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Equivalence of Weighted DT-Moduli of (Co)convex Functions

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Abstract

This paper presents new definitions for weighted DT moduli. Similarly, a general result in an equivalence of moduli of smoothness is obtained. It is known that, for any $r \in \mathbb{N}_\circ$, $0 < p \leq \infty$, $1 \leq \eta \leq r$ and $\phi(x) = \sqrt{1-x^2}$, the equivalences
$\omega^{\phi}_{i+1,r}\ (f^{(r)}, \|\theta_{\mathcal{N}}\|)_{w_{\alpha,\beta},p} \sim \omega^{\phi}_{i,r+1}\ (f^{(r+1)}, \|\theta_{\mathcal{N}}\|)_{w_{\alpha,\beta},p}$
and
$\omega^{\phi}_{i+\eta}\ (f, \|\theta_{\mathcal{N}}\|)_{\alpha,\beta,p} \sim \|\theta_{\mathcal{N}}\|^{-\eta}\omega^{\phi}_{i,2\eta}\ (f^{(2\eta)}, \|\theta_{\mathcal{N}}\|)_{\alpha+\eta,\beta+\eta,p}$
are valid.

1 Introduction

Hierarchy foundations of the moduli of smoothness began modern with the work of Ditzian and Totik (1987), (see [6]), and Kopotun (2006-2019), (see [8, 9, 10, 11, 12, 14, 15, 16, 18]). Ditzian and Totik established better continuous moduli of the function in a normed space. Then Kopotun contributed

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to properties of various moduli of smoothness like univariate piecewise polynomial functions (splines) [16]. He has a significant impact on the hierarchy between moduli of smoothness for the past 14 years including his effect on the kth symmetric difference (see, [9, proof of Lemma 4.1]). Let $\Delta_h^k(f,x)$ be the kth symmetric difference of f [6] given by

$$\Delta_h^k(f,x)=\begin{cases}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}f(x+(\frac{2i-k}{2})h)\ ; & x\pm\frac{kh}{2}\in[-1,1]\ ,\\ 0\ ; & \text{otherwise.}\end{cases}$$

The space $L_p([-1,1])$, $0<p<\infty$, denotes the space of all measurable functions f on $[-1,1]$, [15] such that

$$\|f\|_{L_{p[-1,1]}}=\begin{cases}(\int_{-1}^{1}|f(x)|^p dx)^{\frac{1}{p}}<\infty\ ; & \text{if } 0<p<\infty\ ,\\ \operatorname{esssup}_{x\in[-1,1]}|f(x)|\ ; & \text{if } p=\infty\ .\end{cases}$$

Let $\|.\|_p=\|.\|_{L_{p[-1,1]}}, 0<p\leq\infty$ and $\phi(x)=\sqrt{1-x^2}$. The Ditzian-Totik modulus of smoothness (DTMS) of a function $f\in L_p[-1,1]$, is defined [5] by

$$\omega_{k,r}^{\phi}(f,t)_p=\sup_{0<h\leq t}\|\phi^r\Delta_{h\phi}^k(f,x)\|_p,\quad k,r\in\mathbb{N}_\circ.$$

Also, the kth modulus of smoothness of $f\in L_p[-1,1]$ is defined [6] by

$$\omega_k(f,\delta,[-1,1])_p=\sup_{0<h\leq\delta}\|\Delta_h^k(f,x)\|_p,\quad \delta>0,p\leq\infty.$$

Denote by $AC_{loc}(-1,1)$ and $AC[-1,1]$ the sets of functions which are locally absolutely continuous on $(-1,1)$ and absolutely continuous on $[-1,1]$ respectively. We accept the following:

Definition 1.1. *[18] Let $w_{\alpha,\beta}(x)=(1+x)^\alpha(1-x)^\beta$ be the (classical) Jacobi weight, and let*

$$\alpha,\beta\in J_p=\begin{cases}(-1/p,\infty), & \text{if } p<\infty,\\ [0,\infty), & \text{if } p=\infty.\end{cases}$$

Define

$$\mathbb{L}_p^{\alpha,\beta}=\{f:[-1,1]\longrightarrow\mathbb{R}:\|w_{\alpha,\beta}f\|_p<\infty,\ \text{ and }\ 0<p<\infty\},$$

$$\mathbb{L}_{p,r}^{\alpha,\beta}=\{f:[-1,1]\longrightarrow\mathbb{R}:f^{(r-1)}\in AC_{loc}(-1,1),1\leq p\leq\infty,\|w_{\alpha,\beta}f^{(r)}\|_p<\infty\},$$

and for convenience use $\mathbb{L}_{p,0}^{\alpha,\beta}=\mathbb{L}_p^{\alpha,\beta}$.

Let $f \in \mathbb{L}_{p,r}^{\alpha,\beta}$, we write $\|\cdot\|_{w_{\alpha,\beta},p}$. If $r = 0$, then we use $\|\cdot\|_{\alpha,\beta,p}$.

Definition 1.2. *[20] Let X be a subset of $\mathbb{R}^n$. A function f is called convex on X if*
$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$, for all $x, y \in X$ and $\lambda \in [0,1]$.

Definition 1.3. *[7] Let $Y_s = \{y_i\}_{i=1}^s$, $s \in \mathbb{N}$ be a partition of $[-1,1]$; that is, a collection of s fixed points y_i such that*

$$y_{s+1} = -1 < y_s < \cdots < y_1 < 1 = y_\circ.$$

Let $\Delta^{(2)}(Y_s)$ be the set of continuous functions on $[-1,1]$ that are convex downwards on the segment $[y_{i+1}, y_i]$ if i is even and convex upwards on the same segment if i is odd. The functions from $\Delta^{(2)}(Y_s)$ are called coconvex.

Definition 1.4. *[21] The partition $\hat{T}_\eta = \{t_j\}_{j=0}^{\eta}$, where*

$$t_j = t_{j,\eta} = \begin{cases} -\cos(\frac{j\Pi}{\eta}) \ ; & \text{if } 0 \leq j \leq \eta, \\ -1 \ ; & \text{if } j < 0, \end{cases}$$

and t_j's as the knots of a Chebyshev partition.

Definition 1.5. *[17] A function f is said to be k-monotone, $k \geq 1$ on $[a,b]$, if and only if for all choices of $k+1$ distinct points $x_\circ, \cdots, x_k$ in $[a,b]$ the inequality $f[x_\circ, \cdots, x_k] \geq 0$, holds, where*

$$f[x_\circ, \cdots, x_k] = \sum_{j=0}^{k} \frac{f(x_j)}{\theta'(x_j)} \ , \quad \theta(x_j) = \prod_{j=0}^{k}(x - x_j)$$

denotes the kth divided difference of f at $x_\circ, \cdots, x_k$.

Now, we present the most important Kopotun's methods and some further developments of his contribution to the kth symmetric difference. He stated that [16] "A is equivalent to B, $A \sim B$, if $c^{-1}A \leq B \leq cA$ such that c is a positive constant". Let us recall:

First, for a piecewise polynomial s on a Chebyshev partition of $[-1,1]$, we have [12]:

$$\omega_{k+\eta}^{\phi}(s,t)_p \leq ct^{\eta}\omega_{k,\eta}^{\phi}(s^{\eta},t)_p \ , \quad 0 < p < 1 \ , \quad t > 0 \ ,$$

and

$$\omega_{k-\eta,\eta}^{\phi}(s^{\eta}, n^{-1})_p \sim \omega_k^{\phi}(s^{(\eta)}, n^{-1})_p \ .$$

Secondly, in 2007, Kopotun dedicated his attention on the computation of several results on the equivalence of moduli of smoothness (see [16], for example):

$$n^{\eta}\omega^{\phi}_{k-\eta}(s^{(\eta)}, n^{-1})_p \sim \omega_k(s, n^{-1})_p \ , \quad 1 \le p \le \infty \ , \quad 1 \le \eta \le \min\{k, m+1\} \ . \tag{1.1}$$

Thirdly, in 2009, Kopotun examined the equivalence [14]:

$$\omega_k(f, \delta)_p \le A c_\delta(k, q, p)\|f\|_p \ ,$$

where f is satisfied (1.1), $q < p$ and

$$c_\delta(k,q,p) = \begin{cases} \delta^{\frac{2}{q}-\frac{2}{p}} \ , & \text{if } k \ge 2 \ , \\ \delta^{\frac{2}{q}-\frac{2}{p}} \ , & \text{if } k = 1 \text{ and } p < 2q \ , \\ (\delta\sqrt{|\ln(\delta)|})^{\frac{1}{q}} \ , & \text{if } k = 1 \text{ and } p = 2q \ , \\ \delta^{\frac{1}{q}} \ , & \text{if } k = 1 \text{ and } p > 2q \ . \end{cases}$$

The first to deal with development moduli of smoothness were Kopotun, Leviatan and Shevshuk [8]. They were interested in discussing various properties of the new modulus of smoothness

$$\omega^{\phi}_{k,r}(f^{(r)}, t)_p = \sup_{0<h\le t} \|W^r_{kh}(\cdot)\Delta^k_{h\phi}(f^{(r)}, \cdot)\|_p \ , \tag{1.2}$$

where

$$W^r_{kh}(x) = \begin{cases} ((1 - x - \delta\frac{\phi(x)}{2})(1 + x - \delta\frac{\phi(x)}{2}))^{\frac{1}{2}} \ ; & \text{if } 1 \pm x - \delta\frac{\phi(x)}{2} \in [-1, 1] \ , \\ 0 \ ; & \text{otherwise.} \end{cases}$$

However, they contributed to the kth symmetric difference of modulus by K-functional [9, proof of Lemma 4.1].

The following result was proven by a different method of modulus of smoothness [11]:

Theorem 1.6. *Let $k, n \in \mathbb{N}$, $r \in \mathbb{N}_\circ$, $A > 0$, $0 < p \le \infty$, $\alpha + \frac{r}{2}, \beta + \frac{r}{2} \in J_p$. Let $0 < t \le \varrho n^{-1}$, where ϱ is some positive constant that depends only on α, β, k and q. Then, for any $p_n \in \pi_n$,*

$$\omega^{\phi}_{k,r}(p^{(r)}_n, t)_{\alpha,\beta,p} \sim \Psi^{\phi}_{k,r}(p^{(r)}_n, t)_{\alpha,\beta,p} \sim \Omega^{\phi}_{k,r}(p^{(r)}_n, A, t)_{\alpha,\beta,p} \sim t^k\|w_{\alpha,\beta}\phi^r p^{k+r}_n\|_p \ ,$$

where

$$\Psi^{\phi}_{k,r}(p^{(r)}_n, t)_{\alpha,\beta,p} = \sup_{0\le h\le t} \|w_{\alpha,\beta}\phi^r\Delta^k_{h\phi}(p^{(r)}_n, x)\|_p \ ,$$

$$\Omega^{\phi}_{k,r}(p_n^{(r)}, A, t)_{\alpha,\beta,p} = \sup_{0\le h\le t} \|w_{\alpha,\beta}\phi^r \Delta^k_{h\phi}(p_n^{(r)}, x; T_{A,h})\|_{L_p(T_{A,h})} ,$$

and the equivalence constants depend only on k, r, α, β, A and q.

Definition 1.7. *[10] For $r \in \mathbb{N}_\circ$ and $0 < p \le \infty$, denote $\mathbb{B}^0_p(w_{\alpha,\beta}) = \mathbb{L}^{\alpha,\beta}_p$ and*

$$\mathbb{B}^r_p(w_{\alpha,\beta}) = \{f : f^{(r-1)} \in AC_{loc}(-1,1), \phi^r f^{(r)} \in \mathbb{L}^{\alpha,\beta}_p\}, r \ge 1.$$

In 2018, Kopotun et al. ([10, Lemmas 2.2, 2.3]) proposed a function $f \in \mathbb{B}^{(r)}_p(w_{\alpha,\beta})$ and $\alpha + \frac{r}{2} \ge 0$, $\beta + \frac{r}{2} \ge 0$. Then,

$$\omega^{\phi}_{k,r}(f^{(r)}, t)_{\alpha,\beta,p} \le c\|w_{\alpha,\beta}\phi^r f^{(r)}\|_p , \quad t > 0,$$

and

$$\lim_{t\to 0^+} \omega^{\phi}_{k,r}(f^{(r)}, t)_{\alpha,\beta,p} = 0.$$

2 Notations and Further Results

In this section, we will present the linear space for functions of Lebesgue Stieltjes integrable-i. First, recall the definition of the Lebesgue Stieltjes integrable$-i$ [3]:

Definition 2.1. *Let $\mathbb{D}$ be a measurable set, $f : \mathbb{D} \longrightarrow \mathbb{R}$ be a bounded function, and $\mathcal{L}_i : \mathbb{D} \longrightarrow \mathbb{R}$ be nondecreasing function for $i \in \Lambda$. For a Lebesgue partition $\mathbf{P}$ of $\mathbb{D}$, put $\underline{\mathrm{LS}}(f,\ \mathbf{P},\ \underline{\mathcal{L}}) = \sum_{j=1}^n \prod_{i\in\Lambda}\ m_j\ \mathcal{L}_i(\mu(\mathbb{D}_j))$ and $\overline{\mathrm{LS}}(f,\ \mathbf{P},\ \underline{\mathcal{L}}) = \sum_{j=1}^n \prod_{i\in\Lambda}\ M_j\ \mathcal{L}_i(\mu(\mathbb{D}_j))$ where μ is a measure function of $\mathbb{D}$, $m_j = \inf\{f(x) :\ x \in \mathbb{D}_j\}$, $M_j = \sup\{f(x) :\ x \in \mathbb{D}_j\}$, and $\underline{\mathcal{L}} = \mathcal{L}_1,\ \mathcal{L}_2,\ \cdots$. Also, $\mathcal{L}_i(x_j) - \mathcal{L}_i(x_{j-1}) > 0$, $\underline{\mathrm{LS}}(f,\ \mathbf{P},\ \underline{\mathcal{L}}) \le \overline{\mathrm{LS}}(f,\ \mathbf{P},\ \underline{\mathcal{L}})$, $\prod_{i\in\Lambda} \underline{\int}_i^{\mathbb{D}} f\ \mathrm{d}\underline{\mathcal{L}} = \sup\{\underline{\mathrm{LS}}(f,\ \underline{\mathcal{L}})\}$ and $\prod_{i\in\Lambda} \overline{\int}_i^{\mathbb{D}} f\ \mathrm{d}\underline{\mathcal{L}} = \inf\{\overline{\mathrm{LS}}(f,\ \underline{\mathcal{L}})\}$, where $\underline{\mathrm{LS}}(f,\ \underline{\mathcal{L}}) = \{\underline{\mathrm{LS}}(f,\ \mathbf{P},\ \underline{\mathcal{L}}) :\ \mathbf{P}$ part of set $\mathbb{D}\}$ and $\overline{\mathrm{LS}}(f,\ \underline{\mathcal{L}}) = \{\overline{\mathrm{LS}}(f,\ \mathbf{P},\ \underline{\mathcal{L}}) :$ $\mathbf{P}$ part of set $\mathbb{D}\}$. If $\prod_{i\in\Lambda} \underline{\int}_i^{\mathbb{D}} f\ \mathrm{d}\underline{\mathcal{L}} = \prod_{i\in\Lambda} \overline{\int}_i^{\mathbb{D}} f\ \mathrm{d}\underline{\mathcal{L}}$, where $\mathrm{d}\underline{\mathcal{L}} = \mathrm{d}\mathcal{L}_1 \times\ \mathrm{d}\mathcal{L}_2 \times \cdots$, then f is integral $\int_i$ according to $\mathcal{L}_i$ for $i \in \Lambda$.*

Lemma 2.2. *[2] If f is a function of Lebesgue Stieltjes integral-i, then υf is a function of Lebesgue Stieltjes integral-i, where $\upsilon > 0$ is real number, and*

$$\prod_{i\in\Lambda} \int_i^{\mathbb{D}} \upsilon f\ \mathrm{d}\underline{\mathcal{L}} = \upsilon \prod_{i\in\Lambda} \int_i^{\mathbb{D}} f\ \mathrm{d}\underline{\mathcal{L}} ,$$

holds.

Lemma 2.3. *[2] If the functions f_1, f_2 are integrable on the set $\mathbb{D}$ according to $\mathcal{L}_i$, for $i \in \Lambda$, then $f_1 + f_2$ is the function of integrable according to $\mathcal{L}_i$, for $i \in \Lambda$, such that*

$$\prod_{i\in\Lambda} \underline{\int}_i^{\mathbb{D}} (f_1 + f_2)\, \underline{\mathrm{d}\mathcal{L}} = \prod_{i\in\Lambda} \underline{\int}_i^{\mathbb{D}} f_1\, \underline{\mathrm{d}\mathcal{L}} + \prod_{i\in\Lambda} \underline{\int}_i^{\mathbb{D}} f_2\, \underline{\mathrm{d}\mathcal{L}}.$$

Definition 2.4. *[4] A domain $\mathbb{D}$ of convex polynomial p_n of $\Delta^{(2)}$ is a subset of $X \subseteq \mathbb{R}$, satisfying the following properties:*

1. $\mathbb{D} \in \mathbb{K}^N$ *, where*
   $\mathbb{K}^N = \{\mathbb{D} : \mathbb{D}$ *is a compact subset of* $X\}$
   is the class of all domains of convex polynomials,
2. *there is $t \in X/\mathbb{D}$ such that*
   $|p_n(t)| > \sup\{|p_n(x)| : x \in \mathbb{D}\}$*, and*
3. *there is the function f of $\Delta^{(2)}$, such that*
   $\|f - p_n\| \leq \frac{c}{n^2}\, \omega^{\phi}_{2,2}(f'', \frac{1}{2})$.

Definition 2.5. *[4] A domain $\mathbb{D}$ of coconvex polynomial p_n of $\Delta^{(2)}(Y_s)$ is a subset of X and $X \subseteq \mathbb{R}$, satisfying the following properties:*

1. $\mathbb{D} \in \mathbb{K}^N(Y_s)$ *, where*
   $\mathbb{K}^N(Y_s) = \{\mathbb{D} : \mathbb{D}$ *is a compact subset of X, and p_n changes convexity at* $\mathbb{D}$ $\}$
   is the class of all domains of coconvex polynomials,
2. *y_i's are inflection points such that*
   $|p_n(y_i)| \leq \frac{1}{2},\ i = 1, ..., s,$ *and*
3. *there is the function f of $\Delta^{(2)}(Y_s)$ such that*
   $\|f - p_n\| \leq \frac{c}{n^2}\, \omega^{\phi}_{k,2}(f'', \frac{1}{n})$.

From Definitions 2.1, 2.4 and 2.5, if the function f is convex, then $\mathbb{D}$ is the domain of (co)convex functions of f.

Remark 2.6. *[2] Let I_f be the class of all functions of integrable f satisfying Definition 2.1; i.e.,*

$$\begin{aligned} I_f &= \{f : f \text{ is an integrable function according to } \mathcal{L}_i,\ i \in \Lambda\} \\ &= \{f : \prod_{i\in\Lambda} \underline{\int}_i^{\mathbb{D}} f\, \underline{\mathrm{d}\mathcal{L}} = \prod_{i\in\Lambda} \overline{\int}_i^{\mathbb{D}} f\, \underline{\mathrm{d}\mathcal{L}}\}. \end{aligned}$$

Remark 2.7. *[13] Let* $x_i \in [\frac{x_i+x^{\#}}{2}, \frac{x_i+x^{*}}{2}] \subseteq \theta_{\mathcal{N}}$*. Let*

$$x^{\#} = x_{j(i)+1},\ x_{*} = x_{j(i)-2},$$

where $\theta_{\mathcal{N}} = \theta_{\mathcal{N}}[-1,1] = \{x_i\}_{i=0}^{\mathcal{N}} = \{-1 = x_{\circ} \leq \cdots \leq x_{\mathcal{N}-1} \leq x_{\mathcal{N}} = 1\}$ *and* $\|\theta_{\mathcal{N}}\| = \max_{0\leq i\leq \mathcal{N}-1}\{x_{i+1} - x_i\}$ *is the length of the largest interval in that partition.*

Definition 2.8. *For* $r \in \mathbb{N}_{\circ}$*, the weighted DTMS in* $\mathbb{L}_p^{\alpha,\beta} \cap I_f$*, we define*

$$\Delta_h^i(f,\ x) = \begin{cases} \prod_{i\in\Lambda} \int_i^{\mathbb{D}} f\ \underline{\mathrm{d}\mathcal{L}}\ ; & \textit{if } f \in I_f\ , \\ 0\ ; & \textit{otherwise.} \end{cases} \tag{2.1}$$

By virtue of (2.1) and Definition 1.1, we define

$$\omega_{i,r}^{\phi}\ (f^{(r)},\ \|\theta_N\|, [-1,1])_{w_{\alpha,\beta},p} = \sup\ \{\|w_{\alpha,\beta}\ \phi^r \Delta_{h\phi}^i(f^{(r)},\ x)\|_p\ ,\ 0 < h \leq \|\theta_N\|\},$$

where $\|\theta_N\| < 2(i^{-1}),\ \mathcal{N} \geq 2$.

Definition 2.9. *For* $\alpha, \beta \in J_p$ *,* $r \in \mathbb{N}_{\circ}$ *and* $0 < p \leq \infty$*, we denote*

$$\Phi^{p,r}(w_{\alpha,\beta}) = \{f : f \in \mathbb{L}_{p,r}^{\alpha,\beta} \cap I_f\ \ \textit{and}\ \ \omega_{i,r}^{\phi}\ (f^{(r)},\ \|\theta_N\|, [-1,1])_{w_{\alpha,\beta},p} < \infty\}\ ,$$

and $\Phi^{p,0}(w_{\alpha,\beta}) = \Phi^{p}(w_{\alpha,\beta})$.

We focus on the applications of results that were obtained in [2, Theorems 3.1, 3.3] and [1, Theorem 2.11].

A set of all piecewise polynomial approximation $\mathbb{S}(\hat{T}_\eta, r+2)$ of order $r+2$, with the knots of a Chebyshev partition $\hat{T}_\eta$.

Theorem 2.10. *[2] For* $r \in \mathbb{N}_{\circ}$*,* $\alpha,\ \beta \in J_p$*, there is a constant* $c = c(r, \alpha, \beta, p)$ *such that if* $f \in \Delta^{(2)} \cap \mathbb{L}_{p,r}^{\alpha,\beta}$*, there is a number* $\mathcal{N} = \mathcal{N}(f, \omega_{1,r}^{\phi}\ (f^{(r)}, \|\theta_{\mathcal{N}}\|, I)_{w_{\alpha,\beta},p})$ *for* $n \geq \mathcal{N}$ *and* $S \in \mathbb{S}(\hat{T}_\eta, r+2) \cap \Delta^{(2)} \cap \mathbb{L}_{p,r}^{\alpha,\beta}$ *such that*

$$\|f^{(r)} - S^{(r)}\|_{w_{\alpha,\beta},p} \leq c_{r,\alpha,\beta,p,\omega_{1,r}^{\phi}} \min\{\omega_{i,r}^{\phi}\ (f^{(r)}, \|\theta_{\mathcal{N}}\|, I_\alpha)_{w_{\alpha,\beta},p}\ ,\ \omega_{i,r}^{\phi}\ (f^{(r)}, \|\theta_{\mathcal{N}}\|, I_\beta)_{w_{\alpha,\beta},p}\},$$

where

$$\Delta_{h\phi,\alpha}^i(f^{(r)},\ x) = \int_1^{\mathbb{D}} \int_2^{\mathbb{D}} \cdots \int_i^{\mathbb{D}} \cdots f^{(r)}\ \mathrm{d}\mathcal{L}_{1\mathrm{t},\alpha}\ \mathrm{d}\mathcal{L}_{2\mathrm{t},\alpha} \cdots \mathrm{d}\mathcal{L}_{\mathrm{it},\alpha} \cdots = \prod_{i\in\Lambda} \int_i^{\mathbb{D}} f^{(r)}\ \underline{\mathrm{d}\mathcal{L}_{\mathrm{t}\phi,\alpha}}\ , \tag{2.2}$$

$$\Delta^i_{h\phi,\beta}(f^{(r)},\ x) = \int_1^{\mathbb{D}} \int_2^{\mathbb{D}} \cdots \int_i^{\mathbb{D}} \cdots f^{(r)}\ \mathrm{d}\mathcal{L}_{1\mathrm{t},\beta}\ \mathrm{d}\mathcal{L}_{2\mathrm{t},\beta} \cdots \mathrm{d}\mathcal{L}_{i\mathrm{t},\beta} \cdots = \prod_{i\in\Lambda} \int_i^{\mathbb{D}} f^{(r)}\ \underline{\mathrm{d}\mathcal{L}_{\mathrm{t}\phi,\beta}}\ . \tag{2.3}$$

Moreover, if $r, \alpha, \beta = 0$ *, then*

$$\|f - S\|_p \le c(\omega_1^\phi)\ \omega_i^\phi(f, \|\theta_{\mathcal{N}}\|, I)_p\ .$$

In particular,

$$\|f^{(r)} - S^{(r)}\|_{w_{\alpha,\beta},p} \le c_r \omega^\phi_{1,r}\ (f^{(r)}, \|\theta_{\mathcal{N}}\|, I)_{w_{\alpha,\beta},p}\ .$$

Theorem 2.11. *[2] Let* Δ^k *be the space of all* k*-monotone functions. If* $f \in \Delta^k \cap \mathbb{L}^{\alpha,\beta}_{p,r}$ *is such that* $f^{(r)}(x) = p_n^{(r)}(x)$*, where* $p_n \in \pi_n \cap \Delta^k$*,* $N \ge k \ge 2$ *and* $s \in \mathbb{S}(\hat{T}_\eta, r+2) \cap \Delta^k \cap \mathbb{L}^{\alpha,\beta}_{p,r}$*, then*

$$\|f - s\|_{w_{\alpha,\beta},p} \le c(f, p, k, \alpha, \beta, x_*, x^\#)\omega^\phi_{i,r}(f, \|\theta_{\mathcal{N}}\|, I)_{w_{\alpha,\beta},p}\ .$$

In particular, if f *is a convex function and* p_n *is a convex polynomial or a piecewise convex polynomial, then*

$$\|f - s\|_{w_{\alpha,\beta},p} \le c_k\ \omega^\phi_{i,r}(f, \|\theta_{\mathcal{N}}\|, I)_{w_{\alpha,\beta},p}\ .$$

Definition 2.12. *[1] For* $\alpha, \beta \in J_p$ *and* $f \in I_f$ *, we set*

$$\mathbb{E}_n(f, w_{\alpha,\beta})_{\alpha,\beta,p} = \mathbb{E}_n(f)_{\alpha,\beta,p} = \inf\{\|f - p_n\|_{\alpha,\beta,p}\ ,\quad p_n \in \pi_n \cap I_f\ ,\quad f \in \Delta^{(2)}(Y_s) \cap \Phi^p(w_{\alpha,\beta})\}$$

and

$$\mathcal{E}^{(2)}_n(f, w_{\alpha,\beta}, Y_s)_p = \inf\{\|f - p_n\|_{\alpha,\beta,p}\ ,\quad p_n \in \pi_n \cap \Delta^{(2)}(Y_s) \cap I_f\ ,\quad f \in \Delta^{(2)}(Y_s) \cap \Phi^p(w_{\alpha,\beta})\}$$

respectively which denote the degree of best unconstrained and (co)convex polynomial approximation of f*.*

Theorem 2.13. *[1] Let* $\sigma, m, n \in \mathbb{N}$*,* $\sigma \ne 4$*,* $s \in \mathbb{N}_\circ$ *and* $\alpha, \beta \in J_p$*. If* $f \in \Delta^{(2)}(Y_s) \cap \Phi^p(w_{\alpha,\beta})$*, then*

$$\sup\{n^\sigma \mathcal{E}^{(2)}_n(f, w_{\alpha,\beta}, Y_s)_p : n \ge m\} \le c \sup\{n^\sigma\ \mathbb{E}_n(f)_{\alpha,\beta,p} : n \in \mathbb{N}\}. \tag{2.4}$$

In particular, suppose that $Y_s \in \mathbb{Y}_s$ *and* $s \ge 1$*. Then*

$$\mathcal{E}^{(2)}_n(f, w_{\alpha,\beta}, Y_s)_p \le c\, n^{-\sigma}\ \omega^\phi_{i,r}\ (f^{(r)},\ \|\theta_N\|, I)_{w_{\alpha,\beta},p}\ ,\quad n \ge \|\theta_N\|.$$

Remark 2.14. *If f in I_f is a function of Lebesgue Stieltjes integral $-i$ and f is a differentiable function, then*

$$f' = \frac{df}{dx} = \frac{d}{dx}(\int_0^x \frac{df(u)}{d\ell_{1,\mu,\mathbb{D}_\circ}} \times d\ell_{1,\mu,\mathbb{D}_\circ})$$

$$= \frac{d}{dx}(\int_0^x \int_0^x \frac{d^2 f(u)}{d\ell_{1,\mu,\mathbb{D}_\circ} \times d\ell_{2,\mu,\mathbb{D}_\circ}} \times d\ell_{1,\mu,\mathbb{D}_\circ} \times d\ell_{2,\mu,\mathbb{D}_\circ})$$

$$= \frac{d}{dx}(\int_0^x \int_0^x \cdots \int_0^x \cdots \frac{d^i f(u)}{d\ell_{1,\mu,\mathbb{D}_\circ} \times d\ell_{2,\mu,\mathbb{D}_\circ} \times \cdots \times d\ell_{i,\mu,\mathbb{D}_\circ} \times \cdots} \times d\ell_{1,\mu,\mathbb{D}_\circ} \times d\ell_{2,\mu,\mathbb{D}_\circ} \times \cdots \times d\ell_{i,\mu,\mathbb{D}_\circ} \times \cdots)$$

$$= \frac{d}{dx}(\prod_{i\in\Lambda} \int_i^{I_x} f^{(i)}(u)\, \underline{\mathrm{d}\ell_{\mu,\mathbb{D}_\circ}})\ , \quad x \in I_x = [0,x] \subseteq \mathbb{D}_\circ\ , \quad u \in \mathbb{D}_\circ\ , \quad \textit{and} \quad \ell_{\mu,\mathbb{D}_\circ} = \ell(\mu(\mathbb{D}_\circ))$$

$$= \prod_{i\in\Lambda} \int_i^{I_x} f_x^{(i+1)}\, \underline{\mathrm{d}\ell_{\mu,\mathbb{D}_\circ}}.$$

$$f_x^{(i+1)} = \frac{d^i}{d\ell^i_{i,\mu,\mathbb{D}_\circ}}\, f' = \frac{d}{dx}(\frac{d^i f}{d\ell^i_{i,\mu,\mathbb{D}_\circ}})$$

$$= \frac{d^{i+1} f_x}{dx \times d\ell^i_{i,\mu,\mathbb{D}_\circ}}.$$

Lemma 2.15. *We have*

$$\Phi^{p,r+1}(w_{\alpha,\beta}) = \Phi^{p,r}(w_{\alpha+\frac{1}{2},\beta+\frac{1}{2}})\ .$$

Proof. First, suppose $1 \le p < \infty$, and $w_{\alpha,\beta}(x) = (1+x)^\alpha(1-x)^\beta$. Let $f \in \Phi^{p,r+1}(w_{\alpha,\beta})$ and assume f satisfies Definition 2.8. Next,

$$\|w_{\alpha,\beta}\phi^{r+1}\Delta^i_{h\phi}(f^{(r+1)},x)\|_p = (\int_{-1}^1 |w_{\alpha,\beta}\phi^{r+1}\Delta^i_{h\phi}(f^{(r+1)},x)|^p dx)^{\frac{1}{p}}\ , \quad 0 < h \le \|\theta_{\mathcal{N}}\|$$

$$= (\int_{-1}^1 |w_{\alpha,\beta}\phi^{r+1} \prod_{i\in\Lambda} \int_i^{\mathbb{D}} f^{(r+1)}\, \underline{\mathrm{d}\mathcal{L}_\phi}|^p dx)^{\frac{1}{p}}\ .$$

Next, from [2, proof of Lemma 3.2], [18] and Remark 2.14, we have

$$\|w_{\alpha,\beta}\phi^{r+1}\Delta^i_{h\phi}(f^{(r+1)},x)\|_p = (\int_{-1}^1 |w_{\alpha+\frac{1}{2},\beta+\frac{1}{2}}\phi^r \prod_{i\in\Lambda} \int_{i+1}^{\mathbb{D}} f^{(r)}\, \underline{\mathrm{d}\mathcal{L}_\phi}|^p dx)^{\frac{1}{p}}$$

$$= \|w_{\alpha+\frac{1}{2},\beta+\frac{1}{2}}\phi^r\Delta^{i+1}_{h\phi}(f^{(r)},x)\|_p\ , \quad 0 < h \le \|\theta_{\mathcal{N}}\|\ .$$

□

Remark 2.16. *By virtue of Lemma 2.15, we immediately get*

$$\omega^\phi_{i,r+1}\,(f^{(r+1)},\ \|\theta_N\|)_{w_{\alpha,\beta},p} = \omega^\phi_{i+1,r}\,(f^{(r)},\ \|\theta_N\|)_{w_{\alpha+\frac{1}{2},\beta+\frac{1}{2}},p}\ .$$

3 Main Results for Weighted DT Moduli

Theorem 3.1. *Let $s, r \in \mathbb{N}_\circ$, $0 < p \leq \infty$ and $f \in \Delta^{(2)}(Y_s) \cap \Phi^{p,r}(w_{\alpha,\beta})$. Let $\mathbb{D}$ be defined in Definition 2.1 such that $|\mathbb{D}| \leq \delta_\circ$, for some $\delta_\circ \in \mathbb{R}^+$. Then*

$$\omega^{\phi}_{i+1,r}\ (f^{(r)},\ \|\theta_N\|)_{w_{\alpha,\beta},p} \leq c(\delta_\circ)\omega^{\phi}_{i,r+1}\ (f^{(r+1)},\ \|\theta_N\|)_{w_{\alpha,\beta},p}, \tag{3.1}$$

where the constant c depends on $\delta_\circ$.

Proof. Suppose that $f \in \Delta^{(2)}(Y_s) \cap \Phi^{p,r}(w_{\alpha,\beta})$,

$$\oint_i^{\mathbb{D}_k \cap \mathbb{D}_j} = \prod_{i\in\Lambda} \int_i^{\mathbb{D}_k \cap \mathbb{D}_j} f\ \underline{\mathrm{d}\mathcal{L}_\phi}$$

and

$$\oint_i^{\mathbb{D} \cap \mathbb{D}_j} = \prod_{i\in\Lambda} \int_i^{\mathbb{D} \cap \mathbb{D}_j} f\ \underline{\mathrm{d}\mathcal{L}_\phi}\ .$$

In addition, assume that $\mathbb{D}_j \subset \mathbb{D}$ such that

$$f(x) = \begin{cases} |\mathbb{D}|\ , & \text{if } |\mathbb{D}| \leq \delta_\circ\ , \\ (\oint_i^{\mathbb{D}_k \cap \mathbb{D}_j}) \to (\oint_i^{\mathbb{D} \cap \mathbb{D}_j})\ , & \text{if } \mathbb{D}_k, \mathbb{D}_j \text{ are Lebesgue measurable sets,} \\ 0\ , & \text{otherwise.} \end{cases} \tag{3.2}$$

Then

$$\|w_{\alpha,\beta}\phi^r \Delta^{i+1}_{h\phi}(f^{(r)}, x)\|_p = (\int_{-1}^{1} |w_{\alpha,\beta}\phi^r \prod_{i\in\Lambda} \int_{i+1}^{\mathbb{D}} f^{(r)}\ \underline{\mathrm{d}\mathcal{L}_\phi}|^p dx)^{\frac{1}{p}}$$

$$= \begin{cases} (\int_{-1}^{1} |w_{\alpha,\beta}\phi^r \prod_{i\in\Lambda} \int_{i+1}^{\mathbb{D}} |\mathbb{D}|\ \underline{\mathrm{d}\mathcal{L}_\phi}|^p dx)^{\frac{1}{p}} = I_\circ(x) & \text{if } |\mathbb{D}| \leq \delta_\circ\ ,\ \delta_\circ \in \mathbb{R}^+ \\ (\int_{-1}^{1} |w_{\alpha,\beta}\phi^r \lim_{k\to\infty} \prod_{i\in\Lambda} \int_{i+1}^{\mathbb{D}_k \cap \mathbb{D}_j} f^{(r)}\ \underline{\mathrm{d}\mathcal{L}_\phi}|^p dx)^{\frac{1}{p}} = I_1(x), & \text{if } \mathbb{D}_k, \mathbb{D}_j \text{ are Lebesgue measurable} \\ 0\ , & \text{otherwise.} \end{cases}$$

Therefore, (3.2) implies that

$$I_\circ(x) = (\int_{-1}^{1} |w_{\alpha,\beta}\phi^r \prod_{i\in\Lambda} \int_{i+1}^{\mathbb{D}} |\mathbb{D}|\ \underline{\mathrm{d}\mathcal{L}_\phi}|^p dx)^{\frac{1}{p}} \leq \delta_\circ\ ,$$

for some $\delta_\circ \in \mathbb{R}^+$, while

$$I_1(x) = (\int_{-1}^{1} |w_{\alpha,\beta}\phi^r \lim_{k\to\infty} \prod_{i\in\Lambda} \int_{i+1}^{\mathbb{D}_k \cap \mathbb{D}_j} f^{(r)}\ \underline{\mathrm{d}\mathcal{L}_\phi}|^p dx)^{\frac{1}{p}}$$

$$= \left(\int_{-1}^{1} |w_{\alpha,\beta}\phi^r \lim_{k\to\infty} LS(f^{(r)}, \underline{\mathcal{L}_\phi(\mu(\mathbb{D}_{\mathrm{k}} \cap \mathbb{D}_{\mathrm{j}}))})|^p dx\right)^{\frac{1}{p}}$$

$$= \left(\int_{-1}^{1} |w_{\alpha,\beta}\phi^r LS(f^{(r)}, \underline{\mathcal{L}_\phi(\lim_{\mathrm{k}\to\infty} \mu(\mathbb{D}_{\mathrm{k}} \cap \mathbb{D}_{\mathrm{j}}))})|^p dx\right)^{\frac{1}{p}}$$

$$= \left(\int_{-1}^{1} |w_{\alpha,\beta}\phi^r LS(f^{(r)}, \underline{\mathcal{L}_\phi(\mu((\cup_{\mathrm{k}=1}^{\infty}\mathbb{D}_{\mathrm{k}}) \cap \mathbb{D}_{\mathrm{j}}))})|^p dx\right)^{\frac{1}{p}}$$

$$= \left(\int_{-1}^{1} |w_{\alpha,\beta}\phi^r LS(f^{(r)}, \underline{\mathcal{L}_\phi(\mu(\mathbb{D} \cap \mathbb{D}_{\mathrm{j}}))})|^p dx\right)^{\frac{1}{p}}$$

$$= \left(\int_{-1}^{1} |w_{\alpha,\beta}\phi^r \prod_{i\in\Lambda} \int_{i+1}^{\mathbb{D}\cap\mathbb{D}_j} f^{(r)} \ \underline{\mathrm{d}\mathcal{L}_\phi}|^p dx\right)^{\frac{1}{p}} .$$

By Remark 2.14, we have

$$I_1(x) \leq c\left(\int_{-1}^{1} |w_{\alpha,\beta}\phi^r \prod_{i\in\Lambda} \int_{i}^{\mathbb{D}\cap\mathbb{D}_j} f^{(r+1)} \ \underline{\mathrm{d}\mathcal{L}_\phi}|^p dx\right)^{\frac{1}{p}} .$$

Taking supremum, we obtain (3.1). □

The following corollary is clear.

Corollary 3.2. *Let $s, r \in \mathbb{N}_\circ$, $0 < p \leq \infty$ and $f \in \Delta^{(2)}(Y_s) \cap \Phi^{p,r}(w_{\alpha,\beta})$. Let $\mathbb{D}$ and $\delta_\circ$ be defined in Theorem 3.1. Then*

$$\omega^{\phi}_{i+1,r}\ (f^{(r)},\ \|\theta_N\|)_{w_{\alpha,\beta},p} \leq c(\delta_\circ)\omega^{\phi}_{i+1,r}\ (f^{(r)},\ \|\theta_N\|)_{w_{\alpha+\frac{1}{2},\beta+\frac{1}{2}},p},$$

where the constant c depends on $\delta_\circ$.

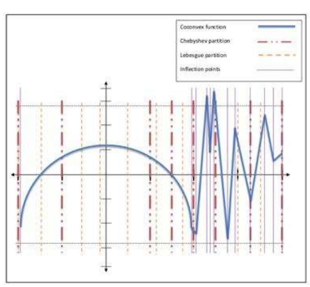

Figure 1: Graph of partitions of the coconvex function on the interval $[-1, 2]$
.

Theorem 3.3. *Let* $s, r \in \mathbb{N}_\circ$, $\alpha, \beta \in J_p$ *and* $0 < p \leq \infty$. *Let* $\mathbf{P}$ *be a Lebesgue partition of* $\mathbb{D}$ *and* $\hat{T}_\eta$ *be a Chebyshev partition with* $\mathbf{P} \cap \hat{T}_\eta \neq \emptyset$, $1 \leq \eta \leq r$. *If* $f \in \Delta^{(2)}(Y_s) \cap \Phi^{p,r}(w_{\alpha,\beta})$, *then there is a constant* c *depending on* η *and* $J_{j,\eta}$ *such that*

$$\omega^{\phi}_{i+\eta}\,(f,\ \|\theta_N\|)_{w_{\alpha,\beta},p} \leq c\|\theta_N\|^{-\eta}\omega^{\phi}_{i,2\eta}\,(f^{(2\eta)},\ \|\theta_N\|)_{w_{\alpha+\eta,\beta+\eta},p}. \tag{3.3}$$

Proof. Recall that $\mathbf{P}$ is a Lebesgue partition of $\mathbb{D}$ and $\hat{T}_\eta$ is a Chebyshev partition. Since $\mathbf{P} \cap \hat{T}_\eta \neq \emptyset$, by virtue of [2, proof of Lemma 2.3], for $\varepsilon > 0$, there is a Lebesgue partition $\mathbf{P}_\varepsilon$ of $\mathbb{D}$ such that $\hat{T}_\eta$ such that $\mathbf{P}_\varepsilon \cup \hat{T}_\eta = \mathbf{P}$. We can construct $J_{j,\eta} = [u_{j-(\eta+i)}, u_{j-(\eta+i)+1}]$ for some $y_i \in \cup_{j=0}^{\eta} J_{j,\eta}$ and y_is inflection points of Y_s, $s \in \mathbb{N}_\circ$, (see Figure 1). Next, if $f \in \Delta^{(2)}(Y_s) \cap \Phi^{p,r}(w_{\alpha,\beta})$, then

$$\omega^{\phi}_{i+\eta}\,(f,\ \|\theta_N\|)^p_{w_{\alpha,\beta},p} = \sup\{\|w_{\alpha,\beta}\phi^r\Delta^{i+\eta}_{h\phi}(f,x)\|^p_p\ ,\quad 0 < h \leq \|\theta_N\|\}$$

$$\leq c\sup\{\sum_{j=0}^{\eta}\|w_{\alpha,\beta}\phi^r\Delta^{i+\eta}_{h\phi}(f - f^{(\eta)} + f^{(\eta)}, x)\|^p_{L_p(J_{j,\eta})}\ ,\quad 0 < h \leq \|\theta_N\|\}\ .$$

By virtue of [19] and Theorem 2.13 or (see [1, proof of Theorem 2.11], we have

$$\|\theta_N\|^\eta\omega^{\phi}_{i+\eta}\,(f,\ \|\theta_N\|)^p_{w_{\alpha,\beta},p} \leq c(\sum_{j=0}^{\eta}(\int_{-1}^{1}|w_{\alpha,\beta}\phi^r(\prod_{i\in\Lambda}\int_{i+\eta}^{J_{j,\eta}}(f - f^{(\eta)} + f^{(\eta)})\ \underline{\mathrm{d}\mathcal{L}_\phi})|^p dx))$$

$$\leq c(\sum_{j=0}^{\eta}(\int_{-1}^{1}|w_{\alpha,\beta}\phi^r(\prod_{i\in\Lambda}\int_{i+\eta}^{J_{j,\eta}}(f - f^{(\eta)})\ \underline{\mathrm{d}\mathcal{L}_\phi} + \prod_{i\in\Lambda}\int_{i+\eta}^{J_{j,\eta}} f^{(\eta)}\ \underline{\mathrm{d}\mathcal{L}_\phi})|^p dx))$$

$$\leq c\sup\{\sum_{j=0}^{\eta}(\|w_{\alpha,\beta}\phi^r\Delta^{i+\eta}_{h\phi}(f - f^{(\eta)}, x)\|^p_{L_p(J_{j,\eta})} + \|w_{\alpha,\beta}\phi^r\Delta^{i+\eta}_{h\phi}(f^{(\eta)}, x)\|^p_{L_p(J_{j,\eta})})\ ,\quad 0 < h \leq \|\theta_N\|\}$$

$$\leq c(\sup\{\sum_{j=0}^{\eta}(\|w_{\alpha,\beta}\phi^r\Delta^{i+\eta}_{h\phi}(f - f^{(\eta)}, x)\|^p_{L_p(J_{j,\eta})}\ ,\quad 0 < h \leq \|\theta_N\|\}$$

$$+\sup\{\sum_{j=0}^{\eta}(\|w_{\alpha,\beta}\phi^r\Delta^{i+\eta}_{h\phi}(f^{(\eta)}, x)\|^p_{L_p(J_{j,\eta})})\ ,\quad 0 < h \leq \|\theta_N\|\})$$

$$\leq c(\eta, J_{j,\eta}) \times \sup\{\sum_{j=0}^{\eta}\|w_{\alpha,\beta}\phi^r\Delta^{i+\eta}_{h\phi}(f^{(\eta)}, x)\|^p_{L_p(J_{j,\eta})}\ ,\quad 0 < h \leq \|\theta_N\|\}$$

$$\leq c(\eta, J_{j,\eta})\omega^{\phi}_{i+\eta,\eta}\ (f^{(\eta)},\ \|\theta_N\|)^p_{w_{\alpha,\beta},p}\ .$$

Now (3.3) foloows from (3.1). □

The following is an immediate of consequence of Theorems 3.1 and 3.3.

Corollary 3.4. *Let* $s, r \in \mathbb{N}_\circ$, $\alpha, \beta \in J_p$, $0 < p \leq \infty$ *and* $f \in \Delta^{(2)}(Y_s) \cap \Phi^{p,r}(w_{\alpha,\beta})$. *Let* $\mathbf{P}$ *be a Lebesgue partition of* $\mathbb{D}$, *and* $\hat{T}_\eta$ *be a Chebyshev partition with* $\mathbf{P} \cap \hat{T}_\eta \neq \emptyset$, $1 \leq \eta \leq r$. *We have*

$$\|w_{\alpha,\beta}\phi^\eta f^{(\eta)}\|_p \geq c(\eta, J_{j,\eta}) \begin{cases} \omega^{\phi}_{i+2\eta,i+\eta}\ (f^{(i+\eta)},\ \|\theta_N\|)_{w_{\alpha,\beta},p}\ , & \textit{if}\ |\mathbb{D}| \leq c(\eta, J_{j,\eta})\ , \\ \omega^{\phi}_{i,i+2\eta}\ (f^{(i+2\eta)},\ \|\theta_N\|)_{w_{\alpha+\frac{\eta}{2},\beta+\frac{\eta}{2}},p}\ , & \textit{if}\ |\mathbb{D}| > c(\eta, J_{j,\eta})\ . \end{cases} \tag{3.4}$$

Proof. Let $s, r \in \mathbb{N}_\circ$, $1 \leq \eta \leq r$, $\phi(x) = \sqrt{1-x^2}$ and $J_{j,\eta} = [u_{j-(\eta+i)}, u_{j-(\eta+i)+1}]$. Let $\mathbf{P}$ be a Lebesgue partition of $\mathbb{D}$, and $\hat{T}_\eta$ be a Chebyshev partition. Assume the function $f \in \Delta^{(2)}(Y_s) \cap \Phi^{p,r}(w_{\alpha,\beta})$ and the constant c depends on ϕ, r and η. Then

$$c \times \|w_{\alpha,\beta}\phi^\eta f^{(\eta)}\|_p^p \geq \|w_{\alpha,\beta}\phi^\eta f^{(\eta)}\|_p^p$$

$$\geq \|w_{\alpha,\beta}\phi^\eta(\frac{\phi^r}{\phi^r})f^{(\eta)}\|_p^p \geq c(\phi^r, \phi^\eta)^{-1}\|w_{\alpha,\beta}\phi^r f^{(\eta)}\|_p^p$$

$$\geq c(\phi^r, \phi^\eta)^{-1}\|w_{\alpha,\beta}\phi^r \prod_{i\in\Lambda, 1\leq\eta\leq r} \int_{i+\eta}^{J^x_{j,\eta}} f^{((i+\eta)+\eta)}\ \underline{\mathrm{d}\mathcal{L}_\phi}\|_p^p$$

$$\geq c(\phi^r, \phi^\eta)^{-1}\|w_{\alpha,\beta}\phi^r \prod_{i\in\Lambda, 1\leq\eta\leq r} \int_{i+\eta}^{J^x_{j,\eta}} f^{(i+2\eta)}\ \underline{\mathrm{d}\mathcal{L}_\phi}\|_p^p$$

$$\geq c(\eta, J_{j,\eta}) \sum_{j=0}^{\eta} \sup \|w_{\alpha,\beta}\phi^r \prod_{i\in\Lambda, 1\leq\eta\leq r} \int_{i+\eta}^{J^x_{j,\eta}} f^{(i+2\eta)} \mathrm{d}\mathcal{L}_\phi\|^{\mathrm{p}}_{\mathrm{L_p(J_{j,\eta})}}$$

$$\geq c(\eta, J_{j,\eta}) \sup\{\sum_{j=0}^{\eta} \|w_{\alpha,\beta}\phi^r \Delta^{i+\eta}_{h\phi}(f^{(i+2\eta)}, x)\|^p_{L_p(J_{j,\eta})}\ , \quad 0 < h \leq \|\theta_N\|\}$$

$$\geq c(\eta, J_{j,\eta})\omega^{\phi}_{i+\eta,i+2\eta}\ (f^{(i+2\eta)},\ \|\theta_N\|)^p_{w_{\alpha,\beta},p}\ .$$

Finally, by virtue of Theorems 3.1 and 3.3, we have

$$\|w_{\alpha,\beta}\phi^\eta f^{(\eta)}\|_p^p \geq c(\eta, J_{j,\eta}) \begin{cases} \omega^{\phi}_{i+2\eta,i+\eta}(f^{(i+\eta)}, \|\theta_N\|)^p_{w_{\alpha,\beta},p}, \text{ if } |\mathbb{D}| \leq c(\eta, J_{j,\eta})\ , \\ \omega^{\phi}_{i,i+2\eta}(f^{(i+2\eta)}, \|\theta_N\|)^p_{w_{\alpha+\frac{\eta}{2},\beta+\frac{\eta}{2}},p}, \text{ if } |\mathbb{D}| > c(\eta, J_{j,\eta}) \end{cases}$$

□

4 Conclusions and Direct Estimates

Theorem 4.1. *Assume that* $s, r \in \mathbb{N}_\circ$, $\alpha, \beta \in J_p$, $0 < p \leq \infty$ *and* $f \in \Delta^{(2)}(Y_s) \cap \Phi^{p,r}(w_{\alpha,\beta})$. *If* $\mathbf{P}$ *is a Lebesgue partition of* $\mathbb{D}$, *and* $\hat{T}_\eta$ *is a Chebyshev partition with* $\mathbf{P} \cap \hat{T}_\eta \neq \emptyset$. *Then, for any constant* c *depending on* η, $J_{j,\eta}$, *and* $|\mathbb{D}| \leq \delta_\circ$, *we have*

$$\omega^{\phi}_{i+1,r}\,(f^{(r)},\ \|\theta_N\|)_{w_{\alpha,\beta},p} \sim c(\delta_\circ)\omega^{\phi}_{i,r+1}\,(f^{(r+1)},\ \|\theta_N\|)_{w_{\alpha,\beta},p} \sim$$

$$c(\delta_\circ) \times \omega^{\phi}_{i+1,r}\,(f^{(r)},\ \|\theta_N\|)_{w_{\alpha+\frac{1}{2},\beta+\frac{1}{2}},p} \sim \|w_{\alpha,\beta}\phi^\eta f^{(\eta)}\|_p \sim$$

$$c(\eta, J_{j,\eta})\{\omega^{\phi}_{i+2\eta,i+\eta}(f^{(i+\eta)}, \|\theta_N\|)_{w_{\alpha,\beta,p}} : |\mathbb{D}| \leq c(\eta, J_{j,\eta})\}$$

and

$$\|\theta_N\|^\eta \times \omega^{\phi}_{i+\eta}\,(f,\ \|\theta_N\|)_{w_{\alpha,\beta},p} \sim c(\eta, J_{j,\eta})\omega^{\phi}_{i,2\eta}\,(f^{(2\eta)},\ \|\theta_N\|)_{w_{\alpha+\eta,\beta+\eta},p} \sim$$

$$\|w_{\alpha,\beta}\phi^\eta f^{(\eta)}\|_p \sim c(\eta, J_{j,\eta})\{\omega^{\phi}_{i,i+2\eta}(f^{(i+2\eta)}, \|\theta_N\|)^p_{w_{\alpha+\frac{\eta}{2},\beta+\frac{\eta}{2}},p} : |\mathbb{D}| > c(\eta, J_{j,\eta})\}.$$

Corollary 4.2. *(*$s = 0$*) For* $r \in \mathbb{N}_\circ$ *and* $\alpha, \beta \in J_p$, *there is a constant* c *depending on* $r, \alpha, \beta, p, \omega^{\phi}_{1,r}$ *and* $r, \alpha, \beta, p, \omega^{\phi}_{1,r}, \eta$ *and* $J_{j,\eta}$ *such that* $f \in \Delta^{(2)}(Y_s) \cap \Phi^{p,r}(w_{\alpha,\beta})$, $J_{j,\eta} = [u_{j-(\eta+i)},\ u_{j-(\eta+i)+1}]$ *and* $1 \leq \eta \leq r$. *Then*

$$\mathcal{E}^{(2)}_n(f, w_{\alpha,\beta}, Y_\circ)_p \leq c\|\theta_N\|^\eta\omega^{\phi}_{i+\eta}\,(f^{(\eta)},\ \|\theta_N\|)_{w_{\alpha,\beta},p}$$

and

$$\mathcal{E}^{(2)}_n(f^{(\eta)}, w_{\alpha,\beta}, Y_\circ)_p \leq c(\eta, J_{j,\eta})\omega^{\phi}_{i,2\eta}\,(f^{(2\eta)},\ \|\theta_N\|)_{w_{\alpha+\eta,\beta+\eta},p}\ .$$

Corollary 4.3. *(*$s \geq 1$*) Suppose that* $Y_s \in \mathbb{Y}_s$, $\sigma, s, n \in \mathbb{N}$ *and* $\sigma \neq 4$. *If* $f \in \Delta^{(2)}(Y_s) \cap \Phi^{p,r}(w_{\alpha,\beta})$, *then*

$$\mathcal{E}^{(2)}_n(f, w_{\alpha,\beta}, Y_s)_p \leq c(\delta_\circ) \times n^{-\sigma}\omega^{\phi}_{i+1,r}\,(f^{(r)},\ \|\theta_N\|)_{w_{\alpha+\frac{1}{2},\beta+\frac{1}{2}},p}$$

and

$$\mathcal{E}^{(2)}_n(f, w_{\alpha,\beta}, Y_s)_p \leq c(\eta, J_{j,\eta}) \times n^{-\sigma}\omega^{\phi}_{i+2\eta,i+\eta}\,(f^{(i+\eta)},\ \|\theta_N\|)_{w_{\alpha,\beta},p}\ .$$

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