mathcal{D}_{z,q}G_q(z)}{G_q(z)} = \frac{1}{(1-q)}\left\{1 - \frac{\left[\frac{e^{qz}}{1+e^{qz}}\right]}{\left[\frac{e^{z}}{1+e^{z}}\right]}\right\} \tag{3.19}$$

in equation (3.17) yields $G_q(z) \in \mathcal{PS}_q$

Example 3.2. *Substituting the following*

$$\frac{z\mathcal{D}^2_{z,q}G_q(z)}{\mathcal{D}_{z,q}G_q(z)} = \frac{1}{q(1-q)}\left\{q - \frac{\left[\frac{e^{q^2z}}{1+e^{q^2z}}\right] - \left[\frac{e^{qz}}{1+e^{qz}}\right]}{\left[\frac{e^{qz}}{1+e^{qz}}\right] - \left[\frac{e^{z}}{1+e^{z}}\right]}\right\}, \tag{3.20}$$

in equation (3.18) yields $G_q(z) \in \mathcal{PC}_q$.

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