International Journal of Mathematics and
Computer Science, **16**(2021), no. 1, 149–157

Three combined sequences related to Jacobsthal sequences

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(Received June 2, 2020, Accepted July 7, 2020)

Abstract

In this paper, we define three combined sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ relate to Jacobsthal sequences.

1 Introduction

The Jacobsthal sequence is an additive sequence similar to the Fibonacci sequence, defined by the recurrence relation $J_n = J_{n-1} + 2J_{n-2}$ with initial terms $J_0 = 0$ and $J_1 = 1$ [1].

Key words and phrases: Jacobsthal sequences, three combined sequences.
AMS (MOS) Subject Classifications: 11B39.
ISSN 1814-0432, 2021, http://ijmcs.future-in-tech.net

In 2018, Atanassov [2] studied two new combined 3-Fibonacci sequences. Later that year, he [3] added two new combined 3-Fibonacci sequences.

In this paper, we generate three combined sequences related to Jacobsthal sequences.

2 Preliminaries

The Jacobsthal sequences is defined by the recurrence relation $J_n = J_{n-1} + 2J_{n-2}$ for $n \geq 2$ with $J_0 = 0$ and $J_1 = 1$.
Its Binet's formula is defined by

$$J_n = \frac{2^n - (-1)^n}{3}.$$

The first thirteen terms of the Jacobsthal sequence J_n are

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	...
J_n	0	1	1	3	5	11	21	43	85	171	341	683	1365	2731	...

The following properties [4] for the Jacobsthal sequences are

(1) $J_n = 2J_{n-1} + (-1)^{n+1}$

(2) $J_n^2 - J_{n-1}^2 = 4(J_{n-1}J_{n-2} + J_{n-2}^2)$

(3) $J_n^2 + 2J_{n-1}^2 = J_{2n-1}$

(4) $J_{n+1}^2 + 2J_n^2 = J_{2n+1}$

(5) $J_{n+1}^2 - 4J_{n-1}^2 = J_{2n}$

(6) $J_n^2 - 4J_{n-1}^2 = (-1)^{n+1}J_{n+1}$.

3 Main Results

Let a, b, c and d be arbitrary real numbers. The first sequence has the form

$$\begin{aligned}\gamma_{n+2} &= \gamma_{n+1} + 2\gamma_n\\ \alpha_{n+1} &= \gamma_{n+1} + 2\beta_n\\ \beta_{n+1} &= \gamma_{n+1} + 2\alpha_n\end{aligned}$$

where $\alpha_0 = a, \beta_0 = b, \gamma_0 = c$ and $\gamma_1 = d$ for integers $n \geq 0$.

The first few members of the sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ with respect to n are in table 1.

Table 1: The first few members of the sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$

n	α_n	γ_n	βn
0		c	
0	a		b
1		d	
1	$d+2b$		$d+2a$
2		$d+2c$	
2	$2c+3d+4a$		$2c+3d+4b$
3		$3d+2c$	
3	$6c+9d+8b$		$6c+9d+8a$
4		$5d+6c$	
4	$18c+23d+16a$		$18c+23d+16b$
5		$11d+10c$	
5	$46c+57d+32b$		$46c+57d+32a$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Theorem 3.1. *For each natural number with the elements of the Jacobsthal sequences.*

(a) $\gamma_n = 2J_{n-1}c + J_n d$

(b) $\alpha_n = 2\alpha_{n-1} + (J_n + (-1)^n)c + J_n d + (-2)^n(a-b)$

(c) $\beta_n = 2\beta_{n-1} + (J_n + (-1)^n)c + J_n d - (-2)^n(a-b)$.

Proof. **(a)** We will prove **(a)** by mathematical induction.
If $n = 1$, then $\gamma_1 = 2J_0 c + J_1 d = d$ thus $n = 1$ is true.

Assume the truth of the statement for some $n-1$ and n; that is,

$$\gamma_{n-1} = 2J_{n-2}c + J_{n-1}d$$

and

$$\gamma_n = 2J_{n-1}c + J_n d.$$

Now consider

$$\begin{aligned}\gamma_{n+1} &= \gamma_n + 2\gamma_{n-1}\\ &= 2J_{n-1}c + J_n d + 2(2J_{n-2}c + J_{n-1}d)\\ &= 2c(J_{n-1} + 2J_{n-2}) + d(J_n + 2J_{n-1})\\ &= 2J_n c + J_{n+1}d,\end{aligned}$$

which is the statement for $n+1$. So, the statement is true for $n = 1$ and its truth for $n-1$ and n implies its truth for $n+1$.
Therefore, it is true for all $n \geq 1$.

(b) We will prove **(b)** by mathematical induction as well.
If $n = 1$ then

$$\begin{aligned}\alpha_1 &= 2\alpha_0 + (J_1 - 1)c + J_1 d - 2(a-b)\\ &= 2a - 2a + 2b + d\\ &= d + 2b,\end{aligned}$$

it is true.
Assume the truth of the statement for some $n-1$ and n; that is,
$\alpha_{n-1} = 2\alpha_{n-2} + (J_{n-1} + (-1)^{n-1})c + J_{n-1}d + (-2)^{n-1}(a-b)$
and
$\alpha_n = 2\alpha_{n-1} + (J_n + (-1)^n)c + J_n d + (-2)^n(a-b)$.
Now consider

$$\begin{aligned}\alpha_{n+1} &= \gamma_{n+1} + 2\beta_n\\ &= (2J_n c + J_{n+1}d) + 2(\gamma_n + 2\alpha_{n-1})\\ &= (2J_n c + J_{n+1}d) + (4J_{n-1}c + 2J_n d)\\ &\quad +2\left[\alpha_n - ((J_n + (-1)^n)c - J_n d - (-2)^n(a-b))\right]\\ &= 2\alpha_n + (4J_{n-1} + 2(-1)^{n+1})c + J_{n+1}d + (-2)^{n+1}(a-b)\\ &= 2\alpha_n + (2(2J_{n-1}) + 2(-1)^{n+1})c + J_{n+1}d + (-2)^{n+1}(a-b)\\ &= 2\alpha_n + (2J_n - 2(-1)^{n+1} + 2(-1)^{n+1})c + J_{n+1}d + (-2)^{n+1}(a-b)\\ &= 2\alpha_n + (J_{n+1} + (-1)^{n+1})c + J_{n+1}d + (-2)^{n+1}(a-b),\end{aligned}$$

which is the statement for $n+1$. So, the statement is true for $n = 1$ and its truth for $n-1$ and n implies its truth for $n+1$. Therefore, it is true for all $n \geq 1$.

(c) The proof of **(c)** is similar to the proof of **(b)**. □

The second sequence has the form

$$\begin{aligned}\gamma_{n+2} &= \gamma_{n+1} + 2\gamma_n \\ \alpha_{n+1} &= \gamma_n + 2\beta_n \\ \beta_{n+1} &= \gamma_n + 2\alpha_n\end{aligned}$$

where $\alpha_0 = a, \beta_0 = b, \gamma_0 = c$ and $\gamma_1 = d$ for natural number $n \geq 0$.
The members of the sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are the following table 2.

Table 2: The members of the sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$

n	α_n	γ_n	β_n
0	a		b
0		c	
1	$c+2b$		$c+2a$
1		d	
2	$2c+d+4a$		$2c+d+4b$
2		$d+2c$	
3	$6c+3d+8b$		$6c+3d+8a$
3		$3d+2c$	
4	$14c+9d+16a$		$14c+9d+16b$
4		$5d+6c$	
5	$34c+23d+32b$		$34c+23d+32a$
5		$11d+10c$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Theorem 3.2. *For each natural number* $n \geq 1$.

(a) $\gamma_n = 2J_{n-1}c + J_n d$

(b) $\alpha_n = 2\alpha_{n-1} + (J_{n-1} + (-1)^{n-1})c + J_{n-1}d + (-2)^n(a-b)$

(c) $\beta_n = 2\beta_{n-1} + (J_{n-1} + (-1)^{n-1})c + J_{n-1}d - (-2)^n(a-b)$.

Proof. The proofs are similar to theorem 3.1. □

The third sequence has the form

$$\gamma_{n+1} = \frac{\alpha_{n+1} + \beta_{n+1}}{2} + 2\gamma_n$$
$$\alpha_{n+1} = \gamma_n + 2\beta_n$$
$$\beta_{n+1} = \gamma_n + 2\alpha_n.$$

where $\alpha_0 = 2a, \beta_0 = 2b$ and $\gamma_0 = c$ for natural number $n \geq 0$.

The members of the sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are the following table 3.

Table 3: The members of the sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$

n	α_n	γ_n	β_n
0	$2a$		$2b$
0		c	
1	$4b + c$		$4a + c$
1		$2a + 2b + 3c$	
2	$10a + 2b + 5c$		$2a + 10b + 5c$
2		$10a + 10b + 11c$	
3	$14a + 30b + 21c$		$30a + 14b + 21c$
3		$42a + 42b + 43c$	
4	$102a + 70b + 85c$		$70a + 102b + 85c$
4		$170a + 170b + 171c$	
5	$310a + 374b + 341c$		$374a + 310b + 341c$
5		$682a + 682b + 683c$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Theorem 3.3. *For each natural number $n \geq 1$.*

(a) $\gamma_{n-1} = (J_{2n-1} - 1)(a + b) + J_{2n-1}c$

(b) $\alpha_n = (J_{n+1}^2 - J_n^2 + 1)(a + b) + (-1)^n J_n a + (-1)^{n+1}(2J_{n+1} + J_n)b + J_{2n}c$

(c) $\beta_n = (J_{n+1}^2 - J_n^2 + 1)(a + b) + (-1)^n J_n b + (-1)^{n+1}(2J_{n+1} + J_n)a + J_{2n}c.$

Proof. **(a)** We prove **(a)** by mathematical induction.
If $n = 1$ then $\gamma_0 = (J_1 - 1)(a + b) + J_1 c = c$ thus $n = 1$ is true.
Assume the truth of the statement for some $n - 2$ and $n - 1$; that is,

$$\gamma_{n-2} = (J_{2n-3} - 1)(a + b) + J_{2n-3}c$$

and

$$\gamma_{n-1} = (J_{2n-1} - 1)(a + b) + J_{2n-1}c.$$

$$\begin{aligned}\gamma_n &= \frac{\alpha_n + \beta_n}{2} + 2\gamma_{n-1} = (\beta_{n-1} + \alpha_{n-1} + \gamma_{n-1}) + 2\gamma_{n-1} \\ &= (2\gamma_{n-1} - 4\gamma_{n-2}) + \gamma_{n-1} + 2\gamma_{n-1} = 5\gamma_{n-1} - 4\gamma_{n-2} \\ &= 5\left[(J_{2n-1} - 1)(a + b) + J_{2n-1}c\right] - 4\left[(J_{2n-3} - 1)(a + b) + J_{2n-3}c\right] \\ &= (5J_{2n-1} - 4J_{2n-3} - 1)(a + b) + (5J_{2n-1} - 4J_{2n-3})c \\ &= (5J_{2n-1} - 2(2J_{2n-3}) - 1)(a + b) + (5J_{2n-1} - 2(2J_{2n-3}))c \\ &= (3J_{2n-1} + (J_{2n} - J_{2n-1}) - 1)(a + b) + (3J_{2n-1} + (J_{2n} - J_{2n-1}))c \\ &= (J_{2n} + 2J_{2n-1} - 1)(a + b) + (J_{2n} + 2J_{2n-1})c \\ &= (J_{2n+1} - 1)(a + b) + J_{2n+1}c.\end{aligned}$$

(b) We prove **(b)** by mathematical induction as well. If $n = 1$, then

$$\begin{aligned}\alpha_1 &= (J_2^2 - J_1^2 + 1)(a + b) + (-1)J_1 a + (-1)^2(2J_2 + J_1)b + J_2 c \\ &= a + b - a + 3b + c = 4b + c,\end{aligned}$$

is true. Assume the truth of the statement for some $n-1$ and n.

$$\begin{aligned}
\alpha_{n+1} &= 2\beta_n + \gamma_n = 2(2\alpha_{n-1} + \gamma_{n-1}) + \gamma_n = 4\alpha_{n-1} + 2\gamma_{n-1} + \gamma_n \\
&= 4\left[(J_n^2 - J_{n-1}^2 + 1)(a+b) + (-1)^{n-1}J_{n-1}a + (-1)^n(2J_n + J_{n-1})b + J_{2n-2}c\right] \\
&\quad + 2\left[(J_{2n-1} - 1)(a+b) + J_{2n-1}c\right] + \left[(J_{2n+1} - 1)(a+b) + J_{2n+1}c\right] \\
&= \left[4(J_n^2 - J_{n-1}^2 + 1 + (-1)^{n-1}J_{n-1}) + 2(J_{2n-1} - 1) + (J_{2n+1} - 1)\right] a \\
&\quad + \left[4(J_n^2 - J_{n-1}^2 + 1 + (-1)^n(2J_n + J_{n-1})) + 2(J_{2n-1} - 1) + (J_{2n+1} - 1)\right] b \\
&\quad + \left[4J_{2n-2} + 2J_{2n-1} + J_{2n+1}\right] c \\
&= \left[4J_n^2 - 4J_{n-1}^2 + 4(-1)^{n-1}J_{n-1} + 2J_{2n-1} + J_{2n+1} + 1\right] a \\
&\quad + \left[4J_n^2 - 4J_{n-1}^2 + 4(-1)^n(2J_n + J_{n-1}) + 2J_{2n-1} + J_{2n+1} + 1\right] b \\
&\quad + \left[2(J_{2n} - J_{2n-1}) + 2J_{2n-1} + J_{2n+1}\right] c \\
&= \left[4J_n^2 - 4J_{n-1}^2 + 4(-1)^{n-1}J_{n-1} + 2(J_n^2 + 2J_{n-1}^2) + (J_{n+1}^2 + 2J_n^2) + 1\right] a \\
&\quad + \left[4J_n^2 - 4J_{n-1}^2 + 8(-1)^n J_n + 4(-1)^n J_{n-1} + 2J_{2n-1} + J_{2n+1} + 1\right] b \\
&\quad + \left[2J_{2n} - 2J_{2n-1} + 2J_{2n-1} + J_{2n+1}\right] c \\
&= \left[8J_n^2 + J_{n+1}^2 + 4(-1)^{n+1}J_{n-1} + 1\right] a \\
&\quad + \left[4J_n^2 - 4J_{n-1}^2 + 8(-1)^n J_n + 4(-1)^n J_{n-1} + (2J_n^2 + 4J_{n-1}^2) + (J_{n+1}^2 + 2J_n^2) + 1\right] b \\
&\quad + (J_{2n+1} + 2J_{2n})c \\
&= \left[8J_n^2 + J_{n+1}^2 + 4J_{n-1}(J_n - 2J_{n-1}) + 1\right] a \\
&\quad + \left[8J_n^2 + J_{n+1}^2 + 8(-1)^n J_n + 4(-1)^n J_{n-1} + 1\right] b + J_{2n+2}c \\
&= \left[8J_n^2 + (4J_nJ_{n-1} + 4J_{n-1}^2 + J_n^2) + 4J_{n-1}J_n - 8J_{n-1}^2 + 1\right] a \\
&\quad + \left[8J_n^2 + J_{n+1}^2 + 8(-1)^{n+2}J_n - 4(-1)^{n+1}J_{n-1} + 1\right] b + J_{2n+2}c \\
&= \left[9J_n^2 + 8J_nJ_{n-1} - 4J_{n-1}^2 + 1\right] a \\
&\quad + \left[8J_n^2 + J_{n+1}^2 + 8J_n(J_{n+1} - 2J_n) - 4J_{n-1}(J_n - 2J_{n-1}) + 1\right] b + J_{2n+2}c \\
&= \left[4J_n^2 + 8J_nJ_{n-1} + 5J_n^2 - 4J_{n-1}^2 + 1\right] a \\
&\quad + \left[8J_n^2 + J_{n+1}^2 + 8J_nJ_{n+1} - 16J_n^2 - 4J_{n-1}J_n + 8J_{n-1}^2 + 1\right] b + J_{2n+2}c \\
&= \left[4J_n(J_n + 2J_{n-1}) + 5J_n^2 - 4J_{n-1}^2 + 1\right] a \\
&\quad + \left[2(J_{n+2}^2 - J_{n+1}^2) + J_{n+1}^2 - 16J_n^2 - 4J_{n-1}J_n + 8J_{n-1}^2 + 1\right] b + J_{2n+2}c \\
&= \left[4J_{n+1}J_n + 5J_n^2 - 4J_{n-1}^2 + 1\right] a \\
&\quad + \left[2J_{n+2}^2 - J_{n+1}^2 - 7J_n^2 - 4J_{n-1}J_n + 4J_{n-1}^2 - 8J_n^2 - (J_n^2 - 4J_{n-1}^2) + 1\right] b + J_{2n+2}c \\
&= \left[(4J_{n+1}J_n + 4J_n^2) + (J_n^2 - 4J_{n-1}^2) + 1\right] a + \\
&\left[2J_{n+2}^2 - J_{n+1}^2 - 7J_n^2 - 4J_{n-1}J_n + (J_{n+1}^2 - J_n^2 - 4J_nJ_{n-1}) - 8J_n^2 - (-1)^{n+1}J_{n+1} + 1\right] b \\
&\quad + J_{2n+2}c \\
&= \left[J_{n+2}^2 - J_{n+1}^2 + (-1)^{n+1}J_{n+1} + 1\right] a + \\
&\quad \left[2J_{n+2}^2 - J_{n+1}^2 - 8J_n^2 - 4J_n(2J_{n-1}) + J_{n+1}^2 - 8J_n^2 - (-1)^{n+1}J_{n+1} + 1\right] b + J_{2n+2}c
\end{aligned}$$

$$
\begin{aligned}
&= \left[J_{n+2}^2 - J_{n+1}^2 + (-1)^{n+1}J_{n+1} + 1\right] a \\
&\quad + \left[2J_{n+2}^2 - J_{n+1}^2 - 8J_n^2 - 4J_n(J_{n+1} - J_n) + J_{n+1}^2 - 8J_n^2 - (-1)^{n+1}J_{n+1} + 1\right] b + J_{2n+2}c \\
&= \left[J_{n+2}^2 - J_{n+1}^2 + (-1)^{n+1}J_{n+1} + 1\right] a \\
&\quad + \left[2J_{n+2}^2 - J_{n+1}^2 - 8J_n^2 - 4J_nJ_{n+1} + 4J_n^2 + J_{n+1}^2 - 8J_n^2 - (-1)^{n+1}J_{n+1} + 1\right] b + J_{2n+2}c \\
&= \left[J_{n+2}^2 - J_{n+1}^2 + (-1)^{n+1}J_{n+1} + 1\right] a \\
&\quad + \left[2J_{n+2}^2 - J_{n+1}^2 - 4(J_n^2 + J_nJ_{n+1}) + J_{n+1}^2 - 8J_n^2 - (-1)^{n+1}J_{n+1} + 1\right] b + J_{2n+2}c \\
&= \left[J_{n+2}^2 - J_{n+1}^2 + (-1)^{n+1}J_{n+1} + 1\right] a \\
&\quad + \left[2J_{n+2}^2 - J_{n+1}^2 - (J_{n+2}^2 - J_{n+1}^2) + J_{n+1}^2 - 8J_n^2 - (-1)^{n+1}J_{n+1} + 1\right] b + J_{2n+2}c \\
&= \left[J_{n+2}^2 - J_{n+1}^2 + (-1)^{n+1}J_{n+1} + 1\right] a \\
&\quad + \left[J_{n+2}^2 - J_{n+1}^2 + 2(J_{n+1}^2 - 4J_n^2) - (-1)^{n+1}J_{n+1} + 1\right] b + J_{2n+2}c \\
&= \left[J_{n+2}^2 - J_{n+1}^2 + (-1)^{n+1}J_{n+1} + 1\right] a \\
&\quad + \left[J_{n+2}^2 - J_{n+1}^2 + 2(-1)^{n+2}J_{n+2} + (-1)^{n+2}J_{n+1} + 1\right] b + J_{2n+2}c \\
&= (J_{n+2}^2 - J_{n+1}^2 + 1)(a+b) + (-1)^{n+1}J_{n+1}a + (-1)^{n+2}(2J_{n+2} + J_{n+1})b + J_{2n+2}c.
\end{aligned}
$$

Therefore, it is true for all $n \geq 1$.

(c) The proof of **(c)** is similar to the proof of **(b)**. □

Acknowledgement. This work was supported by Faculty of Science and Technology, Rajamangala University of Technology Tawan-ok.

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