International Journal of Mathematics and
Computer Science, **16**(2021), no. 1, 23–32

# On functional equations related to generalized Jordan derivations in rings

**Sitthikorn Nakkao, Utsanee Leerawat**

Department of Mathematics
Faculty of Science
Kasetsart University
Bangkok 10900, Thailand

email: sitthikorn.n@ku.th, fsciutl@ku.ac.th

(Received April 16, 2020, Accepted June 1, 2020)

**Abstract**

In this paper, we generalize the notions of Jordan derivation and generalized Jordan derivation to Jordan $(f, g)$ −derivation and generalized Jordan $(f, g)$ −derivation, respectively. Moreover, we investigate additive mappings satisfying some functional equations becomes Jordan $(f, g)$ −derivation and generalized Jordan $(f, g)$ −derivation.

## 1 Introduction

A derivation on a ring $R$ is an additive mapping $D$, which maps a ring $R$ into itself satisfying the product rule $D(xy) = D(x)y + xD(y)$ for all $x$ and $y$ in $R$. In 1957, Posner [2] proved two theorems, namely: (1) In a prime ring of characteristics not 2, if the iterate of two derivations is a derivation, then one of them is zero; (2) If $D$ is a derivation of a prime ring such that, for all elements $a$ of the ring, $aD(a) - D(a)a$ is central, then either the ring is commutative or $D$ is zero. An additive mapping $D$ which maps a ring $R$ into itself is called a Jordan derivation if $D(x^2) = D(x)x + xD(x)$ holds for all $x$ in $R$. It is straight forward to see that every derivation on a ring is a Jordan derivation but the converse need not be true in general. Herstein

**Key words and phrases:** Ring, derivation, Jordan derivation, generalized Jordan derivation.
**AMS (MOS) Subject Classifications**: 16W25, 13N15, 47B47.
**ISSN** 1814-0432, 2021, http://ijmcs.future-in-tech.net
Corresponding author: Utsanee Leerawat

[3] proved that any Jordan derivation on a prime ring of characteristic not 2 is a derivation. Bresar [8] extended this result to semiprime rings. In 1991, Bresar [9] introduced the notion of generalized derivations in rings as follows: an additive mapping $F : R \longrightarrow R$ is said to be a generalized derivation if there exists a derivation $D : R \longrightarrow R$ such that $F(xy) = F(x)y+xD(y)$ holds for all $x, y$ in $R$. An additive mapping $F : R \longrightarrow R$ is said to be a generalized Jordan derivation if there exists a Jordan derivation $D : R \longrightarrow R$ such that $F(x^2) = F(x)x + xD(x)$ holds for all $x, y$ in $R$. Obviously, every generalized derivation is a generalized Jordan derivation but there exists generalized Jordan derivation which is not generalized derivation. Ashraf and Rehman [6] showed that in a 2-torsion free ring, which has a commutator nonzero divisor, every generalized Jordan derivation on $R$ is a generalized derivation. In 2007, Vukman [4] proved that every generalized Jordan derivation on a 2-torsion free semiprime ring is a generalized derivation. An additive mapping $x \longmapsto x^*$ on a ring $R$ is called an involution if $(x^*)^* = x$ and $(xy)^* = y^*x^*$ hold for all $x, y$ in $R$. A ring equipped with an involution is called a ring with involution or $*$-ring. An additive mapping $D : R \longrightarrow R$ is called a $*$ - derivation (resp. Jordan $*$ - derivation) if $D(xy) = D(x)y^* + xD(y)$ (resp. $D(x^2) = D(x)x^* + xD(x)$) for all $x, y$ in $R$. An additive mapping $F : R \longrightarrow R$ is said to be a generalized $*$ - derivation (resp. generalized Jordan $*$ - derivation) if there exists a $*$ - derivation (Jordan $*$ - derivation) $D : R \longrightarrow R$ such that $F(xy) = F(x)y^* + xD(y)$ (resp. $F(x^2) = F(x)x^* + xD(x)$) for all $x, y$ in $R$.

In 2014, Rehman, et al. [11] proved that additive mappings $F$ and $D$, which map an $(n + 1)!$ - torsion free $*$ - ring $R$ into itself, satisfying the relation
$F(x^{n+1}) = F(x)(x^*)^n + xD(x)(x^*)^{n-1} + \ldots + x^nD(x)$,
for all $x, y$ in $R$, implies $D(x^2) = D(x)x^* + xD(x)$ and $F(x^2) = F(x)x^* + xD(x)$ for all $x$ in $R$.

This paper is inspired by the result of Rehman, et al. [11]. We generalize the notions of Jordan derivation and generalized Jordan derivation to Jordan $(f, g)$ – derivation and generalized Jordan $(f, g)$ – derivation respectively. Moreover, we investigate conditions for additive mappings to be Jordan $(f, g)$ – derivation and generalized Jordan $(f, g)$ – derivation.

# 2 Preliminaries

Throughout this paper, $R$ will be a ring. Recall that a ring $R$ is prime if for any $a, b \in R, aRb = 0$ implies $a = 0$ or $b = 0$, and is semiprime if for any $a \in R, aRa = 0$ implies $a = 0$. For an integer $n > 1$, an element $x \in R$ is called $n-$torsion free if $nx = 0$ implies $x = 0$. A ring $R$ is called an $n-$torsion free ring if every element in $R$ is $n-$torsion free. Moreover, a ring $R$ is called an $n!-$torsion free if it is $d-$torsion free for any divisor $d$ of $n!$.

Next, we define generalizations of Jordan derivation and generalized Jordan derivation as follows:

**Definition 2.1.** *Let $R$ be a ring and $f, g$ be endomorphisms of $R$. An additive mapping $D : R \longrightarrow R$ is called a Jordan $(f, g)-$ derivation if*

$$D(x^2) = D(x)f(x) + g(x)D(x),$$

*for all $x \in R$.*

*An additive mapping $F : R \longrightarrow R$ is called a generalized Jordan $(f, g)-$derivation if there exists a Jordan $(f, g)-$ derivation $D$ such that*

$$F(x^2) = F(x)f(x) + g(x)D(x),$$

*for all $x \in R$ and $D$ is said to be an associated Jordan $(f, g)-$ derivation of $F$.*

Obviously, every (generalized) Jordan derivation is a (generalized) Jordan $(f, g)-$derivation.

# 3 Main results

**Theorem 3.1.** *Let $n > 1$ be a fixed positive integer and let $R$ be an $2\,(n-1)!-$torsion free ring with identity $1$. If $D : R \longrightarrow R$ is an additive mapping and $f, g$ are endomorphisms of $R$ satisfying*

$$2D(x^n) = D(x^{n-1})f(x) + g(x^{n-1})D(x) + D(x)f(x^{n-1}) + g(x)D(x^{n-1}) \quad (3.1)$$

*for all $x \in R$, then $D$ is a Jordan $(f, g)-$ derivation.*

*Proof.* For $x = 1$ in (3.1) we have $D(1) = 0$.
Setting $x + k1$ for $x$ in (3.1) , where $k$ is any positive integer, we obtain

$$\begin{aligned} 2D((x+k1)^n) &= D((x+k1)^{n-1})f(x+k1) \\ &\quad +g((x+k1)^{n-1})D(x+k1) \\ &\quad +D(x+k1)f((x+k1)^{n-1}) \\ &\quad +g(x+k1)D((x+k1)^{n-1}), \end{aligned} \tag{3.2}$$

for all $x \in R$.
By expanding (3.2) and using (3.1), we obtain

$$\begin{aligned} 2D\left(\sum_{i=1}^{n-1}\binom{n}{i}k^i x^{n-i}\right) &= D\left(\sum_{i=1}^{n-2}\binom{n-1}{i}k^i x^{n-1-i}\right)f(x) \\ &\quad +D\left(\sum_{i=0}^{n-2}\binom{n-1}{i}k^i x^{n-1-i}\right)kf(1) \\ &\quad +g\left(\sum_{i=1}^{n-2}\binom{n-1}{i}k^i x^{n-1-i}+k^{n-1}1\right)D(x) \\ &\quad +D(x)f\left(\sum_{i=1}^{n-2}\binom{n-1}{i}k^i x^{n-1-i}+k^{n-1}1\right) \\ &\quad +g(x)D\left(\sum_{i=1}^{n-2}\binom{n-1}{i}k^i x^{n-1-i}\right) \\ &\quad +kg(1)D\left(\sum_{i=0}^{n-2}\binom{n-1}{i}k^i x^{n-1-i}\right). \end{aligned}$$

This can be written as

$$kh_1(x,1) + k^2h_2(x,1) + \cdots + k^{n-1}h_{n-1}(x,1) = 0,$$

for all $x \in R$, where $h_i(x,1)$ are the coefficients of $k^i$ for all $i = 1, 2, \ldots, n-1$.
Replacing $k$ by $1, 2, \ldots, n-1$, and expressing the resulting system of $n-1$ homogeneous equations, we get

$$AH = [0],$$

where $A = \begin{bmatrix} 1 & 1^2 & \cdots & 1^{n-1} \\ 2 & 2^2 & \cdots & 2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ n-1 & (n-1)^2 & \cdots & (n-1)^{n-1} \end{bmatrix}$,

$$H = \begin{bmatrix} h_1(x,1) \\ h_2(x,1) \\ \vdots \\ h_{n-1}(x,1) \end{bmatrix}$$ and $[0]$ is $(n-1) \times 1$ zero matrix.

Since $det(A) = \prod_{i=0}^{n-2}(n-i)!$ and $R$ is $2(n-1)!-$torsion free, it follows that the system has only a trivial solution. In particular, $h_{n-2}(x,1) = 0$ implies that

$$2(n-1)D(x^2) = 2(n-1)D(x)f(x) + 2(n-1)g(x)D(x),$$

for all $x \in R$. Since $R$ is $2(n-1)!-$torsion free, it follows that

$$D(x^2) = D(x)f(x) + g(x)D(x),$$

for all $x \in R$.
Therefore, $D$ is a Jordan $(f,g)-$derivation. □

**Theorem 3.2.** *Let $R$ be a $2-$torsion free semiprime ring. Let $f : R \longrightarrow R$ be a homomorphism and $g : R \longrightarrow R$ be an epimorphism. If $D : R \longrightarrow R$ is an additive mapping satisfying*

$$D(xyx) = D(xy)f(x) + g(xy)D(x), \tag{3.3}$$

*for all $x, y \in R$, then $D$ is a Jordan $(f,g)-$derivation.*

*Proof.* Linearization of the equation (3.3) gives

$$D(xyz + zyx) = D(xy)f(z) + D(zy)f(x) + g(xy)D(z) + g(zy)D(x),$$

for all $x, y, z \in R$. In particular, for $z = x^2$, the above equation gives

$$\begin{aligned} D(xyx^2 + x^2yx) &= D(xy)f(x^2) + D(x^2y)f(x) \\ &\quad + g(xy)D(x^2) + g(x^2y)D(x), \end{aligned} \tag{3.4}$$

for all $x, y \in R$.
Replacing $y$ by $xy + yx$ in (3.3) and using (3.3), we get

$$\begin{aligned} D(x^2yx + xyx^2) &= D(x^2y)f(x) + D(xy)f(x^2) + g(xy)D(x)f(x) \\ &\quad + g(x^2y)D(x) + g(xyx)D(x), \end{aligned} \tag{3.5}$$

for all $x, y \in R$. By comparing (3.4) and (3.5), we have

$$g(x)g(y)A(x) = 0 \text{ for all } x, y \in R, \tag{3.6}$$

where $A(x) = D(x^2) - D(x)f(x) - g(x)D(x)$.
Since $g$ is onto, the equation (3.6) can be written

$$g(x)yA(x) = 0 \text{ for all } x, y \in R. \tag{3.7}$$

Right multiplication of (3.7) by $g(x)$ and left multiplication by $A(x)$ yield

$$A(x)g(x)yA(x)g(x) = 0 \text{ for all } x, y \in R.$$

By the semiprimeness of $R$, it follows that

$$A(x)g(x) = 0 \text{ for all } x \in R. \tag{3.8}$$

Replacing $y$ by $A(x)yg(x)$ in (3.7), we have

$$g(x)A(x)yg(x)A(x) = 0 \text{ for all } x, y \in R.$$

Again, by the semiprimeness of $R$, we have

$$g(x)A(x) = 0 \text{ for all } x \in R. \tag{3.9}$$

The linearizating of (3.8) gives

$$B(x,y)g(x) + A(x)g(x) + B(x,y)g(y) + A(x)g(x) = 0, \tag{3.10}$$

for all $x, y \in R$, where
$B(x,y) = D(xy + yx) - D(x)f(y) - D(y)f(x) - g(x)D(y) - g(y)D(x)$.
The substitution $-x$ for $x$ in (3.10) leads to

$$B(x,y)g(x) + A(x)g(y) - B(x,y)g(y) - A(y)g(x) = 0, \tag{3.11}$$

for all $x, y \in R$. The equations (3.10) and (3.11) reduce to

$$2\,(B(x,y)g(x) + A(x)g(y)) = 0, \text{ for all } x, y \in R.$$

Since $R$ is $2-$torsion free, it follows that

$$B(x,y)g(x) + A(x)g(y) = 0 \text{ for all } x, y \in R.$$

In view of the equation (3.9), right multiplication by $A(x)$ yields $A(x)g(y)A(x) = 0$, for all $x, y \in R$. Since $g$ is onto and $R$ is semiprime, it follows that

$$A(x) = 0, \text{ for all } x \in R.$$

Hence $D(x^2) = D(x)f(x) + g(x)D(x)$, for all $x \in R$.
In other words, $D$ is a Jordan $(f, g)-$derivation. □

For the sake of brevity, we omit the proof of the following theorem.

**Theorem 3.3.** *Let $R$ be a $2-$torsion free semiprime rimg. Let $f : R \longrightarrow R$ be an epimorphism and $g : R \longrightarrow R$ be a homomorphism. If $D : R \longrightarrow R$ is an additive mapping satisfying*

$$D(xyx) = D(x)f(yx) + g(x)D(yx),$$

*for all $x, y \in R$, then $D$ is a Jordan $(f, g)-$derivation.*

Let $f$ and $g$ be endomorphisms of a ring $R$. If $F : R \longrightarrow R$ and $D : R \longrightarrow R$ are additive mappings satisfying

$$F(x^{n+1}) = F(x)f(x^n) + \sum_{i=1}^{n} g(x^i)D(x)f(x^{n-i}), \tag{3.12}$$

for all $x \in R$. It is natural to ask the additive mappings satisfying (3.12) implies that

$$F(x^2) = F(x)f(x) + g(x)D(x) \text{ and } D(x^2) = D(x)f(x) + g(x)D(x),$$

for all $x \in R$.

**Theorem 3.4.** *Let $n$ be a fixed positive integer and let $R$ be an $(n+1)!-$torsion free ring with identity $1$. If $F : R \longrightarrow R, D : R \longrightarrow R$ are two additive mappings and $f, g$ are endomorphisms of $R$ satisfying*

$$F(x^{n+1}) = F(x)f(x^n) + \sum_{i=1}^{n} g(x^i)D(x)f(x^{n-i}), \tag{3.13}$$

*for all $x \in R$, then $D$ is a Jordan $(f, g)-$derivation on $R$ and $F$ is a generalized Jordan $(f, g)-$derivation.*

*Proof.* Taking $x = 1$ in (3.13), we obtain $F(1) = F(1) + nD(1)$.
This implies that $nD(1) = 0$ and since $R$ is $(n + 1)!-$torsion free, we get $D(1) = 0$. Replacing $x$ by $x + k1$ in (3.13), where $k$ is any positive integer, we obtain

$$F\left((x + k1)^{n+1}\right) = (F(x) + kF(1))\,(f(x) + k1)^n + \sum_{i=1}^{n} (g(x) + k1)^i\, D(x)\,(f(x) + k1)^{n-i}, \tag{3.14}$$

for all $x \in R$.
By expanding (3.14) and using (3.13), we obtain

$$\begin{aligned}
F\left(\sum_{i=1}^{n}\binom{n+1}{i}k^i x^{n+1-i}\right) &= F(x)\left(\sum_{i=1}^{n-1}\binom{n}{i}k^i f(x^{n-i})+k^n 1\right)\\
&+kF(1)\left(\sum_{i=1}^{n-1}\binom{n}{i}k^i f(x^{n-i})\right)\\
&+g(x)D(x)\left(\sum_{i=1}^{n-2}\binom{n-1}{i}k^i f(x^{n-i})+k^{n-1}1\right)\\
&+kD(x)\left(\sum_{i=0}^{n-2}\binom{n-i}{i}k^i f(x^{n-1-i})+k^{n-1}1\right)\\
&+g(x^2)D(x)\left(\sum_{i=1}^{n-3}\binom{n-2}{i}k^i f(x^{n-2-i})+k^{n-2}1\right)\\
&+2kg(x)D(x)\left(\sum_{i=0}^{n-3}\binom{n-2}{i}k^i f(x^{n-2-i})+k^{n-2}1\right)\\
&+k^2D(x)\left(\sum_{i=0}^{n-3}\binom{n-2}{i}k^i f(x^{n-2-i})+k^{n-2}1\right)\\
&\vdots\\
&+\left(\sum_{i=1}^{n-2}\binom{n-1}{i}k^i g(x^{n-1-i})+k^{n-1}1\right)D(x)f(x)\\
&+\left(\sum_{i=1}^{n-2}\binom{n-1}{i}k^i g(x^{n-1-i})+k^{n-1}1\right)kD(x)\\
&+\left(\sum_{i=1}^{n-1}\binom{n}{i}k^i g(x^{n-i})+k^{n}1\right)D(x).
\end{aligned}$$

Collecting terms, we obtain

$$kh_1(x,1)+k^2h_2(x,1)+\cdots+k^nh_n(x,1)=0,$$

for all $x \in R$, where $h_i(x,1)$ are the coefficients of $k^i$ for all $i = 1,2,\ldots,n$. Replacing $k$ by $1,2,\ldots,n$, and expressing the resulting system of $n$ homogeneous equations, we get

$$AH = [0],$$

where $A = \begin{bmatrix} 1 & 1^2 & \cdots & 1^n \\ 2 & 2^2 & \cdots & 2^n \\ \vdots & \vdots & \ddots & \vdots \\ n & n^2 & \cdots & n^n \end{bmatrix}$,

$H = \begin{bmatrix} h_1(x,1) \\ h_2(x,1) \\ \vdots \\ h_n(x,1) \end{bmatrix}$ and $[0]$ is $n \times 1$ zero matrix.

Since $det(A) = \prod_{i=0}^{n-1}(n-i)!$, and $R$ is $(n+1)!-$torsion free ring, it follows that the system has only a trivial solution. In particular, $h_n(x,1) = 0$ implies that $(n+1)F(x) = F(x) + nF(1)f(x) + nD(x)$ for all $x \in R$. Since $R$ is $(n+1)!-$torsion free, we get

$$F(x) = F(1)f(x) + D(x) \text{ for all } x \in R.$$

Next, $h_{n-1}(x,1) = 0$ implies that

$$\begin{aligned} n(n+1)F(x^2) &= 2nF(x)f(x) + n(n-1)F(1)f(x^2) + n(n+1)g(x)D(x) \\ &\quad + n(n-1)D(x)f(x) \text{ for all } x \in R. \end{aligned}$$

Using $F(x) = F(1)f(x) + D(x)$ and $R$ is $(n+1)!-$torsion free. The above equation reduces to

$$F(x^2) = F(1)f(x^2) + D(x)f(x) + g(x)D(x) \tag{3.15}$$

for all $x \in R$. Replacing $x$ by $x^2$ in $F(x) = F(1)f(x) + D(x)$, yields

$$F(x^2) = F(1)f(x^2) + D(x^2) \text{ for all } x \in R. \tag{3.16}$$

Equating (3.15) and (3.16), we have

$$D(x^2) = D(x)f(x) + g(x)D(x) \text{ for all } x \in R.$$

Thus, $D$ is a Jordan $(f,g)-$derivation on $R$. Now, by (3.15) and using $F(x) = F(1)f(x) + D(x)$, we get

$$F(x^2) = F(x)f(x) + g(x)D(x) \text{ for all } x \in R.$$

Hence, $F$ is a generalized Jordan $(f,g)-$derivation.
The proof of the theorem is complete. □

# 4 Acknowledgement

This research was supported by the Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok, Thailand.

# References

[1] B. Dhara, R. K. Sharma, *On additive mappings in rings with identity element*, Int. Math. Forum, **4,** no. 15, (2009), 727–732.

[2] E. C. Posner *Derivations in prime rings*, Proc. Amer. Math. Soc., **8,** no. 6, (1957), 1093–1100.

[3] I. N. Herstein, *Jordan derivations of prime rings*, Proc. Amer. Math. Soc., **8,** (1957), 1104–1119.

[4] Joso Vukman, *A note on generalized derivations of semiprime rings*, Taiwanese J. Math., **11,** no. 2, (2007), 367–370.

[5] Joso Vukman, Irena Ulbl-Kosi, *A note on derivations in semiprime rings*, Int. J. Math. Sci., **20,** (2005), 3347–3350.

[6] M. Ashraf, N. Rehman, *On Jordan generalized derivations in rings*, Math. J. Okayama Univ., **42,** (2000), 7–10.

[7] M. Ashraf, N. Rehman, *On* $(\sigma,\tau)-$*derivations in prime rings,* Arch. Math., Brno, **38,** (2002), 259–264.

[8] M. Bresar, *Jordan derivations on semiprime ring,* Proc. Amer. Math. Soc., **104,** (1988), 1003–1006.

[9] M. Bresar, *On the distance of the composition of two derivations to the generalized derivation,* Glasgow Math. J., **33,** (1991), 89–93.

[10] N. Aydin, K. Kaya *Some generalizations in prime rings with* $(\sigma,\tau)-$*derivation,* Turk. J. Math., **16,** (1992), 169–176.

[11] N. Rehman, A. Z. Ansari, T. Bano, *On generalized Jordan * - derivation in rings,* J. Egyptian Mathematical Society, **22,** (2014), 11–13.