On the Diophantine Equation
\[ p^x + (p + 20)^y = z^2, \]
where \( p \) and \( p + 20 \) are primes

Rakporn Dokchan\(^1\), Apisit Pakapongpun\(^{1,2}\)*

1Department of Mathematics
Faculty of Science
Burapha University
Chonburi 20131, Thailand

2Centre of Excellence in Mathematics, CHE.Sri
Ayutthaya Road, Bangkok 10400, Thailand

email: rakporn@buu.ac.th
*Corresponding author, email: apisit.buu@gmail.com

(Received June 19, 2020, Accepted July 9, 2020)

Abstract

In this paper, we study the Diophantine equation \( p^x + (p+20)^y = z^2 \) where \( p \) and \( p + 20 \) are primes and show that the equation has no solutions in positive integers \( x, y \) and \( z \).

1 Introduction

In 1844, Catalan [1] conjectured that \( (3, 2, 2, 3) \) is the unique solution \( (a, b, x, y) \) for the Diophantine equation \( a^x - b^y = 1 \) where \( a, b, x \) and \( y \) are integers such that \( \min\{a, b, x, y\} > 1 \).

It took quite a while until 2004 when the Catalan’s conjecture was proved by Mihailescu [2].

Key words and phrases: Exponential Diophantine equation.
AMS (MOS) Subject Classifications: 11D61
In 2018, Burshtein [4] showed that the Diophantine equation \( p^x + (p + 4)^y = z^2 \) when \( p, p + 4 \) are primes has no solution \((x, y, z)\) in positive integers. For the cases when there are solutions he exhibited them [3].

Later that year, Burshtein [5] also studied the Diophantine equation \( p^x + (p + 6)^y = z^2 \) when \( p, p + 6 \) are primes and \( x + y = 2, 3, 4 \). He established that

(i) For the first 10000 primes \( p \) and \( x = y = 1 \), the equation has exactly seven solutions.
(ii) When \( x = 2 \) and \( y = 1 \), the equation has exactly one solution.
(iii) For the other four possibilities, the equation has no solutions.

Still in 2018, Fernando [6] showed that the Diophantine equation \( p^x + (p + 8)^y = z^2 \) when \( p > 3 \) and \( p + 8 \) are primes has no solution \((x, y, z)\) in positive integers.

In this paper, inspired by [3, 4, 5] and [6] we show that the Diophantine equation \( p^x + (p + 20)^y = z^2 \) where \( p \) and \( p + 20 \) are primes has no solutions in positive integers \( x, y \) and \( z \). The way we do this in the next section is by proving a theorem for the case \( p = 3 \) and another one for \( p > 3 \) as the case \( p = 2 \) is not bound to happen because \( p + 20 \) is assumed to be a prime number.

2 Main results

Theorem 2.1. The Diophantine equation \( 3^x + 23^y = z^2 \) has no solutions in positive integers \( x, y \) and \( z \).

Proof. Since \( 3^x + 23^y = z^2 \), \( z \) is even. So \( z^2 \equiv 0 \pmod{4} \).

If \( x = 2k, k \geq 1 \) and \( y = 2s, s \geq 1 \), then \( 3^x \equiv 1 \pmod{4} \) and \( 23^y \equiv 1 \pmod{4} \). Thus \( 3^x + 23^y \equiv 2 \pmod{4} \). This is a contradiction since \( z^2 \equiv 0 \pmod{4} \).

If \( x = 2k + 1, k \geq 0 \) and \( y = 2s + 1, s \geq 0 \), then \( 3^x \equiv -1 \pmod{4} \) and \( 23^y \equiv -1 \pmod{4} \). Thus \( 3^x + 23^y \equiv 2 \pmod{4} \). This is a contradiction since \( z^2 \equiv 0 \pmod{4} \).

If \( x = 2k, k \geq 1 \) and \( y = 2s + 1, s \geq 0 \), then \( 3^{2k} + 23^{2s+1} = z^2 \) or equivalently \( 23^{2s+1} = z^2 - 3^{2k} = (z - 3^k)(z + 3^k) \). Thus, there exist non-negative integers \( \alpha, \beta \) such that \( 23^\alpha = z - 3^k \) and \( 23^\beta = z + 3^k \) where \( \alpha < \beta \) and \( \alpha + \beta = 2s + 1 \). We can see that \( 2 \cdot 3^k = 23^\alpha(23^{\beta-\alpha} - 1) \). This implies
that $\alpha = 0$ and $2 \cdot 3^k = 23^{2s+1} - 1$. For $s = 0$, $2 \cdot 3^k = 22$ so $3^k = 11$ which is impossible. For $s \geq 1$, $2 \cdot 3^k = 22(23^{2s} + 23^{2s-1} + \cdots + 23 + 1)$. So $3^k = 11(23^{2s} + 23^{2s-1} + \cdots + 23 + 1)$ which is impossible.

If $x = 2k + 1, k \geq 0$ and $y = 2s, s \geq 1$, then $3^{2k+1} + 23^s = z^2$ or equivalently $3^{2k+1} = z^2 - 23^s = (z - 23^s)(z + 23^s)$. Thus, there exist non-negative integers $\alpha, \beta$ such that $3^\alpha = z - 23^s$ and $3^\beta = z + 23^s$ where $\alpha < \beta$ and $\alpha + \beta = 2k + 1$. We can see that $2 \cdot 23^s = 3^\alpha(3^\beta - \alpha - 1)$. This implies that $\alpha = 0$ and $2 \cdot 23^s = 3^{2k+1} - 1$. For $k = 0$, $2 \cdot 23^s = 2$ and so $23^s = 1$. But that implies $s = 0$ which contradicts the fact that $s \geq 1$. For $k \geq 1$, $23^s = (3^{2k} + 3^k + \cdots + 3 + 1)$ which is impossible. Hence, The Diophantine equation $3^x + 23^y = z^2$ has no solutions in positive integers $x, y$ and $z$.

**Theorem 2.2.** The Diophantine equation $p^x + (p + 20)^y = z^2$, where $p > 3, p + 20$ are primes, has no solutions in positive integers $x, y$ and $z$.

**Proof.** Since $p^x + (p + 20)^y = z^2$, $z$ is even and so $z^2 \equiv 0 \pmod{4}$. Since $p > 3$ is prime, $p \equiv 1 \pmod{4}$ or $p \equiv -1 \equiv 3 \pmod{4}$.

Case I: For $p \equiv 1 \pmod{4}$, $p^x + (p + 20)^y \equiv 2 \pmod{4}$ which is a contradiction since $z^2 \equiv 0 \pmod{4}$.

Case II: Suppose $p \equiv -1 \pmod{4}$.

If $x = 2k, k \geq 1$ and $y = 2s, s \geq 1$, then $p^x + (p + 20)^y \equiv 2 \pmod{4}$. This is a contradiction since $z^2 \equiv 0 \pmod{4}$.

If $x = 2k + 1, k \geq 0$ and $y = 2s + 1, s \geq 0$, then $p^x + (p + 20)^y \equiv 2 \pmod{4}$. This is a contradiction since $z^2 \equiv 0 \pmod{4}$.

If $x = 2k, k \geq 1$ and $y = 2s + 1, s \geq 0$, then $p^{2k} + (p + 20)^{2s+1} = z^2$ or equivalently $(p + 20)^{2s+1} = z^2 - p^{2k} = (z - p^k)(z + p^k)$. Thus there exist non-negative integers $\alpha, \beta$ such that $(p+20)^\alpha = z - p^k$ and $(p+20)^\beta = z + p^k$, where $\alpha < \beta$ and $\alpha + \beta = 2s + 1$. Therefore $2 \cdot p^k = (p + 20)^\alpha((p + 20)^{\beta - \alpha} - 1)$. This implies that $\alpha = 0$ and $2 \cdot p^k = (p + 20)^{2s+1} - 1$. For $s = 0$, $2 \cdot p^k = p + 19$ and so $p(2p^{k-1} - 1) = 19$. Thus $p = 19$ and $k = 1$ but this contradicts the fact that $p + 20 = 39$ is not prime. For $s \geq 1$, $2 \cdot p^k = (p+19)(p+20)^{2s+1} - 1$. It follows that $p + 19$ is an even positive divisor of $2p^k$; that is, $p + 19 = 2 \cdot p^j$, where $j$ is an integer such that $0 \leq j < k$. For $j = 0$, $p + 19 = 2$ which is impossible. For $1 \leq j < k$, $p + 19 = 2p^j$ and so
\[ p(2p^{j-1} - 1) = 19. \text{ Thus } p = 19 \text{ but } p + 20 = 39 \text{ is not prime.} \]

If \( x = 2k + 1, k \geq 0 \) and \( y = 2s, s \geq 1 \), then \( p^{2k+1} + (p+20)^{2s} = z^2 \) or equivalently \( p^{2k+1} - (p+20)^{2s} = (z - (p+20)^s)(z + (p+20)^s) \). Thus there exist non-negative integers \( \alpha, \beta \) such that \( p^\alpha = z - (p+20)^s \) and \( p^\beta = z + (p+20)^s \), where \( \alpha < \beta \) and \( \alpha + \beta = 2k + 1 \). Therefore \( 2(p+20)^s = p^\alpha(p^\beta - \alpha - 1) \). This implies that \( \alpha = 0 \) and \( 2(p+20)^s = p^{2k+1} - 1 \). For \( k = 0 \), we obtain \( 2(p+20)^s = p - 1 = (p+20) - 21 \) or \( 2(p+20)^s + 21 = p + 20 \) which is impossible. For \( k \geq 1 \), we have \( 2(p+20)^s = p^{2k+1} - 1 = (p - 1)[p^{2k} + p^{2k-1} + \ldots + p + 1] \). It follows that \( p - 1 \) is an even positive divisor of \( 2(p+20)^s \); that is, \( p - 1 = 2(p+20)^j \), where \( j \) is an integer such that \( 0 \leq j < s \). For \( j = 0 \), \( p = 3 \) which contradicts the fact that \( p > 3 \). For \( 1 \leq j < s \), we obtain \( 2(p+20)^j = (p + 20) - 21 \) or \( 2(p+20)^j + 21 = p + 20 \) which is impossible.

Therefore, the Diophantine equation \( p^x + (p + 20)^y = z^2 \), where \( p \geq 3 \) and \( p + 20 \) are primes, has no solutions in positive integers \( x, y \) and \( z \).

**Acknowledgements**

This work is supported by Faculty of Science, Burapha University, Thailand.

**References**


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On the Diophantine Equation $p^x + (p + 20)^y = z^2$,...

