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Achieving Metric Dimension One by Weighting the Edges of a Connected Network

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Abstract

Metric dimension is a cardinal number that can be attached to any metric space. A metric space has metric dimension 1 if and only if for some point in the space, the distances from that point to the other points are all different.

Attaching positive weights to the edges of a connected network naturally defines a metric on the node set of the network. In this paper we consider the following two optimization problems: given a connected network, attach positive integer weights to its edges so that the induced metric space has metric dimension one, and (i) the maximum of the assigned weights is as small as possible; or (ii) the sum of the assigned weights is as small as possible.

1 Introduction

Let (X, ρ) be a metric space. A subset $A \subseteq X$ is *resolving* in (X, ρ) if and only if for any $x, y \in X$, $x \neq y$ implies that, for some $a \in A$, $\rho(x, a) \neq \rho(y, a)$. In fancier language, A is resolving in (X, ρ) if and only if the function from X into $[0, \infty)^A$ that maps $x \in X$ to the $A - tuple$ $[\rho(x, a);\ a \in A]$ is one-to-one.

Clearly X itself is resolving in (X, ρ). The *metric dimension* of (X, ρ), which we will denote $md(X, \rho)$, is the cardinal number:

$$md(X, \rho) = min[|A|;\ \text{A is a resolving set in } (X, \rho)]$$

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Metric dimension was apparently first defined by Blumenthal [2], and it has generated some interest ([1],[3],[4]), primarily for the perspective it provides in geometry. It has also had an impact on graph theory [1]. For every connected graph G there is a naturally defined metric $dist_G$ on $V(G)$: for $u, v \in V(G)$, $dist_G(u, v)$ is the length of (number edge traversals in) a shortest walk in G from either of u, v to the other. (Clearly any shortest such walk is a path; its length is the *number of edges* in the path.) Determining $md(V(G), dist_G)$ for connected graphs G in various classes has been a pleasurable exercise ([1]) with—who knows?—potentially useful results. Here is a warm-up sample for recreational purposes, and later comparison: If $r \geq 2$, $2 \leq p_1 \leq ... \leq p_r$ are integers, and $G = K_{p_1,...,p_r}$ is the complete r-partite graph with parts of sizes $p_1, ..., p_r$, then

$$md(V(G), dist_G) = \sum_{j=1}^{r}(p_j - 1) = |V(G)| - r$$

In this paper the setting will be a class of connected graph-involved metric spaces that contains the spaces $(V(G), dist_G)$, where G is a finite connected graph, but the questions we will ask and try to answer have no analogues that we know of in earlier work on metric dimension in graphs.

In all that follows, G will be a finite connected simple graph with vertex set $V(G)$ and edge set $E(G)$. Let the edges of G be *weighted* by a function $wt : E(G) \to (0, \infty)$. If H is a subgraph of G, let $wt(H) = \sum_{e \in E(H)} wt(e)$.

We define, for $u, v \in V(G)$,

$$dist_{(G,wt)}(u, v) = min[wt(P) : P \text{ is a path in } G \text{ with end vertices } u, v]$$

If $u, v \in V(G)$ and P is a path in G with end-vertices u, v such that $wt(P) = dist_{(G,wt)}(u, v)$, then P is a *minimum* $u - v$ *path in* (G, wt).

The following results are elementary in graph theory; their proofs are omitted.

Proposition 1.1 *For any* $wt : E(G) \to (0, \infty)$,
(a) $(V(G),\ dist_{(G,wt)})$ *is a metric space, and*
(b) *every subpath of a minimum path in* (G, wt) *is a minimum path in* (G, wt).

Remark: it is easy to see that $dist_{(G,wt)}(u, v)$ can be alternatively defined as the smallest sum $\sum_{i=1}^{t} wt(e_i)$ in which $e_1, ...e_t$ are the edges successively traversed in a *walk* in G from u to v. This alternative makes the proof that $dist_{(G,wt)}$ satisfies the triangle inequality, and the proof of proposition 1.1(b), very straightforward.

For any $wt : E(G) \to (0, \infty)$ we use an alternative notation for the metric dimension of $(V(G), dist_{(G,wt)})$:

$$md(V(G),\ dist_{(G,wt)}) = md(G, wt)$$

Thus, $md(V(G), dist_G) = md(G, 1)$, where 1 denote the edge weighting of G with constant value 1.

With wt as above, we will use the standard notation

$$\|wt\|_1 = wt(G) = \sum_{e \in E(G)} wt(e), \text{ and}$$

$$\|wt\|_\infty = \underset{e \in E(G)}{max}\, wt(e).$$

Let $\mathbb{Z}^+$ denote the set of positive integers. It is easy to see that for each $v \in V(G)$ there exists $wt : E(G) \to \mathbb{Z}^+$ such that $md(G, wt) = 1$ and $\{v\}$ is a resolving subset of $V(G)$ in (G, wt). Let

$$A(G, v) = \{wt : E(G) \to \mathbb{Z}^+ \mid md(G, wt) = 1 \text{ and } \{v\} \text{ is resolving in } (G, wt)\}, \text{ and}$$

$$A(G) = \bigcup_{v \in V(G)} A(G, v) = \{wt : E(G) \to \mathbb{Z}^+ \mid md(G, wt) = 1\}$$

Our aim here is to work on two optimization problems, the determination of

$$MD_1(G) = min[\|wt\|_1;\ wt \in A(G)]$$

and

$$MD_\infty(G) = min[\|wt\|_\infty;\ wt \in A(G)],$$

and also to discover the weightings $wt : E(G) \to \mathbb{Z}^+$ that realize these values.

It will be useful to consider, for $v \in V(G)$, $MD_x(G, v) = min[\|wt\|_x;\ wt \in A(G, v)]$, for $x \in \{1, \infty\}$. Clearly $MD_x(G) = \underset{v \in V(G)}{min}\, MD_x(G, v)$.

Finally, for $x \in \{1, \infty\}$ and $v \in V(G)$, we let

$$\hat{A}_x(G, v) = \{wt \in A(G, v) \mid \|wt\|_x = MD_x(G, v)\}$$

and

$$\hat{A}_x(G) = \{wt \in A(G) \mid \|wt\|_x = MD_x(G)\}$$

Proposition 1.2 *Suppose that $wt \in \hat{A}_1(G)$. Then every edge $e = uv \in E(G)$ is a minimum path in (G, wt).*

Proof. If $wt(e) = 1$ then e is a minimum path, since all weights are positive integers. Suppose that $wt(e) > 1$.

If e is not a minimum path in (G, wt), then, by Proposition 1.1(b), e is not an edge in any minimum path in (G, wt). Define $wt' : E(G) \to \mathbb{Z}^+$ by

$$wt'(f) = \begin{cases} wt(f) - 1, & f = e, \\ wt(f) & otherwise \end{cases}$$

Since e is on no minimum path in (G, wt), the distances in (G, wt) are the same as those in (G, wt'). Therefore $wt \in \hat{A}_1(G) \subseteq A(G)$ implies that $wt' \in A(G)$, But then $\|wt'\|_1 = \|wt\|_1 - 1$ contradicts $wt \in \hat{A}_1(G)$.

The reader might wonder why we have confined our weights to the positive integers. There are good reasons! If we allowed all weightings $wt : E(G) \to (0, \infty)$, the minima in the definitions of $MD_x(G, v)$, $x \in \{1, \infty\}$ would not exist, and the corresponding infima would all be 0. In fact, for any non-empty $R \subseteq (0, \infty)$ such that $inf R = r$ is an accumulation point of R, if we confined ourselves to weightings $wt : E(G) \to R$, those minima would not exist and the corresponding infima would all be r, except when $G = K_1$, the unique connected graph with no edges. For the case $G = K_1$, contact the philosophy department.

However, there are interesting choices for discrete sets $R \subseteq (0, \infty)$ other than $\mathbb{Z}^+$ to which we could restrict our weights:

$$R = \{1, 2\}, \;\; R = \{primes\}, \;\; R = \{2^k \mid k = 0, 1, 2...\}.$$

As it is early days on this topic, we are happy to go with $\mathbb{Z}^+$.

2 Values of MD_x, $x \in \{1, \infty\}$, for some graphs

All graphs here are finite and simple, and G, as specified earlier, is connected. In addition, we may as well require that $|V(G)| \geq 2$.

Trees

Let the path with n vertices be denoted P_n. Clearly, for $n \geq 2$, the constant function $wt \equiv 1$ is in $A(P_n)$, with either end-vertex of P_n as a resolving vertex, and with no internal vertex being resolving. Since only weights ≥ 1 are allowed, these observations imply that $MD_1(P_n) = n - 1$ and $MD_\infty(P_n) = 1$.

We shall soon see that the paths are the only graphs G for which $1 \in A(G)$. This is intuitively obvious, but what we need for a clean proof will be useful later.

Proposition 2.1 *Suppose that $|V(G)| \geq 3$, $u \in V(G)$ has degree 1 in G, and v is u's only neighbor in G. Then $MD_1(G, u) = 1 + MD_1(G - u, v)$ and $MD_\infty(G, u) = MD_\infty(G - u, v)$. Further the weightings $wt \in \hat{A}_1(G, u)$ are obtained by extending weightings $wt' \in \hat{A}_1(G - u, v)$*

to $E(G)$ *by setting* $wt(uv) = 1$*, and the weightings* $wt \in \hat{A}_\infty(G, u)$ *are obtained by extending weightings* $wt' \in \hat{A}_\infty(G-u, v)$ *to* $E(G)$ *by setting* $wt(uv) = a$ *for some* $a \in \{1, ..., MD_\infty(G-u)\}$.

Proof. Let $H = G - u$ and let wt' denote the restriction of a weighting $wt : E(G) \to \mathbb{Z}^+$ to $E(H) = E(G) \setminus \{uv\}$. Clearly, for any wt and any $w \in V(H)$, $dist_{(G,wt)}(u, w) = wt(uv) + dist_{(H,wt')}(v, w)$. From this equation it is straightforward to see that $wt \in A(G, u)$ if and only if $wt' \in A(H, v)$, and, from there, to verify the claims of the proposition. □

Corollary 2.2 *If the constant function* $1 \in A(G)$*, then* G *is a path.*

Proof. The proof will be by induction on $n = |V(G)| \geq 2$. If $n \in \{2, 3\}$ then $G = P_n$, so we may assume that $n \geq 4$. Suppose that $1 \in A(G)$. Because no edge weight can be < 1, every single edge in a minimum $(G, 1)$ path must be weighted 1. Let u be a resolving vertex in $(G, 1)$. The distance, in $(G, 1)$, to each of its neighbors is 1. Since u is resolving, it follows that u has exactly one neighbor, which we will call v. By the main observation in the proof of Proposition 2.1 applied to this situation, we have that for every $w \in V(G - u)$, $dist_{(G,1)}(u, w) = 1 + dist_{(G-u,1)}(v, w)$. Since u is resolving in G it follows from this that v is resolving in $G - u$, whence $1 \in A(G - u)$. By the induction hypothesis, $G - u$ is a path. By an earlier remark, since v is a resolving vertex in $(G - u, 1)$, v must be an end-vertex of this path. Therefore, G is a path, with u as an end-vertex. □

P_n is the unique tree on n vertices with the greatest possible diameter, $n - 1$. At the other extreme, when $n \geq 3$, the unique tree on n vertices with the least possible diameter, 2, is the complete bipartite graph $K_{1,n-1}$. This graph is also referred to as the star on n vertices, and, when $n \geq 4$, as the *claw* on n vertices. (The *claw*, without specification of n, is $K_{1,3}$.) The unique (when $n \geq 3$) vertex of $K_{1,n-1}$ of degree $n - 1$ is the *center* of $K_{1,n-1}$.

Lemma 2.3 *Suppose that* $n \geq 3$ *and that* v *is the center of* $G = K_{1,n-1}$*. Then* $MD_1(G, v) = \frac{n(n-1)}{2}$, $MD_\infty(G, v) = n - 1$*, and the only weightings in* $\hat{A}_1(G, v) \bigcup \hat{A}_\infty(G, v)$ *are those that assign each integer in* $\{1, ..., n - 1\}$ *to some edge of* G.

Proof. Since the only paths from v to the other vertices of G are the single edges of G, all incident to v, each $wt \in A(G, v)$ must assign different weights to those edges, Obviously the only way to minimize either $\|wt\|_1$, or $\|wt\|_\infty$, since the weights are required to be integers, is to assign $1, ...n - 1$ to the $n - 1$ edges. □

Proposition 2.4 *Suppose that* $n \geq 3$ *and* $G = K_{1,n-1}$*. For any* $u \in V(G)$ *of degree* 1, $MD_1(G) = MD_1(G, u) = 1 + \frac{(n-1)(n-2)}{2}$ *and* $MD_\infty(G) = MD_\infty(G, u) = n - 2$*. Further,* *the only weightings in* $\hat{A}_1(G,u)$ *assign* 1 *to the single edge incident to* u *and the weights* $1,...,n-2$ *to the other* $n-2$ *edges of* G*. The only weightings in* $\hat{A}_\infty(G,u)$ *assign a weight* $a \in \{1,...,n-2\}$ *to the edge incident to* u *and* $1,...,n-2$ *to the other edges.*

Proof. Clearly it suffices to verify the claims about $\hat{A}_x(G,u),\ x \in \{1,\infty\}$. These are easily verified when $n=3$, since $K_{1,2}=P_3$, so we may as well suppose that $n \geq 4$.

Let v be the center of G and let u be one of the vertices of degree 1. The conclusions of this proposition follow from Proposition 2.1, Lemma 2.3, and the obvious facts that $|V(G-u)| = n-1 \geq 3$, and $G-u = K_{1,n-2}$. □

Suppose that $n \geq 3$ and that T is a tree on n vertices. It is clear that $MD_x(P_n) \leq MD_x(T),\ x \in \{1,\infty\}$, with equality only if $P_n = T$, in each case. We wonder if $MD_x(T) \leq MD_x(K_{1,n-1}),\ x \in \{1,\infty\}$, and if so, for which T does equality hold?

Cycles

Throughout $n \geq 3$. Let the vertices of C_n around the cycle be $v_0, v_1, ..., v_{n-1}$. Clearly finding $\hat{A}_x(C_n, v_0)$ will determine $\hat{A}_x(C_n)$ and thus $MD_x(C_n),\ x \in \{1,2\}$. Keep in mind that for each $i \in \{1,...,n-1\}$ there are exactly two paths in C_n from v_0 to v_i.

Lemma 2.5 *If* $wt \in A(C_n, v_0)$ *then* $wt(v_0v_1) \neq wt(v_0v_{n-1})$.

Proof. Let $G = C_n$. If $wt(v_0v_1) = wt(v_0v_{n-1}) = a$ then the weight of the path $v_0, v_1, ..., v_{n-1}$ is greater than a, so $dist_{(G,wt)}(v_0, v_{n-1}) = a = wt(v_0v_{n-1})$. Similarly, $dist_{(G,wt)}(v_0, v_1) = a$. Thus, v_0 is not a resolving vertex for (G, wt), so $wt \notin A(G, v_0)$. □

The *standard weighting* of $E(C_n)$ with respect to the ordering $v_0, ..., v_{n-1}$ of the vertices assigns 1 to v_0v_1, to $v_{\frac{n}{2}}\ v_{\frac{n}{2}+1}$ if n is even, and to $v_{\frac{n-1}{2}}\ v_{\frac{n+1}{2}}$ if n is odd. All other edges are given weight 2.

Lemma 2.6 *The standard weighting of* C_n with respect to the ordering $v_0, ...v_{n-1}$ *of* $V(G_n)$ *is in* $A(C_n, v_0)$.

The proof is left to the reader.

Corollary 2.7 $MD_\infty(C_n) = 2$ *for all* $n \geq 3$.

This is obvious, by Corollary 2.2 and Lemma 2.6.

The standard weightings are not the only elements of $\hat{A}_\infty(C_n)$. For each standard weighting, one of the weights 1 may be changed to 2 to obtain another, non-standard, element of $\hat{A}_\infty(C_n)$. It seems straightforward to see that these are the only weightings in $\hat{A}_\infty(C_n)$, the standard and the slightly non-standard, but we will not belabor the question.

Observe that if wt is a standard weighting of $E(C_n)$, then $\|wt\|_1 = 2n-2$. We shall soon see that $MD_1(C_n) = 2n-2$. From this it follows (if the claim above about $\hat{A}_\infty(C_n)$ is agreed to) that the standard weightings of $E(C_n)$ are the only elements of $\hat{A}_\infty(C_n) \bigcap \hat{A}_1(C_n)$. But they are not the only weightings in $\hat{A}_1(C_n)$.

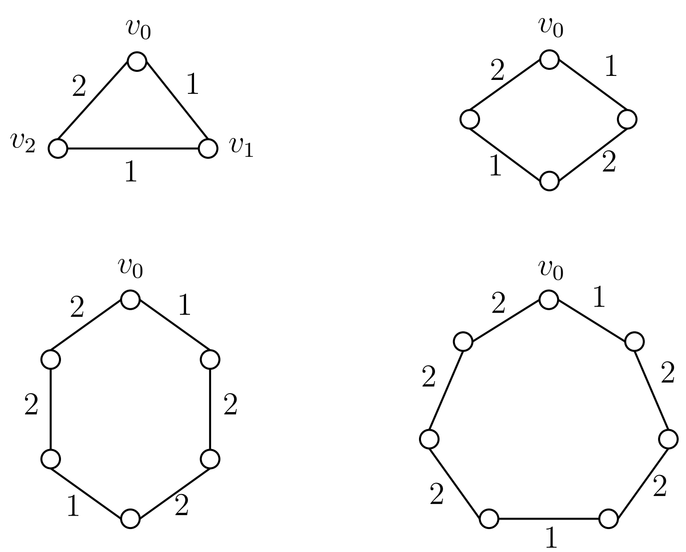

Figure 1. Standard weightings of C_n, $n = 3, 4, 5, 6, 7$.

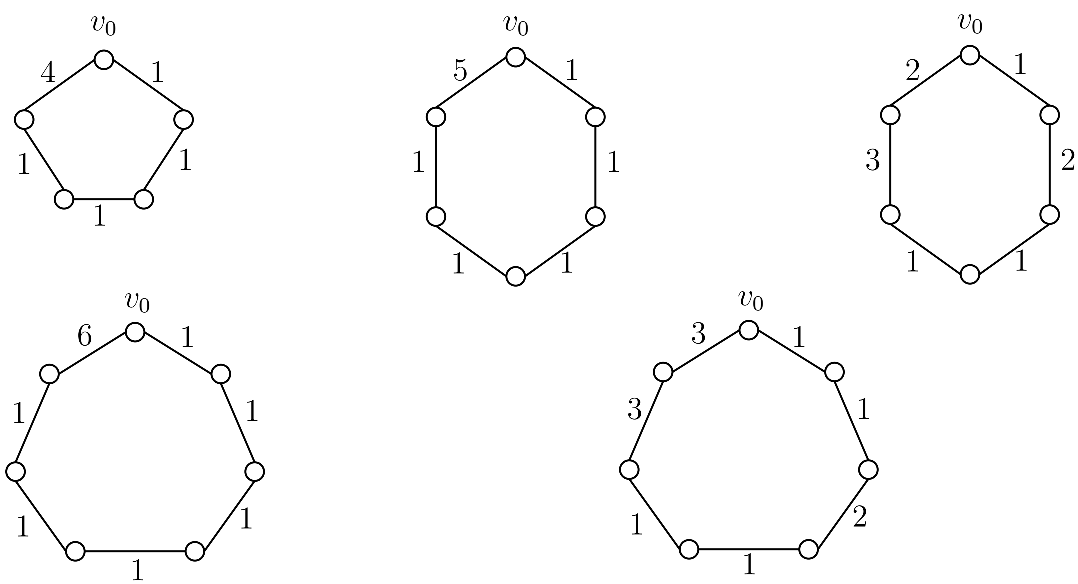

Figure 2. Some non-standard weightings in $\hat{A}_1(C_n, v_0),\ n = 5, 6, 7$.

Now we shall see that $MD_1(C_n) = 2n - 2$. In what follows, for each $i \in 1, ..., n-1$, Let R_i denote the path on C_n with end vertices v_0 and v_i which contains v_1, and let L_i denote the path on C_n with end vertices v_0 and v_i which contains v_{n-1}. Observe that for any weighting wt of $E(C_n)$, for $1 \le i \le n-1,\ dist_{(C_n,wt)}(v_0, v_i) = min[wt(L_i), wt(R_i)]$

Lemma 2.8 *Suppose that* $wt : E(C_n) \to (0, \infty)$ *and that, for some* $q \in \{1, ..., n-1\}$, $wt(L_q) = wt(R_q)$. *Then* $wt(R_q) = \max_{1 \le i \le n-1} dist_{(C_n,wt)}(v_0, v_i)$.

Proof. This result is a straightforward consequence of the remark preceding the lemma statement and the fact that as i increases, $wt(L_i)$ decreases while $wt(R_i)$ increases. □

Lemma 2.9 *Suppose that* $wt \in \hat{A}_1(C_n, v_0)$. *Then either* $wt(v_0v_1) = 1$ or $wt(v_0v_{n-1}) = 1$, *but not both. Further, assuming that* $wt(v_0v_1) = 1$, *if* $q = max[i \in \{1, ..., n-1\};\ wt(L_i) \ge wt(R_i)]$ *then* $wt(L_q) = wt(R_q)$.

Proof. By Lemma 2.5 we have that $a = wt(v_0v_1) \ne wt(v_0v_{n-1}) = b$. If $a, b > 1$ then we define wt' by $wt'(v_0v_j) - 1,\ j \in \{1, n-1\}$ and $wt'(e) = wt(e)$ for all $e \in E(C_n) \setminus \{v_0v_1, v_0v_{n-1}\}$; wt' is positive integer-valued and $wt'(Q) = wt(Q) - 1$ for all $Q \in \{L_i \mid i \in \{1, ..., n-1\}\} \bigcup \{R_i \mid$

$i \in \{1, ..., n-1\}\}$. Therefore the distances of $v_1, ..., v_{n-1}$ from v_0 in (C_n, wt') are distinct, because they are distinct in (C_n, wt). Therefore $wt' \in A(C_n, v_0)$. But then $\|wt'\|_1 = \|wt\|_1 - 2$ contradicts the assumption that $wt \in \hat{A}_1(C_n, v_0)$.

Now suppose that $wt(v_0v_1) = 1$. Then $wt(v_0v_{n-1}) > 1$, so $wt(L_1) > wt(v_0v_{n-1}) > 1 = wt(R_1)$. Thus q is well defined. Suppose that $wt(L_q) > wt(R_q)$.

If $q = n-1$ then $wt(L_{n-1}) = wt(v_0v_{n-1}) > wt(R_{n-1})$ implies that $dist_{(G,wt)}(v_0, v_{n-1}) = wt(R_{n-1}) < wt(v_0v_{n-1})$, so the edge v_0v_{n-1} is not a minimum path in (C_n, wt). Since $wt \in \hat{A}(C_n, v_0) \subseteq \hat{A}(C_n)$, this would contradict Proposition 1.2.

Suppose that $q < n-1$. By the definition of q, we have that $wt(L_{q+1}) < wt(R_{q+1})$. With $wt(L_q) > wt(R_q)$, this implies that for $i \in \{1, ..., q\}$, R_i is the unique minimum $v_0 - v_i$ path in (C_n, wt), and for $j \in \{q+1, ..., n-1\}$, L_j is the unique minimum $v_0 - v_j$ path in (C_n, wt). Therefore, v_qv_{q-1} does not appear on any of these minimum paths with one end at v_0. If $wt(v_qv_{q+1}) \geq 2$ we can reduce the weight on v_qv_{q+1} by 1 to obtain a new weighting $wt' \in A(C_n, v_0)$ with $\|wt'\|_1 = \|wt\|_1 - 1$, contradicting $wt \in \hat{A}(C_n, v_0)$.

If $wt(v_qv_{q+1}) = 1$ then we have:

$$wt(R_q) \leq wt(L_q) - 1 = wt(L_{q+1}) \text{ and}$$

$$wt(L_{q+1}) \leq wt(R_{q+1}) - 1 = wt(R_q), \text{ so}$$

$$dist_{(C_n,wt)}(v_0, v_{q+1}) = wt(L_{q+1}) = wt(R_q) = dist_{(C_n,wt)}(v_0, v_q),$$

which contradicts the assumption that v_0 is a resolving vertex in (C_n, wt). □

Theorem 2.10 *For all* $n \geq 3$, $MD_1(C_n) = 2n - 2$.

Proof. By Lemma 2.6, $MD_1(C_n) \leq 2n - 2$.

Suppose that $wt \in \hat{A}_1(C_n, v_0)$. Since the numbers $dist_{(C_n,wt)}(v_0, v_i)$, $i = 1, ..., n-1$ are distinct positive integers, the largest of them is $\geq n-1$. By Lemmas 2.8 and 2.9, for some $q \in \{1, ..., n-1\}$,

$$wt(L_q) = wt(R_q) = \max_{1 \leq i \leq n-1} dist_{(C_n,wt)}(v_0, v_i).$$

Therefore,
$$MD_1(C_n) = \|wt\|_1 = wt(L_q) + wt(R_q) \geq n - 1 + n - 1 = 2n - 2.$$
□

Remark. Shrewd observers will note that the conclusion in Lemma 2.9 that $min[wt(v_0v_1), wt(v_0v_{n-1})] = 1$ if $wt \in \hat{A}_1(C_n, v_0)$ is unnecessary for the rest of the proofs of Lemma 2.9 and Theorem 2.10. We threw it in because it bears on the question of how to construct weightings in $\hat{A}_1(C_n, z_0)$, a question not completely answered by Lemmas 2.5, 2.6, and 2.9.

Complete graphs

Let the vertices of K_n be $v_0, v_1, ..., v_{n-1}$. Clearly $\hat{A}_x(K_n),\ x \in \{1, \infty\}$, will be determined if we describe $\hat{A}_x(K_n, v_0)$.

The *standard weighting* of $E(K_n)$ with respect to the ordering $v_0, v_1, ..., v_{n-1}$ is defined as follows: $wt(v_0 v_k) = k,\ k = 1, ..., n-1$, and for $1 \le i < j \le n-1$, $wt(v_i v_j) = j - i$. The following proposition is straightforward but tedious to prove; the proof is omitted.

Proposition 2.11 *Suppose that wt is the standard weighting of $E(K_n)$ with respect to the ordering $v_0, v_1, ...v_{n-1}$ of $V(K_n)$.* Then
(i) $dist_{(K_n,wt)}(v_0, v_k) = k,\ k = 1, ..., n-1$; *consequently*
(ii) $wt \in A(K_n, v_0)$;
(iii) $\|wt\|_\infty = n - 1$ *and* $\|wt\|_1 = \frac{n(n^2-1)}{6}$.

It is worth noting that if wt is the standard weighting of $E(K_n)$ with respect to the ordering $v_0, v_1, ...\ v_{n-1}$, then its restriction to $K_n - v_0$ is the standard weighting of K_{n-1} with respect to the ordering $v_1, ..., v_{n-1}$. In addition, the weighting is also the standard weighting of $E(K_n)$ with respect to the ordering $v_n, v_{n-1}, ..., v_1, v_0$.

Theorem 2.12 $MD_\infty(K_n) = n - 1$ *and for every* $wt \in \hat{A}_\infty(K_n, v_0),\ 1, n-1 \in \{wt(v_0 v_j) \mid i = 1, ..., n-1\}$.

Proof. By Proposition 2.11, $MD_\infty(K_n) \le n - 1$. Suppose that $wt \in \hat{A}_\infty(K_n, v_0)$. Then $\|wt\|_\infty = MD_\infty(K_n) \le n - 1$.

Since each edge $v_0 v_i$ is a path with ends v_0 and v_i, we have $n - 1 \ge wt(v_0 v_i) \ge dist_{(K_n,wt)}(v_0, v_i),\ i = 1, ..., n-1$. On the other hand, because v_0 is resolving in (K_n, wt), the positive integers $dist_{(K_n,wt)}(v_0, v_i),\ i = 1, ..., n-1$ are distinct. Therefore, the largest of these numbers is $\ge n - 1$. Therefore, because each is $\le n - 1$, the largest of the distances is $n - 1$ and $\{dist_{(K_n,wt)}(v_0, v_i) \mid i = 1, ..., n\} = \{1, ..., n-1\}$.

Without loss of generality, we can suppose that $dist_{(K_n,wt)}(v_0, v_j) = j,\ j = 1, ..., n-1$. Since $n - 1 \ge wt(v_o v_{n-1}) \ge dist_{(K_n,wt)}(v_0, v_{n-1}) = n - 1$, we have that $n - 1 = wt(v_0 v_{n-1})$. Thus $MD_\infty(K_n) = n - 1$.

If $wt(v_0 v_i) > 1$ for all $i = 1, ..., n-1$ then there can be no path from v_0 to v_1 in (K_n, wt) of weight 1, contradicting $dist_{(K_n,wt)}(v_0, v_1) = 1$. Therefore, $1 \in \{wt(v_o v_i) \mid i = 1, ..., n-1\}$. [In fact, because $wt(v_0 v_i) \ge dist_{(K_n,wt)}(v_0, v_i) = i,\ i = 1, ..., n-1$, it must be that $1 = wt(v_0 v_1) < wt(v_0 v_i),\ i = 2, ..., n-1$.]

□

Theorem 2.12 and its proof put enough restrictions on weightings in $\hat{A}_\infty(K_n)$ that they can be completely described. We shall not do so here. But, in Figure 3, we illustrate how standard weightings of K_n may be modified to obtain nonstandard weightings in $\hat{A}_\infty(K_n)$.

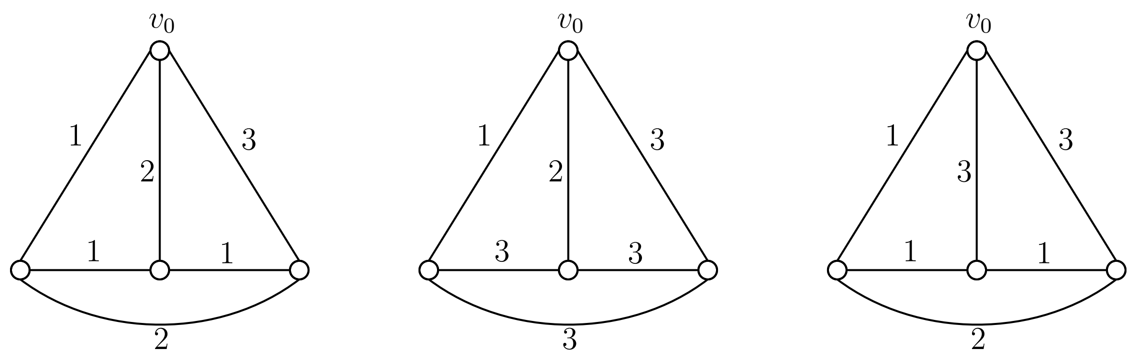

Figure 3. A standard and two nonstandard weightings in $\hat{A}_\infty(K_4, v_o)$.

Theorem 2.13 *$MD_1(K_n) = \frac{n(n^2-1)}{6}$ and the only weightings in $\hat{A}_1(K_n)$ are the standard weightings.*

Proof. Suppose that $wt \in \hat{A}_1(K_n, v_0)$. As in the proof of Theorem 2.12, we have $wt(v_0v_i) \geq dist_{(K_n,wt)}(v_0, v_i) = d_i,\ i = 1, ..., n-1$, and $d_1, ..., d_{n-1}$ are distinct positive integers. Without loss of generality we can suppose that $1 \leq d_1 < ... < d_{n-1}$.

By Proposition 1.2, $d_i = wt(v_0v_i),\ i = 1, ..., n-1$.

If $1 \leq i < j \leq n-1$ then v_0, v_i, v_j are the vertices along a $v_0 - v_j$ path in K_n, whence

$$wt(v_0v_i) + wt(v_iv_j) = d_i + wt(v_iv_j) \geq d_j$$

$$\Rightarrow wt(v_iv_j) \geq d_j - d_i$$

Since $1 \leq d_1 < ... < d_{n-1}$, we have that $wt(v_0v_i) = d_i \geq i$ for $i = 1, ..., n-1$, and $wt(v_iv_j) \geq d_j - d_i \geq j - i$ for $i \leq i < j \leq n-1$. Thus, for each edge in K_n the weight assigned to that edge by wt is greater than or equal to the weight of the edge in the standard weighting with respect to the ordering $v_0, v_1, ..., v_{n-1}$. Now the minimality of $\|wt\|_1$ and claim (ii) of Proposition 2.11 imply the conclusion of the theorem. □

We can exploit the ideas of the optimal weightings of K_n and $K_{1,n-1}$, with respect to $\| \|_1$, to obtain a result about graphs containing a master vertex—a vertex adjacent to every other vertex—which are not complete.

Theorem 2.14 *Suppose that* $n \geq 3$, *G is a graph on vertices* $v_0, ..., v_{n-1}$, $v_0 v_i \in E(G)$, $i = 1, ..., n-1$, *and* $v_1 v_{n-1} \notin E(G)$. *Let* $wt : V(G) \to \mathbb{Z}^+$ *be defined* by:

$$wt(v_0 v_1) = 1, \ wt(v_0 v_i) = i - 1, \ i = 2, ..., n-1;$$

if $2 \leq j \leq n-2$ *and* $v_1 v_j \in E(G)$ *then* $wt(v_1 v_j) = j$; *if* $1 < i < j \leq n-1$ *and* $v_i v_j \in E(G)$ *then* $wt(v_i v_j) = j - i$.

Then $wt \in A(G, v_1)$.

We leave the proof to the reader.

Corollary 2.15 *If* $G \neq K_n$ *is a graph on* $n \geq 3$ *vertices with a master vertex, then* $MD_\infty(G) \leq n - 2$ and $MD_1(G) \leq \frac{(n-1)(n-2)(n+3)}{6}$.

Proof omitted, except: with wt defined as above, $\|wt\|_1 = \frac{(n-1)(n-2)(n+3)}{6}$ when $G = K_n - e$ for a single edge e.

3 Problems

Obviously there is a mother lode of open questions remaining on this optimization problem. Besides the few that we have scattered through the text preceding, we suggest determining $MD_x(G)$, $x \in \{1, \infty\}$, and the optimal weightings of G, when G is

(a) K_n minus the edges of a matching;

(b) a complete multipartite graph;

(c) K_n minus the edges of spanning tree other than $K_{1,n-1}$;

(d) a grid, either plain, $P_m \square P_n$, or cylindrical, $P_m \square C_n$, or toroidal, $C_m \square C_n$; here, $\square$ stands for the Cartesian product of graphs.

Of these, option (d) seems best fitted for possible applications, because grids are fundamental in the design of communication and transportation networks.

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