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On the Diophantine Equation $p^x + (p+20)^y = z^2$, where p and $p+20$ are primes

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Abstract

In this paper, we study the Diophantine equation $p^x+(p+20)^y = z^2$ where p and $p+20$ are primes and show that the equation has no solutions in positive integers x, y and z.

1 Introduction

In 1844, Catalan [1] conjectured that $(3, 2, 2, 3)$ is the unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers such that $\min\{a, b, x, y\} > 1$.

It took quite a while until 2004 when the Catalan's conjecture was proved by Mihailescu [2].

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In 2018, Burshtein [4] showed that the Diophantine equation $p^x + (p + 4)^y = z^2$ when $p, p+4$ are primes has no solution (x, y, z) in positive integers. For the cases when there are solutions he exhibited them [3].
Later that year, Burshtein [5] also studied the Diophantine equation $p^x + (p+6)^y = z^2$ when $p, p+6$ are primes and $x+y = 2, 3, 4$. He established that "(i) For the first 10000 primes p and $x = y = 1$, the equation has exactly seven solutions.
(ii) When $x = 2$ and $y = 1$, the equation has exactly one solution.
(iii) For the other four possibilities, the equation has no solutions."

Still in 2018, Fernando [6] showed that the Diophantine equation $p^x + (p + 8)^y = z^2$ when $p > 3$ and $p + 8$ are primes has no solution (x, y, z) in positive integers.
In this paper, inspired by [3, 4, 5] and [6] we show that the Diophantine equation $p^x + (p + 20)^y = z^2$ where p and $p + 20$ are primes has no solutions in positive integers x, y and z. The way we do this in the next section is by proving a theorem for the case $p = 3$ and another one for $p > 3$ as the case $p = 2$ is not bound to happen because $p + 20$ is assumed to be a prime number.

2 Main results

Theorem 2.1. *The Diophantine equation $3^x + 23^y = z^2$ has no solutions in positive integers x, y and z.*

Proof. Since $3^x + 23^y = z^2$, z is even. So $z^2 \equiv 0 \,(mod\, 4)$.

If $x = 2k, k \geq 1$ and $y = 2s, s \geq 1$, then $3^x \equiv 1 \,(mod\, 4)$ and $23^y \equiv 1 \,(mod\, 4)$. Thus $3^x + 23^y \equiv 2 \,(mod\, 4)$. This is a contradiction since $z^2 \equiv 0 \,(mod\, 4)$.

If $x = 2k + 1, k \geq 0$ and $y = 2s + 1, s \geq 0$, then $3^x \equiv -1 \,(mod\, 4)$ and $23^y \equiv -1 \,(mod\, 4)$. Thus $3^x + 23^y \equiv 2 \,(mod\, 4)$. This is a contradiction since $z^2 \equiv 0 \,(mod\, 4)$.

If $x = 2k, k \geq 1$ and $y = 2s + 1, s \geq 0$, then $3^{2k} + 23^{2s+1} = z^2$ or equivalently $23^{2s+1} = z^2 - 3^{2k} = (z - 3^k)(z + 3^k)$. Thus, there exist non-negative integers α, β such that $23^\alpha = z - 3^k$ and $23^\beta = z + 3^k$ where $\alpha < \beta$ and $\alpha + \beta = 2s + 1$. We can see that $2 \cdot 3^k = 23^\alpha(23^{\beta-\alpha} - 1)$. This implies that $\alpha = 0$ and $2 \cdot 3^k = 23^{2s+1} - 1$. For $s = 0$, $2 \cdot 3^k = 22$ so $3^k = 11$ which is impossible. For $s \geq 1$, $2 \cdot 3^k = 22(23^{2s} + 23^{2s-1} + \cdots + 23 + 1)$. So $3^k = 11(23^{2s} + 23^{2s-1} + \cdots + 23 + 1)$ which is impossible.

If $x = 2k+1, k \geq 0$ and $y = 2s, s \geq 1$, then $3^{2k+1} + 23^{2s} = z^2$ or equivalently $3^{2k+1} = z^2 - 23^{2s} = (z - 23^s)(z + 23^s)$. Thus, there exist non-negative integers α, β such that $3^\alpha = z - 23^s$ and $3^\beta = z + 23^s$ where $\alpha < \beta$ and $\alpha + \beta = 2k+1$. We can see that $2 \cdot 23^s = 3^\alpha(3^{\beta-\alpha} - 1)$. This implies that $\alpha = 0$ and $2 \cdot 23^s = 3^{2k+1} - 1$. For $k = 0$, $2 \cdot 23^s = 2$ and so $23^s = 1$. But that implies $s = 0$ which contradicts the fact that $s \geq 1$. For $k \geq 1$, $23^s = (3^{2k} + 3^{2k-1} + \cdots + 3 + 1)$ which is impossible. Hence, The Diophantine equation $3^x + 23^y = z^2$ has no solutions in positive integers x, y and z.

Theorem 2.2. *The Diophantine equation* $p^x + (p+20)^y = z^2$*, where* $p > 3, p+20$ *are primes, has no solutions in positive integers* x, y *and* z*.*

Proof. Since $p^x + (p+20)^y = z^2$, z is even and so $z^2 \equiv 0\,(mod\,4)$. Since $p > 3$ is prime, $p \equiv 1\,(mod\,4)$ or $p \equiv -1 \equiv 3\,(mod\,4)$.

Case I: For $p \equiv 1\,(mod\,4)$, $p^x + (p+20)^y \equiv 2\,(mod\,4)$ which is a contradiction since $z^2 \equiv 0\,(mod\,4)$.

Case II: Suppose $p \equiv -1\,(mod\,4)$.
If $x = 2k, k \geq 1$ and $y = 2s, s \geq 1$, then $p^x + (p+20)^y \equiv 2\,(mod\,4)$. This is a contradiction since $z^2 \equiv 0\,(mod\,4)$.

If $x = 2k+1, k \geq 0$ and $y = 2s+1, s \geq 0$, then $p^x + (p+20)^y \equiv 2\,(mod\,4)$. This is a contradiction since $z^2 \equiv 0\,(mod\,4)$.

If $x = 2k, k \geq 1$ and $y = 2s+1, s \geq 0$, then $p^{2k} + (p+20)^{2s+1} = z^2$ or equivalently $(p+20)^{2s+1} = z^2 - p^{2k} = (z - p^k)(z + p^k)$. Thus there exist non-negative integers α, β such that $(p+20)^\alpha = z - p^k$ and $(p+20)^\beta = z + p^k$, where $\alpha < \beta$ and $\alpha + \beta = 2s+1$. Therefore $2 \cdot p^k = (p+20)^\alpha((p+20)^{\beta-\alpha} - 1)$. This implies that $\alpha = 0$ and $2 \cdot p^k = (p+20)^{2s+1} - 1$. For $s = 0$, $2 \cdot p^k = p + 19$ and so $p(2p^{k-1} - 1) = 19$. Thus $p = 19$ and $k = 1$ but this contradicts the fact that $p + 20 = 39$ is not prime. For $s \geq 1$, $2 \cdot p^k = (p+19)\left[(p+20)^{2s} + (p+20)^{2s-1} + \cdots + (p+20) + 1\right]$. It follows that $p + 19$ is an even positive divisor of $2p^k$; that is, $p + 19 = 2 \cdot p^j$, where j is an integer such that $0 \leq j < k$. For $j = 0$, $p + 19 = 2$ which is impossible. For $1 \leq j < k$, $p + 19 = 2p^j$ and so

$p(2p^{j-1}-1)=19$. Thus $p=19$ but $p+20=39$ is not prime.

If $x=2k+1, k\geq 0$ and $y=2s, s\geq 1$, then $p^{2k+1}+(p+20)^{2s}=z^2$ or equivalently $p^{2k+1}=z^2-(p+20)^{2s}=(z-(p+20)^s)(z+(p+20)^s)$. Thus there exist non-negative integers α,β such that $p^\alpha=z-(p+20)^s$ and $p^\beta=z+(p+20)^s$, where $\alpha<\beta$ and $\alpha+\beta=2k+1$. Therefore $2(p+20)^s=p^\alpha(p^{\beta-\alpha}-1)$. This implies that $\alpha=0$ and $2(p+20)^s=p^{2k+1}-1$. For $k=0$, we obtain $2(p+20)^s=p-1=(p+20)-21$ or $2(p+20)^s+21=p+20$ which is impossible. For $k\geq 1$, we have $2(p+20)^s=p^{2k+1}-1=(p-1)\left[p^{2k}+p^{2k-1}+\cdots+p+1\right]$. It follows that $p-1$ is an even positive divisor of $2(p+20)^s$; that is, $p-1=2(p+20)^j$, where j is an integer such that $0\leq j<s$. For $j=0$, $p=3$ which contradicts the fact that $p>3$. For $1\leq j<s$, we obtain $2(p+20)^j=(p+20)-21$ or $2(p+20)^j+21=p+20$ which is impossible.

Therefore, the Diophantine equation $p^x+(p+20)^y=z^2$, where $p\geq 3$ and $p+20$ are primes, has no solutions in positive integers x,y and z.

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References

[1] E. Catalan, Note extraite d'une lettre adressée à l'éditeur par Mr. E. Catalan, Répétiteur à l'école polytechnique de Paris, Journal für die reine und angewandte Mathematik, **27**, (1844), 192–192.

[2] P. Mihailescu, Primary cycolotomic units and a proof of Catalan's conjecture, J. Reine Angew. Math., **572**, (2004), 167–195.

[3] N. Burshtein, All the solutions of the Diophantine Equation $p^x+(p+4)^y=z^2$, where $p,p+4$ are primes and $x+y=2,3,4$, Annals of Pure and Applied Mathematics, **16**, no. 1, (2018), 241–244.

[4] N. Burshtein, The Diophantine Equation $p^x + (p+4)^y = z^2$, when $p > 3, p+4$ are primes, is insovable in positive integers x, y, z, Annals of Pure and Applied Mathematics, **16**, no. 2, (2018), 283–286.

[5] N. Burshtein, Solutions of the Diophantine Equation $p^x + (p+6)^y = z^2$, when $p, p+6$ are primes and $x+y = 2, 3, 4$, Annals of Pure and Applied Mathematics, **17**, no. 1, (2018), 101–106.

[6] N. Fernando, On the Solvability of the Diophantine Equation $p^x + (p+8)^y = z^2$, when $p > 3$ and $p+8$ are primes, Annals of Pure and Applied Mathematics, **18**, no. 1, (2018), 9–13.