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Analysis of Quasistatic Problem for an Elastic Material with Friction and Wear

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Abstract

This paper presents a quasistatic problem of an elastic body in frictional contact with a moving foundation. The model takes into account wear of the contact surface of the body caused by the friction. Existence and uniqueness results are proved by using arguments of elliptic variational inequalities. This makes the problem evolutionary and leads to a new and nonstandard mathematical model which couples a time-dependent variational inequality with an integral equation. Frictional contact between a deformable body and a foundation is a phenomenon that occurs in various forms in different physical settings. In any particular case, different factors can influence the behavior of the body. Such factors include, for example, constitutive law of the body, friction law that describes the contact with the foundation, the influence of the temperature or piezoelectricity effects. This is why different models have been developed in the field of contact mechanics. We provide the unique weak solvability of the model by using a

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fixed point argument. Next, we present a fully discrete scheme for numerical approximation with an error estimation of a solution to the problem.

1 Introduction

Frictional contact processes between deformable bodies or between a deformable body and a foundation abound in industry and everyday life. Their modeling is rather complex and, usually, leads to strongly nonlinear boundary value problems. Basic reference in the field includes $[1-5]$, the mathematical analysis of various models of contact is provided, including existence and uniqueness results of the solution. The references $[2,3]$ deal also with the numerical analysis of various models of contact, including the study of fully discrete schemes, error estimates and numerical simulations. Contact processes are accompanied by a number of phenomena among which the main one is the friction. Moreover, frictional contact is associated with heat generation, material damage, wear and adhesion of contacting surfaces. As the contact process evolves, the contacting surfaces evolve too, via their wear. To model the contacting surfaces the wear function $w = w(x,t)$ is introduced. Therefore, it measures the change in the surface geometry and represents the cumulative amount of material removed per unit surface area in the neighborhood of the point x up to time t. Since the amount of material removed is small, as an approximation, one may treat it as a change in the gap. It is usually assumed that the rate of wear of the surface is proportional to the contact pressure and to the relative slip rate;that is. to the dissipated frictional power. This leads to the rate form of Archard's law of surface wear $\dot{w} = k|\sigma_v|v$, where k is the wear coefficient, a very small positive constant in practice. Also, σ_v represents the normal stress on the contact surface and v denotes the relative slip rate. The initial condition is $w(x,0) = w_0(x)$ and $w_0(x) = 0$ when the surface is new or the initial shape is used as the reference configuration. The wear implies the evolution of contacting surfaces and these changes affect the contact process. Thus, due to its crucial role, there exists a large engineering and mathematical literature devoted to this topic [8].

The paper is structured as follows. In Section 2, we present the notation and some preliminary material. In Section 3, we introduce the model of sliding frictional contact with wear, list the assumptions on the data. In section 4, we derive its variational formulation. The unique weak solvability of the contact problem is presented in Section 5. There, we state and prove our main existence and uniqueness result whose proof is based on arguments on time-dependent variational inequalities and fixed point. In Section 6, we analyze a fully discrete scheme for the problem. We obtain an error estimate for this scheme. Finally, under appropriate regularity assumptions on the exact solution, we obtain an optimal-order error estimate.

2 Notations and Preliminaries

In this section, we present the notation we shall use and some preliminary material. We denote by $\mathcal{S}^d$ the space of second-order symmetric tensors on $\mathbb{R}^d$. Moreover, the inner product and norm on $\mathbb{R}^d$ and $\mathcal{S}^d$ are defined by

$$\begin{aligned} &u\cdot v = u_i v_i \quad , \qquad \|v\| = (v\cdot v)^{\frac{1}{2}} \,, \forall u, v \in \mathbb{R}^d \\ &\sigma\cdot\tau = \sigma_{ij}\tau_{ij} \quad , \qquad \|\sigma\| = (\sigma\cdot\sigma)^{\frac{1}{2}}, \forall \sigma, \tau \in \mathcal{S}^d. \end{aligned} \tag{2.1}$$

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded domain with Lipschitz continuous boundary $\Gamma$and let Γ_1, Γ_2, and Γ_3 be three measurable parts of such that $meas(\Gamma_1) > 0$. We use the notation $x = (x_i)$ for a typical point in $\Omega\cup\Gamma$ and we denote by $v = (v_i)$ the outward unit normal at Γ. Also, we use standard notation for the Lebesgue and Sobolev spaces associated to it. In particular, we recall that the inner products on the Hilbert spaces $L^2(\Omega)^d$ and $L^2(\Gamma)^d$ are given by

$$(u, v)_{L^2(\Omega)^d} = \int_\Omega u\cdot v \ dx \quad ; \quad (u, v)_{L^2(\Gamma)^d} = \int_\Gamma u\cdot v \ da \tag{2.2}$$

and the associated norms will be denoted by $\|\cdot\|_{L^2(\Omega)^d}$ and $\|\cdot\|_{L^2(\Gamma)^d}$, respectively. Moreover, we consider the spaces

$$V = \left\{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1 \right\} \tag{2.3}$$

$$Q = \left\{ \tau = (\tau_{ij}) \in L^2(\Omega)^d : \tau_{ij} = \tau_{ji} \right\} \tag{2.4}$$

These are real Hilbert spaces endowed with the inner products

$$(u, v)_V = \int_\Omega \varepsilon(u)\cdot\varepsilon(v) dx \quad , \quad (\sigma, \tau)_Q = \int_\Omega \sigma\cdot\tau dx \tag{2.5}$$

and the associated norms $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively.

Here ε is the deformation operator given by

$$\varepsilon(v) = (\varepsilon_{ij}(v)) \quad ; \quad \varepsilon_{ij}(v) = \frac{1}{2}(v_{ij} + v_{ji}) \ ; \forall v \in H^1(\Omega)^d \tag{2.6}$$

Recall that the completeness of the space $(V, \|\cdot\|_V)$ follows from the assumption $meas(\Gamma_1) > 0$ which allows the use of Korn's inequality.

For an element $v \in V$. we still write v for the trace of v on the boundary. We denote by v_v and v_τ the normal and the tangential component of v on Γ, respectively, defined by

$$v_v = v{\cdot}\nu, v_\tau = v - v_v\nu.$$

By the Sobolev trace theorem, there exists a positive constant c_0 which depends on Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3)^d} \leq c_0 \|v\|_V \ , \forall v \in V. \tag{2.7}$$

Then, the classical formulation of the contact problem under consideration is the following.

3 Mechanical contact problem

An elastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a Lipschitz continuous boundary Γ, divided into three measurable parts Γ_1, Γ_2, and Γ_3 with the latter being a plane. The body is subject to the action of body forces of density f_0. It is fixed on Γ_1 and surfaces tractions of density f_2 act on Γ_2. On Γ_3, the body is in frictional contact with a moving obstacle, the so-called foundation. We denote by v^* the velocity of the foundation, which is supposed to be a non-vanishing time-dependent function in the plane of Γ_3. The friction implies the wear of the foundation that we model with a surface variable, the wear function. Its evolution is governed by a simplified version of Archard's law that we shall describe below. Moreover, we assume that the foundation is deformable and, therefore, its penetration is allowed. We model the contact with a normal compliance condition with unilateral constraint which takes into account the wear of the foundation.We associate this condition to a sliding version of Coulomb's law of dry friction. We adopt the framework of the small strain theory and we assume that the contact process is quasistatic and it is studied in the interval of time
$\mathbb{R}_+ = [0, \infty)$.

Then, the classical formulation of the contact problem under consideration is the following:

Problem 1. *Find a stress field* $\sigma : \Omega\times \mathbb{R}_+ \to \mathcal{S}^d$*, a displacement field* $u : \Omega\times \mathbb{R}_+ \to \mathbb{R}^d$*, and a wear function* $w : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\sigma(t) = \mathcal{F}\varepsilon(u(t)) \qquad \text{in } \ \Omega \times \mathbb{R}_+ \tag{3.1}$$

$$Div(\sigma) + f_0(t) = 0 \quad \text{in } \Omega \times \mathbb{R}_+ \tag{3.2}$$

$$u(t) = 0 \qquad \text{on } \Gamma_1 \times \mathbb{R}_+ \tag{3.3}$$

$$\sigma(t)v = f_2(t) \qquad \text{on } \Gamma_2 \times \mathbb{R}_+ \tag{3.4}$$

$$\left\{ \begin{array}{l} u_v(t) \leq g \ , \ \sigma_v(t) + p(u_v(t) - w(t)) \leq 0 \\ (u_v(t) - g)\,(\sigma_v(t) + p(u_v(t) - w(t))) = 0 \end{array} \right. \quad \text{on } \Gamma_3 \times \mathbb{R}_+ \tag{3.5}$$

$$-\sigma_\tau(t) = \mu p(u_v(t) - w(t))n^*(t) \ \text{ on } \Gamma_3 \times \mathbb{R}_+ \tag{3.6}$$

$$\dot{w}(t) = \alpha(t)p(u_v(t) - w(t)) \qquad \text{on } \Gamma_3 \ \times \mathbb{R}_+ \tag{3.7}$$

$$w(0) = 0 \quad \text{on } \Gamma_3 \tag{3.8}$$

Here and below, for simplicity, we do not indicate explicitly the dependence of various functions on the spatial variable x. Moreover, the functions $n^*$and α are given by

$$n^*(t) = -\frac{v^*(t)}{\|v^*(t)\|} \quad , \ \alpha(t) = k\,\|v^*(t)\| \ \ , \forall t \in \mathbb{R}_+ \tag{3.9}$$

where k represents the wear coefficient. We now provide a brief explanation for the equations and conditions in Problem $\mathcal{P}$.

First, Equation (3.1) represents the elastic constitutive law of the material in which $\mathcal{F}$ denotes a given nonlinear operator. Equation (3.2) is the equilibrium equation in which Div represents the divergence operator for tensor-valued functions. Conditions (3.3) and (3.4) are the displacement and traction boundary conditions, respectively. Next, condition (3.5) represents the contact condition in which $g > 0$ and p is a positive Lipschitz continuous increasing function which vanishes for a negative argument. This condition can be derived in the following way.

First, we assume that the obstacle is made of a hard material covered by a layer of soft material of thickness g. Thus, at each moment t the normal stress has an additive decomposition of the form

$$\sigma_v(t) = \sigma_v^R(t) + \sigma_v^S(t) \quad \text{on } \Gamma_3 \tag{3.10}$$

in which the function $\sigma_v^R(t)$ describes the reaction to penetration of the hard material and $\sigma_v^S(t)$ describes the reaction of the soft material. The hard material does not wear and is perfectly rigid.

Therefore, the penetration is limited by the bound g and σ_v^R satisfies the Signorini condition in the form with a gap function; i.e.,

$$u_v(t) \leq g \ , \ \sigma_v^R(t) \leq 0 \ , \ \sigma_v^R(t)(u_v(t) - g) = 0 \text{ on } \Gamma_3 \tag{3.11}$$

The soft material is elastic and could wear. Therefore, we assume that $\sigma_v^S(t)$ satisfies a normal compliance contact condition with wear; that is,

$$-\sigma_v^S(t) = p(u_v(t) - w(t)) \quad \text{on } \Gamma_3 \tag{3.12}$$

This condition shows that, at each moment t, the reaction of the soft layer depends on the current value of the penetration, represented by $u_v(t)- w(t)$. Indeed, we assume that a wear process of the soft layer of the foundation takes place and the debris are immediately removed from the system.

Thus, the penetration becomes $u_v(t)- w(t)$ instead of $u_v(t)$ as in the case without wear. Condition (3.12) describes the fact that the surface geometry of foundation is affected by wear.

We now combine (3.10) and (3.12) to see that

$$\sigma_v^R(t) = \sigma_v(t) + p(u_v(t) - w(t)) \text{ on } \Gamma_3. \tag{3.13}$$

We then substitute equality (3.13) into (3.11) to obtain the contact condition (3.5). We now describe the frictional contact condition (3.6).

First, we recall the classical Coulomb's law of dry friction

$$\left.\begin{array}{c} \|\sigma_\tau\| \leq \mu\,|\sigma_\nu| \\ -\sigma_\tau = \mu\,|\sigma_\nu|\,\dfrac{\dot{u}_\tau(t) - v^*(t)}{\|\dot{u}_\tau(t) - v^*(t)\|} \quad \text{if } \dot{u}_\tau(t) - v^*(t) \neq 0 \end{array}\right\} \text{ on } \Gamma_3 \tag{3.14}$$

Here μ represents the friction coefficient, $\dot{u}_\tau(t)$ is the tangential velocity, and $\dot{u}_\tau(t) - v^*(t)$ represents the relative tangential velocity or the relative slip rate. We assume that at each moment t the velocity of the foundation $v^*(t)$ is large in comparison with the tangential velocity $\dot{u}_\tau(t)$ and, for this reason, we approximate the relative slip rate by $v^*(t)$.

Therefore, using the approximations

$$\dot{u}_\tau(t) - v^*(t) \approx -v^*(t) \neq 0 \quad , \quad \|\dot{u}_\tau(t) - v^*(t)\| \approx v^*(t)$$

the friction law (3.14) implies that

$$\sigma_\tau(t) = \mu\,|\sigma_\nu|\,\frac{v^*(t)}{\|v^*(t)\|} \quad \text{on} \quad \Gamma_3$$

Therefore, using the definition (3.9) of the vector $n^*(t) = -\frac{v^*(t)}{\|v^*(t)\|}$, we have

$$-\sigma_\tau(t) = \mu\,|\sigma_\nu|\,n^*(t) \quad \text{on } \Gamma_3 \tag{3.15}$$

Next, we note that as far as the contact of the elastic body is in the status of normal compliance (i.e., $u_v(t) < g$), condition (3.5) shows that

$$-\sigma_\nu(t) = p(u_v(t) - w(t)) \quad \text{on } \Gamma_3 \tag{3.16}$$

and, therefore, substituting this equality in (3.15) we deduce that (3.6) holds. We extend this condition to the case when the contact is unilateral; i.e., when $u_v(t) = g$. In this way, we fully justify the friction law (3.6).

To obtain the differential equation (3.7), we start from the Archard's law

$$\dot{w}(t) = k|\sigma_v(t)|\,\|\dot{u}_\tau(t) - v^*(t)\| \quad \text{on } \Gamma_3. \tag{3.17}$$

Then, using again the approximation $\dot{u}_\tau(t) - v^*(t) \approx -v^*(t)$, equation (3.17) leads to

$$\dot{w}(t) = k|\sigma_v(t)|\,\|v^*(t)\| \quad \text{on } \Gamma_3$$

We now use the definition (3.9) of the function α to obtain

$$\dot{w}(t) = \alpha(t)|\sigma_v(t)| \quad \text{on } \Gamma_3 \tag{3.18}$$

Next, we note that as far as the contact of the elastic body is in the status of normal compliance (3.16) holds and, therefore, substituting this equality in (3.18) we deduce (3.7). We extend this equality in the case of the unilateral contact; i.e., in the case when $u_v(t) = g$. In this way, we fully justify the differential Equation (3.7) which governs the evolution of the wear function. Finally (3.8) represents the initial condition for the wear function which shows that at the initial moment the foundation is new.

We note that considering an arbitrary contact surface Γ_3 and a thickness $g = g(x)$ depending on the spatial variable does not cause additional mathematical difficulties in the analysis of Problem $\mathcal{P}$.

4 Variational formulation

We now turn to the variational formulation of Problem $\mathcal{P}$. To this end, we list the assumptions on the data.

1. We assume that the elasticity operator $\mathcal{F} : \Omega\times\ \mathcal{S}^d \to \mathcal{S}^d$ satisfy the following condition:

$$\left\{\begin{array}{c} (a)\text{ There exists } L_{\mathcal{F}}\ > 0 \text{ such that } \|\mathcal{F}(x,\varepsilon_1) - \mathcal{F}(x,\varepsilon_1)\| \leq L_{\mathcal{F}}\,\|\varepsilon_1 - \varepsilon_2\| \\ \forall\varepsilon_1,\varepsilon_2 \in \mathcal{S}^d\ ,\ x\in\Omega \\ (b)\text{ There exists } m_F > 0 \text{ such that} \\ (\mathcal{F}(x,\varepsilon_1) - \mathcal{F}(x,\varepsilon_1))\cdot(\varepsilon_1-\varepsilon_2) \geq m_F\,\|\varepsilon_1-\varepsilon_2\|^2,\ \forall\varepsilon_1,\varepsilon_2 \in \mathcal{S}^d\ ,\ x\in\Omega \\ (c)\text{ The mapping } x \to F(x,\varepsilon) \text{ is measurable on },\forall\varepsilon\in\mathcal{S}^d\ ,\ x\in\Omega \\ (d)\text{ The mapping } x\to F(x,0) \text{ belongs to } Q. \end{array}\right. \tag{4.1}$$

2. The normal compliance function $p : \Gamma_3\times\mathbb{R} \to \mathbb{R}+$ satisfy the condition:

$$\left\{\begin{array}{c} (a)\text{ There exists } Lp > 0 \text{ such that } |p(x,r_1) - p(x,r_2)| \leq Lp\,|r_1 - r_2| \\ \forall r_1,r_2\in\mathbb{R},\text{a.e.}x\in\Gamma_3 \\ (b)\quad (p(x,r_1)-p(x,r_2))(r_1-r_2) \geq 0\ ,\quad \forall r_1,r_2\in\mathbb{R},\ \text{a.e .}x\in\Gamma_3 \\ (c)\text{ The mapping } x\to p(x,r)\ \text{ is measurable on }\Gamma_3,\forall r\in R \\ (d)\ \ p(x,r) = 0 \text{ for all } r\leq 0, a.e.x\in\Gamma_3 \end{array}\right. \tag{4.2}$$

3. The densities of body forces and surface tractions have the regularity

$$f_0 \in C(\mathbb{R}_+; L^2(\Omega)^d)\ \ ,\quad f_2 \in C(\mathbb{R}_+; L^2(\Gamma_2)^d). \tag{4.3}$$

4. The friction coefficient, the wear coefficient, and the foundation velocity satisfy

$$\mu \in L^\infty(\Gamma_3), \mu(x) \geq 0 \text{ a.e.}x\in\Gamma_3 \tag{4.4}$$

$$k \in L^{\infty}(\Gamma_3)\ ,\ k(x) \geq 0 \text{ a.e. } x \in \Gamma_3 \tag{4.5}$$

$$v^* \in C(\mathbb{R}_+; R^d) \text{ and there exists } v > 0 \text{ such that } v^*(t) \geq v, \forall t \in \mathbb{R}_+ \tag{4.6}$$

Note that assumption (4.6) is compatible with the physical setting described above since, at each time moment, the velocity of the foundation is assumed to be large enough. In addition, (3.9), (4.5), and (4.6) imply that

$$n^* \in C(\mathbb{R}_+; R^d), \alpha \in C(\mathbb{R}_+; L^{\infty}(\Gamma_3)) \tag{4.7}$$

Moreover,

$$\alpha(t) \geq 0 \text{ a.e .on } \Gamma_3, \text{ for all } t \in \mathbb{R}_+ \tag{4.8}$$

We introduce now the set of admissible displacements fields defined by

$$U = \{v_v \in V : v_v \leq g \text{ on } \Gamma_3\} \tag{4.9}$$

In addition, we use the Riesz representation to define the function $f : \mathbb{R}_+ \to V$

$$(f(t), v)_V = (f_0(t), v)_{L^2(\Omega)^d} + (f_2(t), v)_{L^2(\Gamma_2)^d} \tag{4.10}$$

for all $v \in V$ and $t \in \mathbb{R}_+$.

By assumption (4.3) it follows that f has the regularity

$$f \in C(\mathbb{R}_+; V) \tag{4.11}$$

In what follows, assume that (σ, u, w) are sufficiently regular functions which satisfy (3.1) – (3.8) and let $v \in U$ and $t > 0$ be given.

We use Green's formula (2.2) and the equilibrium Equation (3.2) to obtain

$$\int_\Omega \sigma(t)(\varepsilon(v) - \varepsilon(u(t))dx - \int_\Omega f_0(t)(v - u(t))dx = \int_\Gamma \sigma(t)v\cdot(v - u(t))da\ \ , \forall v \in V$$

Next, we split the boundary integral over Γ_1, Γ_2, and Γ_3.

Since $v - u(t) = 0$ on Γ_1, $\sigma(t)v = f_2(t)$ on Γ_2, taking into account (4.10), we deduce that

$$(\sigma(t)(\varepsilon(v) - \varepsilon(u(t)))_Q = (f(t), v - u(t))_V + \int_{\Gamma_3} \sigma(t)v \cdot (v - u(t))da \tag{4.12}$$

Note that

$$\sigma(t)v\cdot(v-u(t))=\sigma_v(t)(v_v-u_v(t))+\sigma_\tau(t)\cdot(v_\tau-u_\tau(t)) \text{ on } \Gamma_3 \quad (4.13)$$

and, using contact condition (3.5) and the definition (4.9) of the set U, we have

$$\begin{gathered}\sigma_v(t)(v_v-u_v(t))=(\sigma_v(t)+p(u_v(t)-w(t)))(v_v-g)\\ +(\sigma_v(t)+p(u_v(t)-w(t)))(g-u_v(t))-p(u_v(t)-w(t))(v_v-u_v(t))\\ \geq -p(u_v(t)-w(t))(v_v-u_v(t)) \quad \text{, on } \Gamma_3\end{gathered} \quad (4.14)$$

Therefore, taking into account identity (4.13), inequality (4.14), and the friction law (3.6) we obtain that

$$\begin{gathered}\int_{\Gamma_3}\sigma(t)v\cdot(v-u(t))da \geq -\int_{\Gamma_3}p(u_v(t)-w(t))(v_v-u_v(t))da\\ -\int_{\Gamma_3}\mu p(u_v(t)-w(t))n^*(t)\cdot(v_\tau-u_\tau(t))da.\end{gathered} \quad (4.15)$$

We now combine (4.12) and (4.15) to obtain

$$\begin{gathered}(\sigma(t),\varepsilon(v)-\varepsilon(u(t)))_Q+\int_{\Gamma_3}p(u_v(t)-w(t))(v_v-u_v(t))da\\ +\int_{\Gamma_3}\mu p(u_v(t)-w(t))n^*(t)\cdot(v_\tau-u_\tau(t))da \geq (f(t),v-u(t))_V \text{ , } \forall v\in U.\end{gathered} \quad (4.16)$$

In addition, we note that the boundary condition (3.3), the first inequality in (3.5) and (4.9) imply that $u(t)\in U$.

Finally, we integrate the differential Equation (3.7) with the initial condition (3.8) to obtain that

$$w(t)=\int_0^t \alpha(s)p(u_v(s)-w(s))ds. \quad (4.17)$$

We now gather the constitutive law (3.1), the variational inequality (4.16), and the integral e quation (4.17) to obtain the following variational formulation of the contact problem $\mathcal{P}$.

Problem $\mathcal{P}^{\mathcal{V}}$. Find a stress field $\sigma:\mathbb{R}_+\to Q$, a displacement field $u:\mathbb{R}_+\to U$, and a wear function $w:\mathbb{R}_+\to L^2(\Gamma_3)$ such that

$$\sigma(t)=\mathcal{F}\varepsilon(u(t)) \quad (4.18)$$

$$(\sigma(t), \varepsilon(v) - \varepsilon(u(t)))Q + \int_{\Gamma_3} p(u_v(t) - w(t))(v_v - u_v(t))da \tag{4.19}$$
$$+ \int_{\Gamma_3} \mu p(u_v(t) - w(t))n^*(t)\cdot(v_\tau - u_\tau(t))da$$
$$\geq (f(t), v - u(t))_V \ ; \forall v \in U$$

$$w(t) = \int_0^t \alpha(s)p(u_v(s) - w(s))ds \ \ , t \in \mathbb{R}_+ \tag{4.20}$$

The unique solvability of Problem $\mathcal{P}^{\mathcal{V}}$ will be proved in the next section. A triple (σ, u, w) which satisfy $(4.18) - (4.19)$ is called weak solution of Problem $\mathcal{P}$.

5 Existence and uniqueness result

In this section, we state and prove the unique solvability of Problem $\mathcal{P}^{\mathcal{V}}$.

Theorem 5.1. *Assume that* $(4.1) - (4.6)$ *hold. Then there exists a constant* μ_0 *depending only on* Ω, Γ_1, Γ_3, $\mathcal{F}$, *and* p *such that if*

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0, \tag{5.1}$$

then Problem $\mathcal{P}^{\mathcal{V}}$ *has a unique solution. Moreover, the solution has the regularity*

$$\sigma \in C(\mathbb{R}_+; Q) \ , u \in C(\mathbb{R}_+; U) \ , w \in C^1(\mathbb{R}_+; L^2(\Gamma_3)) \tag{5.2}$$

and, in addition,

$$w(t) \geq 0 \ a.e. on \ \Gamma_3 \ \ for \ all \ t \in \mathbb{R}_+ \tag{5.3}$$

The proof of theorem 2 will be carried out in several steps, we assume in the rest of this section that $(4.1) - (4.6)$ hold.

In the first step, we consider a given wear function $w \in C(\mathbb{R}_+; L^2(\Gamma_3))$ and we construct the following intermediate variational problem.

Problem 2. $\mathcal{P}_w^{\mathcal{V}}$ *Find a displacement field* $u_w : \mathbb{R}_+ \to U$ *such that*

$$(\mathcal{F}\varepsilon(u_w(t)), \varepsilon(v) - \varepsilon(u_w(t)))Q + \int_{\Gamma_3} p(u_{wv}(t) - w(t))(v_v - u_{wv}(t))da$$
$$+ \int_{\Gamma_3} \mu p(u_{wv}(t) - w(t))n^*(t)\cdot(v_\tau - u_{w\tau}(t))da \geq (f(t), v - u_w(t))_V \ , \forall v \in U, t \in \mathbb{R}_+ \tag{5.4}$$

In the study of Problem $\mathcal{P}_w^{\mathcal{V}}$, we have the following existence and uniqueness result.

Lemma 5.2. *There exists a constant μ_0 which depends only on Γ_1,Γ_3, $\mathcal{F}$, and p such that if (4.1) holds, then there exists a unique solution to Problem $\mathcal{P}_w^{\mathcal{V}}$ which satisfies $u_w \in C(\mathbb{R}_+; U)$.*

Proof. Let $t \in \mathbb{R}_+$ and consider the operator $A_{wt} : V \to V$ defined by

$$(A_{wt}u, v)_V = (\mathcal{F}\varepsilon(u), \varepsilon(v))_Q + \int_{\Gamma_3} p(u_v - w(t))v_v da$$
$$+ \int_{\Gamma_3} \mu p(u_v - w(t))n^*(t)\cdot v_\tau da \ \ ; \ \forall u, v \in V. \tag{5.5}$$

We use assumptions (4.1), (4.2), (4.4), and inequality (2.1) to see that the operator A_{wt} is Lipschitz continuous; i.e., it satisfies the inequality

$$\|A_{wt}u_1 - A_{wt}u_2\|_V \leq \left(L_{\mathcal{F}} + c_0^2 L_p(1 + \|\mu\|_{L^\infty(\Gamma_3)})\right) \|u_1 - u_2\|_V \ , \text{for all } u_1, u_2 \in V \tag{5.6}$$

Next, we introduce the constant μ_0 defined by

$$\mu_0 = \frac{m_{\mathcal{F}}}{c_0^2 L_p} \tag{5.7}$$

and note that it depends only on Γ_1, Γ_3, $\mathcal{F}$, and p. Assume that (5.1) holds. Then

$$c_0^2 L_p \|\mu\|_{L^\infty(\Gamma_3)} < m_{\mathcal{F}} \tag{5.8}$$

We use again assumptions (4.1), (4.2), and inequalities (2.1) and (5.8) to deduce that the operator A_{wt} is strongly monotone; i.e., it satisfies the inequality

$$(A_{wt}u_1 - A_{wt}u_2, u_1 - u_2)_V \geq (m_{\mathcal{F}} - c_0^2 L_p \|\mu\|_{L^\infty(\Gamma_3)}) \|u_1 - u_2\|_V^2 \ , \forall u_1, u_2 \in V \tag{5.9}$$

Together with Theorem 1, we deduce that there exists a unique element $u_{wt} \in U$ such that

$$(A_{wt}u_{wt}, v - u_{wt})_V \geq (f(t), v - u_{wt})_V \ , \forall v \in U \tag{5.10}$$

Denoting $u_{wt} = u_w(t)$, it follows from (5.10) and (5.5) that the element $u_w(t) \in U$ is the unique element which solves the variational inequality (5.4).

We now prove the continuity of the function $t \to u_w(t) : \mathbb{R}_+ \to V$.

To this end, let

$$u_i = u_w(t_i), w_i = w(t_i), f_i = f(t_i), n_i^* = n^*(t_i) \ , t_1, t_2 \in \mathbb{R}_+; \text{ for } i = 1, 2.$$

We use standard arguments in (5.4) to find that

$$(\mathcal{F}\varepsilon(u_1) - \mathcal{F}\varepsilon(u_2), \varepsilon(u_1) - \varepsilon(u_2))_Q \leq (f_1 - f_2, u_1 - u_2)_V$$

$$+ \int_{\Gamma_3} [p(u_{1v} - w_1) - p(u_{2v} - w_2)][(u_{2v} - w_2) - (u_{1v} - w_1)]da$$

$$+ \int_{\Gamma_3} [p(u_{1v} - w_1) - p(u_{2v} - w_2)](w_2 - w_1)da$$

$$+ \int_{\Gamma_3} \mu[p(u_{1v} - w_1)n_1^* - p(u_{2v} - w_2)n_1^*]\cdot(u_{2\tau} - u_{1\tau})da$$

$$+ \int_{\Gamma_3} \mu[p(u_{2v} - w_2)n_1^* - p(u_{2v} - w_2)n_2^*]\cdot(u_{2\tau} - u_{1\tau})da.$$

Therefore (4.1), (4.2), (4.4), and (2.1) yield

$$(m_{\mathcal{F}} - c_0^2 L_p \left\|\mu\right\|_{L^\infty(\Gamma_3)}) \left\|u_1 - u_2\right\|_V^2 \leq$$

$$c_0 L_p(1 + \left\|\mu\right\|_{L^\infty(\Gamma_3)}) \left\|w_1 - w_2\right\|_{L2(\Gamma_3)} + \left\|f_1 - f_2\right\|_V$$

$$+ c_0 p(g) \left\|\mu\right\|_{L^\infty(\Gamma_3)} \left\|n_1^* - n_2^*\right\|) \left\|u_1 - u_2\right\|_V + L_p \left\|w_1 - w_2\right\|_{L^2(\Gamma_3)}^2$$

We now use (5.8) and

$$x, y, z \geq 0 \text{ and } x^2 \leq yx + z \Rightarrow x^2 \leq y^2 + 2z$$

to deduce that

$$\|u_1 - u_2\|_V^2 \le a\left(\|w_1 - w_2\|_{L^2(\Gamma_3)} + \|f_1 - f_2\| + \|n_1^* - n_2^*\|^2\right) + b\,\|w_1 - w_2\|_{L^2(\Gamma_3)}^2 \tag{5.11}$$

where a and b denote two positive constants which do not depend on t_1 and t_2. This inequality combined with $(4.7), (4.11)$ and the regularity $w \in C(\mathbb{R}_+; L^2(\Gamma_3))$ show that $u_w \in C(\mathbb{R}_+; V)$.

Thus, we conclude the existence part in Lemma 1. The uniqueness part follows from the unique solvability of (4.10) for each $t \in \mathbb{R}_+$.

We assume in what follows that $(5.1), (5.7)$ hold and we consider the operator Λ defined by

$$\Lambda : C(\mathbb{R}_+; L^2(\Gamma_3)) \to C(\mathbb{R}_+; L^2(\Gamma_3)) \tag{5.12}$$

$$\Lambda w(t) = \int_0^t \alpha(s) p(u_{wv}(s) - w(s)) ds,$$

for all $w \in C(\mathbb{R}_+; L^2(\Gamma_3))$, where u_w is the unique solution of Problem $\mathcal{P}_w^{\mathcal{V}}$.

We have the following fixed point result which represents the second step in the proof of Theorem 1.

Lemma 5.3. *The operator Λ has a unique fixed point $w^* \in C(\mathbb{R}_+; L^2(\Gamma_3))$.*

We now have all ingredients needed to provide the proof of our main existence and uniqueness result.

Proof of Theorem 1

Existence. Let $w^* \in C(\mathbb{R}_+; L^2(\Gamma_3))$ be the unique fixed point of the operator and let u^*, σ^* defined by

$$u^*(t) = u_w^*(t), \ \text{, for all } t \in \mathbb{R}_+ \tag{5.13}$$

$$\sigma^*(t) = \mathcal{F}\varepsilon(u^*(t)) \ \text{, for all } t \in \mathbb{R}_+ \tag{5.14}$$

We recall that $w^* = \Lambda w^*$ and using (5.12) and (5.13) we deduce

$$w^*(t) = \int_0^t \alpha(s) p(u_w^* v(s) - w^*(s)) ds, \tag{5.15}$$

for all $t \in \mathbb{R}_+$. We show that the triple (σ^*, u^*, w^*) satisfies $(4.18)-(4.20)$.

First, we note that (4.18) is a direct consequence of (5.14). Then, writing the inequality (5.14) for $w = w^*$ and using $(4.13), (4.14)$ we see that

(4.19) holds. Finally, (4.20) follows from (4.15). We conclude that the triple (σ^*, u^*, w^*) represents a solution of Problem $\mathcal{P}^{\mathcal{V}}$ as claimed. The regularity expressed in (5.2) is a direct consequence of the Lemma 1 combined with assumption (4.1) and formula (5.15).

Finally, condition (5.3) follows from (5.15) since α and p are positive functions, as it results from (4.8) and (4.2).

*Uniqueness***.** The uniqueness of the solution follows from the unique solvability of Problem $\mathcal{P}^{\mathcal{V}}_w$, provided in Lemma 1, combined with the uniqueness of the fixed point of operator Λ defined by (5.12).

6 Discrete approximation of the problem

We introduce and analyze a fully discrete approximation scheme for the problem. Here and below, we denote $W = L^2(\Gamma_3)$. Let $V_h \subset V$ and $W_h \subset W$ be two families of finite dimensional subspaces with a discretization parameter $h > 0$.

Let $U_h = U \cap V_h$. Introduce the time step $k = T/N$ for $N \in \mathbb{N}$, $N > 0$ and $t_n = nk, n = 0, 1, .., N$. We also use notation $g_j = g(t_j)$ for any $g \in C([0, T]; X)$, where X is any introduced function space.

We make the following additional assumptions on the solution u to Problem P and the velocity of the foundation v^*, with the regularity

$$u \in H^1(0, T; V), v^* \in W^{1,\infty}(0, T; \mathbb{R}^d)$$

Now we define some operators and functions in order to present a fully discrete approximation of the problem.

Let $F : V \to V^*$, $f : [0, T] \to V^*$ and $\varphi : [0, T] \times L^2(\Gamma_3) \times V \times V \to \mathbb{R}$ be defined for all $u, v \in V, w \in L^2(\Gamma_3), t \in [0, T]$ as follows:

$$\langle Fu, v\rangle_{V^*\times V} = (\mathcal{F}\varepsilon(u), \varepsilon(v))_{\mathcal{H}}$$

$$\langle f(t), \mathbf{v}\rangle_{V^*\times V} = \int_\Omega f_0(t)\cdot\mathbf{v}dx + \int_{\Gamma_2} f_2(t)\cdot\mathbf{v}da$$

$$\varphi(t, w, u, v) = \int_{\Gamma_3} p(u_\nu - w)v_\nu da + \int_{\Gamma_3} \mu p(u_\nu - w)n^*(t)\cdot v_\tau da.$$

We now present the following fully discrete scheme.

Problem 3. $\mathcal{P}^{hk}$.*Find* $u^{hk} = \{u_j^{hk}\}_{j=0}^N \subset U^h$ *and* $w^{hk} = \{w_j^{hk}\}_{j=0}^N \subset W^h$ *such that*

$$\left\langle F\ u_j^{hk}, v^h - u_j^{hk}\right\rangle_{V^*\times V} + \varphi_j(w_j^{hk}, u_j^{hk}, v_j^{hk}) \geq \left\langle f_j, v^h - u_j^{hk}\right\rangle_{V^*\times V}\ ,\ v^h \in U^h. \tag{6.1}$$

$$w_j^{hk} = k\sum_{m=0}^{j-1} \alpha_m p((u_m^{hk}) - w_m^{hk}), j \in \{0, ..., N\} \tag{6.2}$$

$$w_0^{hk} = 0 \tag{6.3}$$

We observe that existence of the unique solution to Problem $\mathcal{P}^{hk}$ follows from application of discrete version of theorem 2.

The next lemma presents Gronwall's inequality in the following form

Lemma 6.1. *Let* T *be given. For* $N > 0$, *we define* $k = T/N$. *Let* $\{g_n\}_{n=1}^N, \{e_n\}_{n=1}^N$ *be two nonnegative sequences satisfying for* $c > 0$ *and for all* $n \in \{1, ..., N\}$

$$e_n \leq cg_n + c\sum_{j=1}^{n} ke_j.$$

Then

$$\max_{1\leq n\leq N} e_n \leq \tilde{c} \max_{1\leq n\leq N} g_n\ ,\ \tilde{c} > 0$$

We now present the following theorem dealing with error estimation of the introduced numerical scheme.

Theorem 6.2. *Under the assumptions of Theorem 1, for the unique solution* $(u, w) \in C([0, T]; V) \times C^1([0, T]; W)$ *of Problem* $\mathcal{P}^{\mathcal{V}}$ *and the unique solution* $(u^{hk}, w^{hk}) \subset U^h \times W^h$ *of Problem* $\mathcal{P}^{hk}$, *there exists a constant* $\tilde{c} > 0$ *such that*

$$k\sum_{j=1}^{N}(\| u_j - u_j^{hk} \|_V^2 + \| w_j - w_k^{jh} \|_W^2) \leq$$

$$\tilde{c} \inf_{v^h\in U^h} \left\{ k^2 + k\left\|u_0 - u_0^{hk}\right\|_V^2 + k\sum_{j=1}^{N}\left\|u_j - u_j^{hk}\right\|_V^2 + k\sum_{j=1}^{N}\left|R_j(w_j, u_j, v_j^h)\right| \right\} \tag{6.4}$$

where

$$R_j(w_j, u_j, v_j^h) = \left\langle Fu_j, v_j^h - u_j\right\rangle_{V^*\times V} + \varphi_j(w_j, u_j, v_{jh}) - \varphi_j(w_j, u_j, u_j) - \left\langle f_j, v_j^h - u_j\right\rangle_{V^*\times V}.$$

We finish this section by providing a sample error estimate under the additional assumption on the solution regularity.

We consider a polygonal domain Ω and a space of continuous piecewise affine functions V^h.

Theorem 6.3. *Under the hypotheses of Theorem 2 and assuming the solution regularity* $u \in C([0,T]; H^2(\Omega)^d)$ *,we have the following error estimate:*

$$k \sum_{j=1}^{n} (\| u_j - u_j^{hk} \|_V^2 + \| w_j - w_j^{hk} \|_W^2) \leq \tilde{c}(k^2 + kh^2 + h) \text{ , with } \tilde{c} > 0 \quad (6.5)$$

Proof. We fix any $t = t_j$, $j \in \{0, 1, ..., N\}$ and denote by $\Pi^h u_j \in U^h$ the finite element interpolant of u_j.

By the standard finite element interpolation error bounds, we have

$$\| \eta - \Pi^h \eta \|_V \leq ch \| \eta \|_{H^2(\Omega)^d} \text{ for all } \eta \in H^2(\Omega)^d. \quad (6.6)$$

We obtain

$$|R_j(w_j, u_j, v_j^h)| \leq c \| u_j - v_j^h \|_V . \quad (6.7)$$

Using inequality (5.15), we get

$$k \sum_{j=1}^{n} (\| u_j - u_j^{hk} \|_V^2 + \| w_j - w_j^{hk} \|_W^2) \leq$$

$$c(k^2 + k \| u_0 - \Pi^h u_0 \|_V^2 + k \sum_{j=1}^{n} \| u_j - \Pi^h u_j \|^2 + k \sum_{j=1}^{n} \left| R_j(w_j, u_j, \Pi^h u_j) \right|$$

$$\leq ck^2 + ckh^2 + ckNh^2 + ckNh. \quad (6.8)$$

Since $kN = T$, we obtain the desired result.

Conclusion. In this paper, we focused on the study of a problem of mechanical contact with wear modeled by Archard's law. In this model,the friction between the body and the foundation can cause the wear of the contact surface of the body over time. The proof of the existence and the uniqueness of the weak solution to the problem was presented in the first part using fixed point arguments. Next, we presented a fully discrete numerical approximation scheme with an error estimation of the solution.

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