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Approximation Solution for Backward Fuzzy Delay Stochastic Differential equations

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Abstract

In this article, we consider the fuzzy stochastic integrals. Then we propose some conditions and definitions of backward stochastic fuzzy differential equations with coefficients delay and discuss the numerical convergence.

1 Introduction

Backward stochastic differential equations "BSDEs" have been first interested in Pardoux and Peng [1] in order to prove existence and uniqueness of the adapted solutions. A backward stochastic differential equation is an equation of the following form

$$\Upsilon_t = \xi + \int_t^T f(s, \Upsilon_s)ds - \int_t^T \Psi_s dW_s \tag{1}$$

where ξ is a terminal condition such that $E|\xi|^2 < \infty$, with Brownian motion $\{W_t\}_{0\le t\le T}$ defined on the complete probability space (Ω, Γ, P) with the natural filtration $\{\Gamma_t\}_{0\le t\le T}$.

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Numerous mathematicians were interested in approximating the solution of BSDEs [2, 3, 4, 5, 6]. Buckwar [7] presented the new numerical method for stochastic delay differential equations with It$\hat{o}$ form and gave a numerical convergence for explicit single-step methods. He discussed numerical convergence for SDEs with time delayed coefficients and studied the problems of numerical solution of SDDEs. Delong and Imkeller [8, 9] discussed the existence and uniqueness of solution of BSDEs with time delayed coefficients. Delong [10, 11] was interested in applications of BSDEs with time delayed coefficients. Moreover, he investigated the solutions of BSDDEs. Malrnowski and Michta [12] established the continuous dependence on initial condition and stability properties in addition to existence and uniqueness of solutions for SFDDEs. Furthermore, Malrnowski [13] studied the strong uniqueness for strong solutions. Michta [14] discussed approaches of SFDE with a semimartingale integrator and existence of fuzzy solution.

In this work, we propose some definitions and basic concepts to study the fuzzy stochastic differential delay equation. Moreover, we discuss the numerical convergence of BFSDDEs with

$$\Upsilon_t = \xi + \int_t^T f(s, \Upsilon_s, \Psi_s, \Upsilon(s), \Psi(s))ds - \int_t^T \Psi_s dW_s \tag{2}$$

where W_t, $0 \leq t \leq T$, is a Brownian motion defined on the complete probability space (Ω, Γ, P). The coefficient f at time s and the terminal condition ξ depend on the past values of a solution $(\Upsilon(s), \Psi(s)) = (\Upsilon_{s+v}, \Psi_{s+v})_{-T \leq v \leq 0}$.

2 Preliminaries and basic assumptions

In this section, we present some assumptions and spaces used in the sequel. We consider the standard d-dimensional $\{W_t, 0 \leq t \leq 1\}$ defined on the complete probability space (Ω, Γ, P) with $\{\Gamma_t\}_{0 \leq t \leq 1}$ denoting the natural filtration of σ-algebra P with Γ_t-progressively measurable subsets of $\Omega \times [0, 1]$. We consider a backward stochastic differential equation as follows

$$\Upsilon_t = \xi + \int_t^1 f(s, \Upsilon_s, \Psi_s, \Upsilon(s), \Psi(s))ds - \int_t^1 \Psi_s dW_s \tag{3}$$

where W_t, $0 \leq t \leq 1$ is a Brownian motion defined on the complete probability space (Ω, Γ, P), ξ is a given Γ_1-measurable random variable, where $E|\xi|^2 < \infty$. The coefficient f is a mapping from $\Omega \times [0, 1] \times \mathbb{R}^p \times \mathbb{R}^{p \times d}$ into

$\mathbb{R}^p$. If $\Upsilon \in \rho^2([0,1], \mathbb{R}^n)$, we define the following norm

$$||\Upsilon||^2 = \sum_{i=1}^{n} E[\sup_{0 \le t \le 1} |\Upsilon_t^i|^2].$$

If $\Psi \in \rho^2([0,1] \times \Omega, \mathbb{R}^{n \times d})$, then we also define the following norm

$$||\Psi||^2 = \sum_{i=1}^{n} \sum_{j=1}^{d} E[| \int_0^1 \Psi_t^i dW_t^j|^2].$$

We consider the following spaces

i) Let $L^2_{-T}(\mathbb{R}^{p \times d})$ be the space of measurable function $\Psi : [-T, 0] \to \mathbb{R}^{p \times d}$ such that $\int_{-T}^0 |\Psi_t|^2 \mathrm{dt} < \infty$.

ii) Let $L^\infty_{-T}(\mathbb{R}^p)$ be the space of measurable function $\Upsilon : [-T, 0] \to \mathbb{R}^p$ such that $\sup_{-T \le t \le 0} |\Upsilon_t|^2$
$< \infty$.

iii) Let $H^2_T(\mathbb{R}^n)$ be the space of Γ-predictable processes $\Upsilon : \Omega \times [0,T] \to \mathbb{R}^n$ such that $E \int_0^T |\Upsilon_t|^2 \mathrm{dt} < \infty$.

iv) Let $Q^2_T(\mathbb{R}^p)$ is the space of Γ-adapted, product measurable processes $\Upsilon : \Omega \times [0,T] \to \mathbb{R}^p$ such that $E[\sup_{0 \le t \le T} |\Upsilon_t|^2] < \infty$.

We consider the norms $||\Upsilon||^2_{Q^2_T} = E[\sup_{0 \le t \le T} |\Upsilon_t|^2]$ and $||\Psi||^2_{H^2_T} = E \int_0^T |\Psi_t|^2 dt$ with the spaces $Q^2_T(\mathbb{R}^p)$ and $H^2_T(\mathbb{R}^{p \times d})$, respectively. We present the past values of the solution $\Upsilon(s) = \Upsilon_{s+\alpha}$ and $\Psi(s) = \Psi_{s+\alpha}$ for $-T \le \alpha \le 0$ with the terminal condition ξ and the set $\Upsilon_t = \Upsilon_0$ and $\Psi_t = 0$ for $t < 0$. Suppose that $B(\mathbb{R}^p)$ is the family of all nonempty compact and convex subsets of $\mathbb{R}^p$ and $\Theta(\mathbb{R}^p)$ is the fuzzy set space of $\mathbb{R}^p$. The set of functions $\lambda : \mathbb{R}^p \to [0,1]$ such that $[\lambda]^v \in B(\mathbb{R}^p)$ for every $v \in [0,1]$, where $[\lambda]^v = \{b \in \mathbb{R}^p : \lambda_b \ge v\}$ for $v \in [0,1]$ and $[\lambda]^0 = \{b \in \mathbb{R}^p : \lambda_b > 0\}$. A mapping $\Upsilon : \Omega \to \Theta(\mathbb{R}^p)$ is said to be a fuzzy random variable if $[\Upsilon]^v : \Omega \to B(\mathbb{R}^p)$ is a Γ-measurable multifunction for all $v \in [0,1]$.

Definition 2.1. *Let (Ω, Γ, P) be a complete probability space. A mapping $\Upsilon : \Omega \to \Gamma(\mathbb{R}^p)$ is a function random variable if for all $v \in [0,1], [\Upsilon]^v : \Omega \to B(\mathbb{R}^p)$ is Γ-measurable.*

Definition 2.2. *A mapping $\Upsilon : [0,1] \times \Omega \to \Gamma(\mathbb{R}^p)$ is said to be a fuzzy stochastic process if the mapping $\Upsilon(t,\cdot) = \Upsilon_t : \Omega \to \Gamma(\mathbb{R}^p)$ is a fuzzy random variable.*

Definition 2.3. *A fuzzy stochastic process is d_∞-continuous if the mappings $\Upsilon(\cdot, W) : [0,1] \to \Gamma(\mathbb{R}^p)$ are d_∞-continuous functions.*

In this work, we consider the BFSDE with time delayed coefficients as follows

$$d\Upsilon_t = f(t, \Upsilon_t, \Psi_t, \Upsilon(t), \Psi(t))dt - \Psi_t dW_t, 0 \leq t \leq T$$

$$\Upsilon_T = \xi(\Upsilon_T, \Psi_T), -T \leq t \leq 0,$$

where $f : \Omega \times [0,T] \times \mathbb{R}^p \times \mathbb{R}^{p\times d} \times \Theta(\mathbb{R}^p) \to \Theta(\mathbb{R}^p)$ is Borel-measurable function at time set depend on the past values of the solution $\Upsilon(s) = \Upsilon_{s+v}$ and $\Psi(s) = \Psi_{s+v}$ for $-T \leq v \leq 0$, and $\Psi_t = 0$, $\Upsilon_t = \Upsilon(0)$ for $t < 0$. We present the following conditions

(H1): $f(\cdot, 0, 0) \in \rho^2([0,1] : \mathbb{R}^p)$.

(H2): $\forall t \in [0,1]$, $\lambda \in [-\varepsilon, 0]$ and $(\Upsilon^1, \Psi^1), (\Upsilon^2, \Psi^2) \in \mathbb{R}^p \times \mathbb{R}^{p\times d}$ then

$$|f(t, \Upsilon^1, \Psi^1, \Upsilon^1(t), \Psi^1(t)) - f(t, \Upsilon^2, \Psi^2, \Upsilon^2(t), \Psi^2(t))|^2 \leq \mu(t, |\Upsilon^1 - \Upsilon^2|^2) + q|\Psi^1 - \Psi^2|^2$$
$$+ k\left(\int_{-T}^{0} |\Upsilon^1_{t+v} - \Upsilon^2_{t+v}|^2 \lambda(dv) + \int_{-T}^{0} |\Psi^1_{t+v} - \Psi^2_{t+v}|^2 \lambda(dv)\right) \quad (4)$$

where $q, k > 0$ and $\mu(t,c)$ is continuous function in both t and c, which is concave and nondecreasing as a function of t, such that for all $t \in [0,1]$, $\mu(t,0) = 0$. The ordinary differential equation $\acute{c} = -\mu(t,c)$, $c(1) = 0$ has a unique solution $c(t) = 0$ for all $t \in [0,1]$.

(H3): There exist $\theta(t) \geq 0$ and $\beta(t) \geq 0$ such that $\mu(t,c) \leq \theta(t) + \beta(t)c$, and $\int_0^1 \theta(t)dt < \infty$, $\int_0^1 \beta(t)dt < \infty$.

3 Numerical Scheme for BFSDDEs

In this section, we introduce a numerical scheme of BFSDDE. For all integers $l, r > 1$ and $t \in [0,T]$, let $-\varepsilon = t_{-r} < t_{-r+1} < \cdots < 0 = t_0 < t_1 < \cdots < t_l = T$ be a partition of interval $[-\varepsilon, T]$, and denote $\delta = \Delta_{i+1} - t_i = \frac{T}{l}$, $1 \leq i \leq l$, $\Delta W_{i+1} = W_{i+1} - W_i$, where $i = 0, 1, \cdots, l-1$ and $\Delta_t = \max_{-\varepsilon \leq i \leq l-1} \Delta_{t_i}$. Let ρ^l denote the approximating binomial random walk with natural filtration

Γ^l. If Γ is a functional defined on Ω, then we shall identify $\Gamma(\rho_0^l, \cdots, \rho_1^l)$ and $\Gamma(\rho^l)$ as the same. The discrete version of the BFSDDE is

$$\check{\Upsilon}_{t_i}^l = \Gamma(\rho^l) + \sum_{j=i}^{l-1} f(t_j, \check{\Upsilon}_{t_j}^l, \check{\Psi}_{t_j}^l, \check{\Upsilon}^l(t_j), \check{\Psi}^l(t_j))\Delta t_j - \sum_{j=i}^{l-1} \check{\Psi}_{t_j}^l \Delta W_{t_j}. \tag{5}$$

From Euler scheme, we deduce that

$$\check{\Upsilon}_{t_i}^l = \check{\Upsilon}_{t_{i+1}}^l + \frac{1}{l} f(t_i, \check{\Upsilon}_{t_i}^l, \check{\Psi}_{t_i}^l, \check{\Upsilon}^l(t_i), \check{\Psi}^l(t_i)) - \check{\Psi}_{t_i}^l \Delta W_{t_i}. \tag{6}$$

Without loss of generality, generator f is bounded by G. By taking expectation and analyzing the error between the approximate solution and the exact solution, the error is bounded by

$$|\check{\Upsilon}_{t_i}^l - \hat{\Upsilon}_{t_i}^l| \leq |E[\check{\Upsilon}_{t_{i+1}}^l - \hat{\Upsilon}_{t_{i+1}}^l / \Gamma_{t_i}^l]| + \frac{1}{l} |f(t_i, \check{\Upsilon}_{t_i}^l, \check{\Psi}_{t_i}^l, \check{\Upsilon}^l(t_i), \check{\Psi}^l(t_i)) - f(t_i, \hat{\Upsilon}_{t_i}^l, \hat{\Psi}_{t_i}^l, \hat{\Upsilon}^l(t_i), \hat{\Psi}^l(t_i))|$$

and

$$|\check{\Psi}_{t_i}^l - \hat{\Psi}_{t_i}^l| \leq l |E[\check{\Upsilon}_{t_{i+1}}^l - \hat{\Upsilon}_{t_{i+1}}^l \Delta\rho_{t_i}^l / \Gamma_{t_i}^l]|.$$

Ma et al. [4] presented the following assumptions

$$\sup_{\omega} |\check{\Upsilon}_{t_i}^l - \hat{\Upsilon}_{t_i}^l| \leq \frac{G(e^{2L} - 1)}{l}$$

and

$$\sup_{\omega} |\check{\Psi}_{t_i}^l - \hat{\Psi}_{t_i}^l| \leq \frac{G(e^{2L} - 1)(2 + \frac{L}{l})}{\sqrt{l}} a.s.,$$

where L is Lipschitz constant.

4 Main Results

This section is devoted to the discussion of numerical convergence of BFSDDE.

Theorem 4.1. *For all $t \in [0, 1]$, the approximate solution $\{\hat{\Upsilon}^l, \hat{\Psi}^l\}$ of equation (3) converges to $\{\check{\Upsilon}^l, \check{\Psi}^l\}$.*

$$\lim_{l\to\infty} E|\check{\Upsilon}_t^l - \hat{\Upsilon}_t^l|^2 = 0 \text{ and } \lim_{l\to\infty} E\int_t^1 |\check{\Psi}_s^l - \hat{\Psi}_s^l|^2 ds = 0.$$

Proof. We need to prove

$$\lim_{l\to\infty} E|\check{\Upsilon}^l_t - \hat{\Upsilon}^l_t|^2 = 0.$$

From the inequality

$$|\check{\Upsilon}^l_{t_i} - \hat{\Upsilon}^l_{t_i}|$$

$$\leq |E[\check{\Upsilon}^l_{t_{i+1}} - \hat{\Upsilon}^l_{t_{i+1}}/\Gamma^l_{t_i}]| + \frac{1}{l}|f(t_i, \check{\Upsilon}^l_{t_i}, \check{\Psi}^l_{t_i}, \check{\Upsilon}^l(t_i), \check{\Psi}^l(t_i)) - f(t_i, \hat{\Upsilon}^l_{t_i}, \hat{\Psi}^l_{t_i}, \hat{\Upsilon}^l(t_i), \hat{\Psi}^l(t_i))|$$

we get

$$|\check{\Upsilon}^l_{t_i} - \hat{\Upsilon}^l_{t_i}|^2$$

$$\leq (|E[\check{\Upsilon}^l_{t_{i+1}} - \hat{\Upsilon}^l_{t_{i+1}}/\Gamma^l_{t_i}]| + \frac{1}{l}|f(t_i, \check{\Upsilon}^l_{t_i}, \check{\Psi}^l_{t_i}, \check{\Upsilon}^l(t_i), \check{\Psi}^l(t_i)) - f(t_i, \hat{\Upsilon}^l_{t_i}, \hat{\Psi}^l_{t_i}, \hat{\Upsilon}^l(t_i), \hat{\Psi}^l(t_i))|)^2.$$

By using the inequality $|\alpha + \beta|^2 \leq 2(|\alpha|^2 + |\beta|^2)$, we deduce that

$$|\check{\Upsilon}^l_{t_i} - \hat{\Upsilon}^l_{t_i}|^2 \leq 2|E[\check{\Upsilon}^l_{t_{i+1}} - \hat{\Upsilon}^l_{t_{i+1}}/\Gamma^l_{t_i}]|^2 + \frac{2}{l^2}|f(t_i, \check{\Upsilon}^l_{t_i}, \check{\Psi}^l_{t_i}, \check{\Upsilon}^l(t_i), \check{\Psi}^l(t_i))$$
$$-f(t_i, \hat{\Upsilon}^l_{t_i}, \hat{\Psi}^l_{t_i}, \hat{\Upsilon}^l(t_i), \hat{\Psi}^l(t_i))|^2.$$

Taking the expectation, we have

$$E|\check{\Upsilon}^l_{t_i} - \hat{\Upsilon}^l_{t_i}|^2 \leq 2E[|E[\check{\Upsilon}^l_{t_{i+1}} - \hat{\Upsilon}^l_{t_{i+1}}/\Gamma^l_{t_i}]|^2 + \frac{2}{l^2}|f(t_i, \check{\Upsilon}^l_{t_i}, \check{\Psi}^l_{t_i}, \check{\Upsilon}^l(t_i), \check{\Psi}^l(t_i))$$
$$-f(t_i, \hat{\Upsilon}^l_{t_i}, \hat{\Psi}^l_{t_i}, \hat{\Upsilon}^l(t_i), \hat{\Psi}^l(t_i))|^2] \leq 2E[|E[\check{\Upsilon}^l_{t_{i+1}} - \hat{\Upsilon}^l_{t_{i+1}}/\Gamma^l_{t_i}]|^2]$$
$$+\frac{2}{l^2}E[|f(t_i, \check{\Upsilon}^l_{t_i}, \check{\Psi}^l_{t_i}, \check{\Upsilon}^l(t_i), \check{\Psi}^l(t_i)) - f(t_i, \hat{\Upsilon}^l_{t_i}, \hat{\Psi}^l_{t_i}, \hat{\Upsilon}^l(t_i), \hat{\Psi}^l(t_i))|^2].$$

Applying condition (H2), we obtain

$$E|\check{\Upsilon}^l_{t_i} - \hat{\Upsilon}^l_{t_i}|^2 \leq 2E[|E[\check{\Upsilon}^l_{t_{i+1}} - \hat{\Upsilon}^l_{t_{i+1}}/\Gamma^l_{t_i}]|^2] + \frac{2}{l^2}E[\mu(t, |\check{\Upsilon}^l_{t_i} - \hat{\Upsilon}^l_{t_i}|^2) + cE|\check{\Psi}^l_{t_i} - \hat{\Psi}^l_{t_i}|^2]$$
$$+\frac{2}{l^2}E[\int_0^T k\int_{-T}^0 |\check{\Upsilon}^{(i)l}_{s+v} - \hat{\Upsilon}^{(i)l}_{s+v}|^l\lambda(dv)ds + \int_0^T\int_{-T}^0 |\check{\Psi}^{(i)l}_{s+v} - \hat{\Psi}^{(i)l}_{s+v}|^2\lambda(dv)ds].$$

Similarly from theorem (4.1) in [15], we have

$$E|\check{\Upsilon}^l_{t_i} - \hat{\Upsilon}^l_{t_i}|^2 \leq 2E[|E[\check{\Upsilon}^l_{t_{i+1}} - \hat{\Upsilon}^l_{t_{i+1}}/\Gamma^l_{t_i}]|^2] + \frac{2}{l^2}E[\mu(t, |\check{\Upsilon}^l_{t_i} - \hat{\Upsilon}^l_{t_i}|^2) + cE|\check{\Psi}^l_{t_i} - \hat{\Psi}^l_{t_i}|^2]$$

$$+\frac{2}{l^2}[k_1E\int_0^T|\check{\Upsilon}_s^{(i)l}-\hat{\Upsilon}_s^{(i)l}|^2ds+k_2E\int_0^T|\check{\Psi}_s^{(i)l}-\hat{\Psi}_s^{(i)l}|^2ds]$$

where k_1, k_2 are positive constants. By taking the term

$$\frac{2}{l^2}[k_1E\int_0^T|\check{\Upsilon}_s^{(i)l}-\hat{\Upsilon}_s^{(i)l}|^2ds+k_2E\int_0^T|\check{\Psi}_s^{(i)l}-\hat{\Psi}_s^{(i)l}|^2ds]$$

$$\leq\frac{2}{l^2}[\frac{k_1G(e^{2L}-1)}{l}+\frac{k_2G(e^{2L}-1)(2+\frac{L}{l})}{\sqrt{l}}].$$

From the inequality

$$E|\check{\Psi}_{t_i}^l-\hat{\Psi}_{t_i}^l|\leq l|E[\check{\Upsilon}_{t_{i+1}}^l-\hat{\Upsilon}_{t_{i+1}}^l\Delta\rho_{t_i}^l/\Gamma_{t_i}^l]|,$$

we have

$$E|\check{\Psi}_{t_i}^l-\hat{\Psi}_{t_i}^l|^2\leq 2l^2E|E[(\check{\Upsilon}_{t_{i+1}}^l-\hat{\Upsilon}_{t_{i+1}}^l)\Delta\rho_{t_i}^l/\Gamma_{t_i}^l]|^2.$$

Now,

$$E|\check{\Upsilon}_{t_i}^l-\hat{\Upsilon}_{t_i}^l|^2\leq 2E[|E[\check{\Upsilon}_{t_{i+1}}^l-\hat{\Upsilon}_{t_{i+1}}^l/\Gamma_{t_i}^l]|^2]+\frac{2}{l^2}E[\mu(t,|\check{\Upsilon}_{t_i}^l-\hat{\Upsilon}_{t_i}^l|^2)$$

$$+2l^2E|E[(\check{\Upsilon}_{t_{i+1}}^l-\hat{\Upsilon}_{t_{i+1}}^l)\Delta\rho_{t_i}^l/\Gamma_{t_i}^l]|^2+\frac{2}{l^2}[\frac{k_1G(e^{2L}-1)}{l}+\frac{k_2G(e^{2L}-1)(2+\frac{L}{l})}{\sqrt{l}}]$$

$$\leq 2E|\check{\Upsilon}_{t_{i+1}}^l-\hat{\Upsilon}_{t_{i+1}}^l|^2+\frac{2}{l^2}E\mu(t,|\check{\Upsilon}_{t_i}^l-\hat{\Upsilon}_{t_i}^l|^2)+4cE|\check{\Upsilon}_{t_{i+1}}^l-\hat{\Upsilon}_{t_{i+1}}^l|^2E|\Delta\rho_{t_i}^l|^2$$

$$+\frac{2}{l^2}[\frac{k_1G(e^{2L}-1)}{l}+\frac{k_2G(e^{2L}-1)(2+\frac{L}{l})}{\sqrt{l}}]\leq 2[\frac{G(e^{2L}-1)}{l}]^2+\frac{2}{l^2}E\mu(t,|\check{\Upsilon}_{t_i}^l-\hat{\Upsilon}_{t_i}^l|^2)$$

$$+\frac{4c}{l}[\frac{G(e^{2L}-1)}{l}]^2+\frac{2}{l^2}[\frac{k_1G(e^{2L}-1)}{l}+\frac{k_2G(e^{2L}-1)(2+\frac{L}{l})}{\sqrt{l}}].$$

Using condition (H3), we get

$$E|\check{\Upsilon}_{t_i}^l-\hat{\Upsilon}_{t_i}^l|^2\leq 2[\frac{G(e^{2L}-1)}{l}]^2+\frac{2}{l^2}[Ea(t)+Eb(t)|\check{\Upsilon}_{t_i}^l-\hat{\Upsilon}_{t_i}^l|^2]+\frac{4c}{l}[\frac{R(e^{2L}-1)}{l}]^2$$

$$\leq 2[\frac{G(e^{2L}-1)}{l}]^2+\frac{2}{l^2}Ea(t)+\frac{2}{l^2}Eb(t)[\frac{G(e^{2L}-1)}{l}]^2+\frac{4c}{l}[\frac{G(e^{2L}-1)}{l}]^2$$

$$=\frac{2l^2(G(e^{2L}-1))^2+2l^2Ea(t)+2Eb(t)(G(e^{2L}-1))^2+4cl(G(e^{2L}-1))^2}{l^4}.$$

Now,

$$E|\check{\Upsilon}^l_{t_i} - \hat{\Upsilon}^l_{t_i}|^2 \leq \frac{2l^2(G(e^{2L}-1))^2 + 2l^2 Ea(t) + 2Eb(t)(G(e^{2L}-1))^2}{l^4}$$

$$+\frac{4cl(G(e^{2L}-1))^2 + l2k_1G(e^{2L}-1) + 2k_2G(e^{2L}-1)(2l+L)}{l^4}.$$

Therefore,

$$E|\check{\Upsilon}^l_{t_i} - \hat{\Upsilon}^l_{t_i}|^2 \leq \frac{C1}{.}l,$$

Letting $l \to \infty$, it follows that, for all $t \in [0,1]$, $\lim_{l\to\infty} E|\check{\Upsilon}^l_{t_i} - \hat{\Upsilon}^l_{t_i}|^2 = 0$.

Consequently,

$$\lim_{l\to\infty} E\int_t^1 |\check{\Psi}^l_s - \hat{\Psi}^l_s|^2 ds = 0$$

for all $t \in [0,1]$.

□

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