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Hardy-type Inequalities for Convex Functions

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Abstract

In this note, we obtain a new class of Hardy's integral inequalities by using a fairly elementary analysis. This result simplifies the proofs of some existing results and two possible constants are shown in the limit Hardy-type version.

1 Introduction

Attention of researchers were drawn to the following famous inequalities:

Theorem 1.1. *Let $a_1, a_2, ..., a_n$ be non-negative real numbers. Set $A_n = \sum_{k=1}^{n} a_k$. If $1 < p < \infty$, then*

$$\sum_{n=1}^{\infty} \left|\frac{1}{n} A_n\right|^p \leq C_p \sum_{n=1}^{\infty} |a_n|^p. \tag{1.1}$$

Theorem 1.2. *Let $f(x) \geq 0, p > 1$, f is integrable over any finite interval $(0, x)$ and f^p is integrable over $(0, \infty)$. Then*

$$\int_0^{\infty} \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{1-p}\right)^p \int_0^x f^p(x)dx. \tag{1.2}$$

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Theorem 1.3. *Let $p > 1$ and $\epsilon < p - 1$. For all measurable non-negative functions f, we have*

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p x^\epsilon dx \le \left(\frac{p}{1-p-\epsilon}\right)^p \int_0^x f^p(x)x^\epsilon dx. \tag{1.3}$$

The constants in (1.1), (1.2) and (1.3) are sharp. Hardy [5] (see also [8]) obtained (1.1) in his attempt to provide a solution to the Hilbert double series theorem:

Theorem 1.4. *If $\sum_{m=1}^\infty a_m^2 < \infty$ and $\sum_{n=1}^\infty b_n^2 < \infty$, where $a_m \ge 0$ and $b_n \ge 0$, then the double series $\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n}$ converges. In particular,*

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} \le \pi \Big(\sum_{m=1}^\infty a_m^2\Big)^{\frac{1}{2}} \Big(\sum_{n=1}^\infty b_n^2\Big)^{\frac{1}{2}}. \tag{1.4}$$

Hardy announced (1.2) without proof in [5] and finally proved it in [6]. He also established (1.3) which is an extension of (1.2) in [7]. Many researchers have extensively devoted much effort and time on the improvement and generalization of those inequalities. Most of this research appear in [1] and [9]-[12].
Motivated by this research is the work of Barza, et al., [3], where the following inequalities were deduced:

$$\begin{aligned} &\alpha^{p-1}\left(\int_0^1 f(x)dx\right)^p + \alpha^p \int_0^1 \left[\log\frac{e}{x}\right]^{\alpha p-1}\left(\int_0^x f(y)dy\right)^p \frac{dx}{x} \\ &\le \int_0^1 x^p \left[\log\frac{e}{x}\right]^{(\alpha+1)p-1} f^p(x)\frac{dx}{x}. \end{aligned} \tag{1.5}$$

$$\begin{aligned} &\alpha^{p-1}\left(\int_0^1 f(x)dx\right)^p + \alpha^p \int_0^1 \left[\log\frac{e}{x}\right]^{-\alpha p-1}\left(\int_x^1 f(y)dy\right)^p \frac{dx}{x} \\ &\le \int_0^1 x^p \left[\log\frac{e}{x}\right]^{(1-\alpha)p-1} f^p(x)\frac{dx}{x}, \end{aligned} \tag{1.6}$$

for all non-negative measurable function f on [0, 1]. Both constants α^p and α^{p-1} in (1.5) and (1.6) are sharp. The above results are refinements of those in [4]. The definition of superquadratic or subquadratic function is given in [2]. The focus of this paper is to complement the results in [3] which will be useful in applications by using the derivation of subquadratic, superquadratic and limit Hardy-type inequalities with two possible constants through which a new class of inequalities was achieved. Except otherwise stated, φ is taken to be a convex and superquadratic function with $\varphi(t) = t^q, \quad q \ge 2$.

2 Main Results

The following lemmas from [2] are in order with the next two deduced theorems.

Lemma 2.1. *Let (Ω, μ) be a probability measure space. The inequality*

$$\varphi\left(\int_\Omega f(s)d\mu(s)\right) \leq \int_\Omega \varphi(f(s))d\mu(s) - \int_\Omega \varphi\left(\left|f(s) - \int_\Omega f(s)d\mu(s)\right|\right) d\mu(s), \tag{2.7}$$

holds for all probability measures μ and all non negative μ-integrable functions f if and only if φ is superquadratic.

Moreover, (2.7) holds in the reverse direction if and only if φ is subquadratic.

Lemma 2.2. *Let $\varphi : [0, \infty) \longrightarrow \mathbb{R}$ be a continuously differentiable and $\varphi(0) \leq 0$. If φ' is superadditive or $\frac{\varphi'(x)}{x}$ is nondecreasing, then φ is superquadratic.*

Theorem 2.3. *Let $a, b \in \mathbb{R}$. Suppose $\eta : \mathbb{R} \longrightarrow [0, \infty)$ is rd-continuous and $\varphi : [0, \infty) \longrightarrow \mathbb{R}$. Then*

$$\left(\frac{1}{b-a}\int_a^b \eta(t)d\mu(t)\right)^q \leq \frac{1}{b-a}\int_a^b \left(\eta(x)^q - \left|\eta(x)^q - \frac{1}{b-a}\int_a^b \eta(t)d\mu(t)\right|^q\right) d\mu(x). \tag{2.8}$$

The proof of the next theorem is sufficient for Theorem 2.3.

Theorem 2.4. *Let $u, \upsilon \in \mathbb{R}$ be non negative functions such that the $\mu-$integral $\int_t^b \frac{u(x)\eta(x)}{(b-a)\sigma(x)-a}d\mu(x) < \infty$ and define the weight function $\upsilon(\tau)$ by*

$$\upsilon(t) = (t-a)\int_t^b \frac{u(x)\eta(x)}{(x-a)(\sigma(x)-a)}d\mu(x), \quad t \in (a, b).$$

(i) If φ is defined on (a, b) and $0 < a < b < \infty$, then

$$\begin{aligned}
&\int_a^b u(x)\eta(x)\left(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} \eta(t)d\mu(t)\right)^q \frac{d\mu(x)}{x-a} \\
&+ \int_a^b \int_t^b \left|(\eta(t))^q - \left(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} \eta(t)d\mu(t)\right)\right|^q \frac{u(x)\eta(x)}{(x-a)(\sigma(x)-a)}d\mu(x)d\mu(t) \\
&\leq \int_a^b u(x)(\eta(x))^{q+1}\frac{d\mu(x)}{x-a}
\end{aligned} \tag{2.9}$$

holds for all $\mu-$integrable functions $\eta \in \mathbb{R}$ on (a,b) and any real function $\sigma(x)$.
(ii) If the real valued function φ is defined on (a,b), $0 < a < c \leq \infty$, then (2.9) holds in the reverse direction.

Proof:
(i) Applying Jensen′s inequality and Fubini theorem, we have

$$\begin{aligned}
&\int_a^b u(x)\eta(x)\left(\tfrac{1}{\sigma(x)-a}\int_a^{\sigma(x)}\eta(t)d\mu(t)\right)^q \tfrac{d\mu(x)}{x-a}\\
&\leq \int_a^b \tfrac{u(x)\eta(x)}{(x-a)(\sigma(x)-a)}\int_a^{\sigma(x)}\eta(t)^q d\mu(x)d\mu(t)\\
&-\int_a^b \tfrac{u(x)\eta(x)}{(x-a)(\sigma(x)-a)}\int_t^b \left|\eta(t)^q - \tfrac{1}{\sigma(x)-a}\int_a^{\sigma(x)}\eta(t)d\mu(t)\right|^q d\mu(x)\\
&=\int_a^b \eta(t)^q \int_t^b \tfrac{u(x)\eta(x)}{(x-a)(\sigma(x)-a)}d\mu(x)d\mu(t)\\
&-\int_a^b\int_t^b \left|\eta(t)^q - \tfrac{1}{\sigma(x)-a}\int_a^{\sigma(x)}\eta(t)d\mu(t)\right|^q \tfrac{u(x)\eta(x)}{(x-a)(\sigma(x)-a)}d\mu(x)d\mu(t)\\
&=\int_a^b u(x)\eta(x)\eta(t)^q \tfrac{d\mu(t)}{t-a}\\
&-\int_a^b\int_t^b \left|\eta(t)^q - \tfrac{1}{\sigma(x)-a}\int_a^{\sigma(x)}\eta(t)d\mu(t)\right|^q \tfrac{u(x)\eta(x)}{(x-a)(\sigma(x)-a)}d\mu(x)d\mu(t)
\end{aligned} \tag{2.10}$$

□.

(ii) The proof is similar to (i) above with the inequality sign reversed. This proves the theorem.
Further simplification of (2.10) with $u(x) = x - a, a < b < \infty$ and $u(x) = 1$ yields

$$\begin{aligned}
&\int_a^b \eta(x)\left(\frac{1}{\sigma(x)-a}\int_a^b \eta(t)\right)d\mu(x) - \int_a^b \eta(x)\eta(x)^q d\mu(x)\\
&\leq \int_a^b \eta(x)^{q+1}d\mu(x) - \int_a^b\int_t^b \left(\left|\eta(t) - \left(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)}\eta(t)d\mu(t)\right)\right|\right)\frac{\eta(x)}{\sigma(x)-a}d\mu(x)d\mu(t).
\end{aligned} \tag{2.11}$$

If $b = \infty$, then the above inequality becomes:

$$\begin{aligned}
&\int_a^\infty \eta(x)\left(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)}\eta(t)\right)^q \frac{d\mu(x)}{x-a} - \int_a^\infty \eta(t)\frac{\eta(x)^q}{x-a}d\mu(x)\\
&\leq \int_a^\infty\int_t^\infty \left|\eta(x) - \frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)}\eta(t)d\mu(t)\right|^q \frac{\eta(x)}{\sigma(x)-a}d\mu(x)d\mu(t)
\end{aligned} \tag{2.12}$$

(ii) Inequality (2.12) holds in the reverse direction if φ is subquadratic.

Theorem 2.5. *Let $\alpha, q > 0$, n be a non-negative integer and f be a non-negative and measurable function on [0, 1]. Then the following statements are valid:*
(i) If $q > 1$, then

$$\left(\int_0^x f(y)dg(y)\right)^{q+n} + \alpha^{q+n-1}\left(\int_0^1 f(x)dg(x)\right)^{q+n} + \alpha^{q+n}\int_0^1 \left[\log\frac{e}{x}\right]^{\alpha(q+n)-1} \left(\int_0^x f(y)dg(y)\right)^{q+n}\frac{dg(x)}{x} \le \int_0^1 x^{q+n}\left[\log\frac{e}{x}\right]^{(\alpha+1)(q+n-1)} f^{q+n}(x)\frac{dg(x)}{x} + \left(\int_0^x f(y)dg(y)\right)^{q+n} \tag{2.13}$$

and

$$\left(\int_0^x f(y)dg(y)\right)^{q+n} + \alpha^{q+n-1}\left(\int_0^1 f(x)dg(x)\right)^{q+n} + \alpha^{q+n}\int_0^1 \left[\log\frac{e}{x}\right]^{-(\alpha(q+n)-1)} \left(\int_0^x f(y)dg(y)\right)^{q+n}\frac{dg(x)}{x} \le \int_0^1 x^{q+n}\left[\log\frac{e}{x}\right]^{(\alpha+1)(q+n-1)} f^{q+n}(x)\frac{dg(x)}{x} + \left(\int_0^x f(y)dg(y)\right)^{q+n} \tag{2.14}$$

constants α^{q+n} and α^{q+n-1} in (2.13) and (2.14) are sharp. Equality is never attained unless f is identically zero and for any measurable function f with $a > 0$.
(ii) If $0 < q+n < 1$, then both (2.13) and (2.14) hold in the reverse direction and the constants in both inequalities are sharp. Equality is never attained unless f is identically zero.

Proof:
(i) Let $q > 1$. Let $x \in (0, 1]$. Consider the function

$$G(x;n,q,\alpha) = \int_0^x y^{q+n}\left(\log_y e\right)^{(1+\alpha)(q+n-1)} f^{q+n}(y)\frac{dg(y)}{y} + \left(\int_0^x f(y)dg(y)\right)^{q+n} - \alpha^{q+n}\int_0^x \left(\log_y e\right)^{\alpha(q+n)-1}\left(\int_0^y f(s)dg(s)\right)^{q+n}\frac{dg(y)}{y} - \left(\int_0^x f(y)dg(y)\right)^{q+n} - \alpha^{q+n-1}\left(\log_y e\right)^{\alpha(q+n)}\left(\int_0^x f(y)dg(y)\right)^{q+n}.$$

Then

$G^{'}(x;n,q,\alpha) =$

$$\begin{aligned}
&\frac{x^{q+n}}{x}\left(\log_y e\right)^{(1+\alpha)(q+n-1)} f^{q+n}(x) + (q+n)\left(\int_0^x f(y)dg(y)\right)^{q+n-1} f(x) \\
&- \frac{\alpha^{q+n}}{x}\left(\log_y e\right)^{\alpha(q+n)-1}\left(\int_0^y f(s)dg(s)\right)^{q+n} \\
&- (q+n)\alpha^{q+n-1} f(x)\left(\log_y e\right)^{\alpha(q+n)}\left(\int_0^x f(y)dg(y)\right)^{q+n-1} \\
&- (q+n)\left(\int_0^x f(y)dg(y)\right)^{q+n-1} f(x) \\
&- \frac{\alpha^{(\alpha+1)(q+n)-1}}{x}\left(\log_y e\right)^{\alpha(q+n)-1}\left(\int_0^x f(y)dg(y)\right)^{q+n} \\
&- (q+n)\alpha^{q+n-1}\left(\log_y e\right)^{\alpha(q+n)}\left(\int_0^x f(y)dg(y)\right)^{q+n-1} f(x).
\end{aligned} \tag{2.15}$$

For any $h > 0$, we have $h^q - (q+n)h - q - n - 1 \geq 0$ if $q \geq 2$ and $h^q - (q+n)h - q - n - 1 \leq 0$ for $0 \leq q+n \leq 1$. Equality holds if and only if $h = 1$. We obtain $G'(x;n,q) =$

$$\left(\log_y e\right)^{\alpha(q+n)-1}\frac{\alpha^{q+n}}{x}\left(\int_0^x f(s)dg(s)\right)^{q+n}\left[h^{(q+n)}(x;\alpha) - h(q+n)(x;\alpha) - q - n - 1\right]$$

We assume without restriction that $f(t) > 0$ and $t > 0$. We use a limit argument with

$$h(x;y,\alpha) = \frac{x\left(\log_y e\right) f(x)}{\alpha\int_0^x f(y)dg(y)}.$$

Hence, $G^{'}(x;n,q,\alpha) > 0$ which implies that $G(x;n,q,\alpha)$ is strictly increasing. In particular, $G(1;n,q,\alpha) \geq \lim_{x\longrightarrow 0^+} G(x;n,q,\alpha)$. Hence,

$$\lim_{x\longrightarrow 0^+} G(x;n,q,\alpha) = 0. \tag{2.16}$$

If $q = p$ and $n = 0$, then we have the results in [3] and [4]. Moreover, by applying Hölder's inequality with exponential indices $q+n$ and $\frac{q+n}{q+n-1}$, we get

$$\int_0^x f(y)dg(y) \leq \int_0^x \left(y^{1-\frac{1}{q+n}}(\log_y e)^{\alpha+1-\frac{1}{q+n}} f(y)\right)\left(y^{-1+\frac{1}{q+n}}(\log_y e)^{-\alpha-1+\frac{1}{q+n}}\right)dg(y)$$
$$\leq \left(\int_0^x y^{q+n-1}(\log_y e)^{(\alpha+1)(q+n-1)} f^{q+n}(y)dg(y)\right)^{\frac{1}{q+n}} \left(\int_0^x \frac{1}{y}(\log_y e)^{-\frac{\alpha(q+n)}{q+n-1}-1}dg(y)\right)^{\frac{q+n-1}{q+n}}$$
$$= \left(\int_0^x y^{q+n-1}(\log_y e)^{(\alpha+1)(q+n-1)} f^{q+n}(y)dg(y)\right)^{\frac{1}{q+n}} \left(\frac{q+n-1}{\alpha(q+n)}(\log_y e)^{-\frac{\alpha(q+n)}{q+n-1}}\right)^{\frac{q+n-1}{q+n}}$$
$$= \left(\int_0^x y^{q+n-1}(\log_y e)^{(\alpha+1)(q+n-1)} f^{q+n}(y)dg(y)\right)^{\frac{1}{q+n}} \left(\frac{q+n-1}{\alpha(q+n)}\right)^{\frac{q+n-1}{q+n}}(\log_y e)^{-\alpha}.$$

We observe that

$$0 < (\log_x e)^{\alpha(q+n)}\left(\int_0^x f(y)dg(y)\right)^{q+n}$$
$$\leq \left(\frac{q+n-1}{\alpha(q+n)}\right)^{q+n-1}\int_0^x y^{q+n-1}(\log_y e)^{(\alpha+1)(q+n-1)} f^{q+n}(y)dg(y)$$

and

$$\lim_{x\longrightarrow 0^+}(\log_x e)^{\alpha(q+n)}\left(\int_0^x f(y)dg(y)\right)^{q+n} = 0.$$

This shows that

$$\lim_{x\longrightarrow 0^+} G(x;n,q,\alpha) = 0$$

and

$$G(1;n,q,\alpha) \geq \lim_{x\longrightarrow 0^+} G(x;n,q,\alpha) = 0$$

which proves (2.13) for all continuous functions and the sharpness of the inequality. By standard approximation arguments, (2.13) holds for all measurable functions. Let $\{H_i\}, i \in \mathbb{N}$ be an increasing sequence. Consider

$$H_1\left(\int_0^1 f(x)dg(x)\right)^{q+n} + H_2\int_0^1 \frac{dg(x)}{x}(\log_y e)^{\alpha(q+n)-1}\left(\int_0^x f(y)dg(y)\right)^{q+n}$$
$$\leq \int_0^1 \frac{dg(x)}{x}x^{q+n}\left(\log_y e\right)^{(\alpha+1)(q+n-1)} f^{q+n}(x), \tag{2.17}$$

and adopt the method of a test function to show the sharpness of the latter inequality as follows: Let $\alpha > 1$ and $q+n > 0$ and

$$f_\epsilon(x) = \frac{1}{x}\left(\log_y e\right)^{(\epsilon+\alpha+1)}, \qquad \epsilon > 0 \tag{2.18}$$

$$\left(\int_0^1 f(x)dg(x)\right)^{q+n} = \frac{1}{(\alpha+\epsilon)^{q+n}}$$

$$\int_0^1 \frac{dg(x)}{x}(\log_y e)^{\alpha(q+n)-1}\left(\int_0^x f(y)dg(y)\right)^{q+n} = \frac{1}{\epsilon(q+n)(\alpha+\epsilon)^{q+n}} \quad (2.19)$$

and

$$\int_0^1 \frac{dg(x)}{x}x^{q+n}\left(\log_y e\right)^{(\alpha+1)(q+n-1)} f^{q+n}(x) = \frac{1}{\epsilon(q+n)}. \quad (2.20)$$

Combining and adding (2.18) to (2.20) yields

$$\frac{H_1}{(\alpha+\epsilon)^{q+n}} + \frac{\alpha^{q+n}}{\epsilon(q+n)(\alpha+\epsilon)^{q+n}} \leq \frac{1}{\epsilon(q+n)}$$

which implies

$$H_1 \leq \frac{(\alpha+\epsilon)^{q+n} - \alpha^{q+n}}{\epsilon(q+n)}.$$

Thus, the limit of H_1 cannot be greater than α^{q+n-1} as ϵ goes to zero and hence $H_1 = \alpha^{q+n-1}$.
(ii) The proof of (2.14) is similar.

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