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Fixed Variables Generalized Hypersubstitutions

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Abstract

In this paper, we generalize the concept of a term of a fixed variable which was introduced by Wattanatripop and Changphas [8]. A term of a fixed variable is a spacial term of type τ. It turns out that the set of all terms of a fixed variable of type τ forms clone with the universe $W_\tau^{fv}(X)$ and generalized superposition operations. Moreover, generalized hypersubstitution of a fixed variable and the related strong hyperidentities of a fixed variable and strongly solid of a fixed variable are studied.

1 Preliminaries

Let $X := \{x_1, x_2, \ldots\}$ be a countably infinite set of symbols called *variables*. We refer to these variables as letters, to X as an alphabet, and to the set $X_n := \{x_1, x_2, \ldots, x_n\}$ as an n-element alphabet. Let $(f_i)_{i\in I}$ be an indexed set which is disjoint from X. Each f_i is called an n_i-*ary operation symbol*, where $n_i \geq 1$ is a natural number. Let τ be a function which assigns to every f_i the number n_i as its arity. The function τ, on the values of τ written as $(n_i)_{i\in I}$, is called a *type*.

An n-ary term of type τ is defined inductively as follows:

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(i) The variables $x_1, \ldots, x_n$ are n-ary terms.

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term.

We denote by $W_\tau(X_n)$ the smallest set which contains $x_1, \ldots, x_n$ and is closed under a finite number of applications of (ii). Then the set $W_\tau(X) := \cup_{n=1}^{\infty} W_\tau(X_n)$ is the set of all terms of type τ over the alphabet X.

We define the first concept of a generalized superposition of terms

$$S^n : W_\tau(X)^{n+1} \longrightarrow W_\tau(X)$$

by the following steps:

for any term $t \in W_\tau(X)$,

(i) if $t = x_j, 1 \leq j \leq n$, then $S^n(x_j, t_1, \ldots, t_n) := t_j$,

(ii) if $t = x_j, n < j \in \mathbb{N}$, then $S^n(x_j, t_1, \ldots, t_n) := x_j$,

(iii) if $t = f_i(s_1, \ldots, s_{n_i})$, then
$S^n(t, t_1, \ldots, t_n) := f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_{n_i}, t_1, \ldots, t_n))$.

Now, we consider the algebra $clone_g(\tau) := (W_\tau(X); S^n, (x_i)_{i\in\mathbb{N}})$ with the universe $W_\tau(X)$, with the $(n+1)$-ary operation S^n and with infinitely many nullary operations. In 2003, Denecke [1] proved that $clone_g(\tau)$ satisfies (Cg1) – (Cg4):

(Cg1) $\widetilde{S}^n(X_0, \widetilde{S}^n(Y_1, X_1, \ldots, X_n), \ldots, \widetilde{S}^n(Y_n, X_1, \ldots, X_n)) \approx$
$\widetilde{S}^n(\widetilde{S}^n(X_0, Y_1, \ldots, Y_n), X_1, \ldots, X_n)$;

(Cg2) $\widetilde{S}^n(\lambda_i, X_1, \ldots, X_n) \approx X_i$, for $1 \leq i \leq n$;

(Cg3) $\widetilde{S}^n(\lambda_i, X_1, \ldots, X_n) \approx \lambda_i$, for $i > n$;

(Cg4) $\widetilde{S}^n(X_1, \lambda_1, \ldots, \lambda_n) \approx X_1$,

where $\widetilde{S}^n$ are operation symbols corresponding to the operations S^n of $clone_g(\tau)$, $\lambda_1, \ldots, \lambda_n$ are nullary operation symbols, and $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are variables.

A generalized hypersubstitution of type τ is a mapping $\sigma : \{f_i \mid i \in I\} \longrightarrow W_\tau(X)$ which does not necessarily preserve arities. We denote the set of all generalized hypersubstitutions of type τ by $Hyp_G(\tau)$.

Then the generalized hypersubstitution σ can be extended to a mapping

$$\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$$

by the following steps:

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i where $\hat{\sigma}[t_j]$, $1 \leq j \leq n_i$, are already defined.

We define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Let σ_{id} be the hypersubstitution mapping which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$. It turns out that $(Hyp_G(\tau); \circ_G, \sigma_{id})$ is a monoid and the monoid $(Hyp(\tau); \circ_h, \sigma_{id})$ of all arity preserving hypersubstitutions of type τ forms a submonoid of $(Hyp_G(\tau); \circ_G, \sigma_{id})$.

For $t \in W_\tau(X)$, the set of all variables of term t will be denoted by $var(t)$. Therefore, we introduce n-ary terms of a fixed variable of type τ as follows:

Definition 1.1. [8] An n-ary terms of a fixed variable of type τ is inductively defined by:

(i) $x_i \in X_n$ are n-ary terms of a fixed variable; and

(ii) if $t_1, \ldots, t_{n_i}$ are n-ary terms of a fixed variable, and if $var(t_j) = var(t_k)$ for all $1 \leq j \leq k \leq n_i$, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of a fixed variable.

Let $W_\tau^{fv}(X_n)$ be the set of all n-ary terms of a fixed variable of type τ; i.e., $W_\tau^{fv}(X_n)$ contains $x_1, \ldots, x_n$ and is closed under the finite applications of (ii). Then the set $W_\tau^{fv}(X) := \cup_{n=1}^{\infty} W_\tau^{fv}(X_n)$ is the set of all terms of a fixed variable of type τ.

Example 1.1. Let us consider the type $\tau = (2)$ with one binary operation symbol f and set of variable X_2. Then some examples of binary of a fixed variable of type (2) are:

$x_1, x_2, f(x_1, x_1), f(x_2, x_2), f(f(x_1, x_1), x_1), f(f(x_2, x_2), f(x_2, x_2))$.

Consider the type $\tau = (3)$ with one ternary operation symbol f and set of variable X_3. Then some examples of ternary of a fixed variable of type (3) are: $x_1, x_2, x_3, f(x_1, x_1, x_1), f(x_2, x_2, x_2), f(x_1, f(x_1, x_1, x_1), x_1), f(x_2, f(x_2, x_2, x_2), f(x_2, x_2, x_2))$.

Remark 1.2. Observe that, for $t \in W_\tau^{fv}(X)$, $var(t) = \{x_j\}$ for some $x_j \in X$.

Then we show that $W_\tau^{fv}(X)$ is closed under the generalized superposition operations.

Lemma 1.3. For $n \in \mathbb{N}$, if $f_i(t_1, \ldots, t_{n_i}), s_1, \ldots, s_n \in W_\tau^{fv}(X)$, then

$$S^n(f_i(t_1, \ldots, t_{n_i}), s_1, \ldots, s_n) \in W_\tau^{fv}(X).$$

Proof. By the definition of operation S^n, we have

$$S^n\left(f_i(t_1, \ldots, t_{n_i}), s_1, \ldots, s_n\right) = f_i\left(S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_{n_i}, s_1, \ldots, s_n)\right).$$

Then we have to show that equations (i) and (ii) hold:

(i) $S^n(t_1, s_1, \ldots, s_n) \in W_\tau^{fv}(X), \ldots, S^n(t_{n_i}, s_1, \ldots, s_n) \in W_\tau^{fv}(X)$; and

(ii) $var\left(S^n(t_1, s_1, \ldots, s_n)\right) = \cdots = var\left(S^n(t_{n_i}, s_1, \ldots, s_n)\right).$

With the other proofs in (i) done similarly, we will only show that $S^n(t_1, s_1, \ldots, s_n) \in W_\tau^{fv}(X)$.

If $t_1 \in X_n$ and without loss of generality, we put $t_1 = x_1$, then

$$S^n(t_1, s_1, \ldots, s_n) = s_1 \in W_\tau^{fv}(X).$$

If $t_1 \in X \setminus X_n$ and we put $t_1 = x_j$ for $j > n$, then

$$S^n(t_1, s_1, \ldots, s_n) = x_j \in W_\tau^{fv}(X).$$

Assume that $t_1 = f_i\left(p_1, \ldots, p_{n_i}\right)$, where $p_1, \ldots, p_{n_i} \in W_\tau^{fv}(X)$ such that

$$S^n\left(p_1, s_1, \ldots, s_n\right) \in W_\tau^{fv}(X), \ldots, S^n\left(p_{n_i}, s_1, \ldots, s_n\right) \in W_\tau^{fv}(X).$$

Since

$$\begin{aligned} S^n(t_1, s_1, \ldots, s_n) &= S^n(f_i\left(p_1, \ldots, p_{n_i}\right), s_1, \ldots, s_n) \\ &= f_i\left(S^n\left(p_1, s_1, \ldots, s_n\right), \ldots, S^n\left(p_{n_i}, s_1, \ldots, s_n\right)\right) \end{aligned}$$

and

$$var(p_1) = \cdots = var(p_{n_i}),$$

we have

$$var(S^n\left(p_1, s_1, \ldots, s_n\right)) = \cdots = var(S^n\left(p_{n_i}, s_1, \ldots, s_n\right)).$$

Therefore,

$$S^n(t_1, s_1, \ldots, s_n) = S^n(f_i\left(p_1, \ldots, p_{n_i}\right), s_1, \ldots, s_n) \in W_\tau^{fv}(X).$$

From

$$var(t_1) = \cdots = var(t_{n_i}),$$

we have that

$$var\left(S^n(t_1, s_1, \ldots, s_n)\right) = \cdots = var\left(S^n(t_{n_i}, s_1, \ldots, s_n)\right).$$

Thus we complete the proof (ii). □

Using Lemma 1.3, we have mappings

$$S^n : W_\tau^{fv}(X)^{n+1} \longrightarrow W_\tau^{fv}(X)$$

for $n \in \mathbb{N}$. Thus we obtain the algebra

$$clone_{gfv}(\tau) := \left(W_\tau^{fv}(X); S^n, (x_i)_{i\in\mathbb{N}}\right).$$

Moreover, since $W_\tau^{fv}(X) \subseteq W_\tau(X)$, the following theorem follows.

Theorem 1.4. $clone_{gfv}(\tau)$ satisfies $(Cg1) - (Cg4)$.

2 Variable Fixed Generalized Hypersubstitutions

Definition 2.1. A generalized hypersubstitution $\sigma \in Hyp_G(\tau)$ is called an fv-*generalized hypersubstitution* of type τ if, for all $i \in I$, $\sigma(f_i) \in W_\tau^{fv}(X)$.

Let $Hyp_G^{fv}(\tau)$ be the set of all fv-generalized hypersubstitutions of type τ.

Lemma 2.2. If $\sigma \in Hyp_G^{fv}(\tau)$, then $\hat{\sigma} : W_\tau^{fv}(X) \longrightarrow W_\tau^{fv}(X)$.

Proof. Let $\sigma \in Hyp_G^{fv}(\tau)$ and let $t \in W_\tau^{fv}(X)$. We want to show that $\hat{\sigma}[t] \in W_\tau^{fv}(X)$. It is clear that for $t \in X$, $\hat{\sigma}[t] = \hat{\sigma}[x] = x \in W_\tau^{fv}(X)$. Assume that $t = f_i(t_1, \ldots, t_{n_i})$, where $t_1, \ldots, t_{n_i} \in W_\tau^{fv}(X)$ such that $\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}] \in W_\tau^{fv}(X)$.
Since $var(t_1) = \cdots = var(t_{n_i})$, we have

$$var\left(\hat{\sigma}[t_1]\right) = \cdots = var\left(\hat{\sigma}[t_{n_i}]\right).$$

From $\sigma(f_i) \in W_\tau^{fv}(X)$, we have

$$\hat{\sigma}[t] = \hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] = S^n\left(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]\right)$$

is a term of fixed variable. ☐

Theorem 2.3. The extension of $\sigma \in Hyp_G^{fv}(\tau)$ is an endomorphism on $clone_{gfv}(\tau)$.

Proof. Let $\sigma \in Hyp_G^{fv}(\tau)$. By Lemma 2.2, we have

$$\hat{\sigma} : W_\tau^{fv}(X) \longrightarrow W_\tau^{fv}(X).$$

Let $t, s_1, \ldots, s_n \in W_\tau^{fv}(X)$. We want to show by induction on the complexity of t that

$$\hat{\sigma}\left[S^n(t, s_1, \ldots, s_n)\right] = S^n\left(\hat{\sigma}[t], \hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_n]\right).$$

If $t = x_i$ for $1 \leq i \leq n$, then

$$\begin{aligned}
\hat{\sigma}\left[S^n(t, s_1, \ldots, s_n)\right] &= \hat{\sigma}\left[S^n(x_i, s_1, \ldots, s_n)\right] \\
&= \hat{\sigma}[s_i]; \\
S^n\left(\hat{\sigma}[t], \hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_n]\right) &= S^n\left(\hat{\sigma}[x_i], \hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_n]\right) \\
&= S^n\left(x_i, \hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_n]\right) \\
&= \hat{\sigma}[s_i].
\end{aligned}$$

If $t = x_j$ for $j > n$, then

$$\begin{aligned}
\hat{\sigma}\left[S^n(t, s_1, \ldots, s_n)\right] &= \hat{\sigma}\left[S^n(x_j, s_1, \ldots, s_n)\right] \\
&= \hat{\sigma}[x_j] \\
&= x_j; \\
S^n\left(\hat{\sigma}[t], \hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_n]\right) &= S^n\left(\hat{\sigma}[x_j], \hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_n]\right) \\
&= S^n\left(x_j, \hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_n]\right) \\
&= x_j.
\end{aligned}$$

If $t = f_i(t_1, \ldots, t_{n_i})$ where $t_1, \ldots, t_{n_i} \in W_\tau^{fv}(X)$ such that

$$\hat{\sigma}\left[S^n(t_k, s_1, \ldots, s_n)\right] = S^n\left(\hat{\sigma}[t_k], \hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_n]\right)$$

for all $1 \leq k \leq n_i$, then $\hat{\sigma}\left[S^n(t, s_1, \ldots, s_n)\right]$

$$\begin{aligned}
&= \hat{\sigma}\left[S^n(f_i(t_1, \ldots, t_{n_i}), s_1, \ldots, s_n)\right] \\
&= \hat{\sigma}\left[f_i(S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_{n_i}, s_1, \ldots, s_n))\right] \\
&= S^{n_i}\left(\sigma(f_i), \hat{\sigma}\left[S^n(t_1, s_1, \ldots, s_n)\right], \ldots, \hat{\sigma}\left[S^n(t_{n_i}, s_1, \ldots, s_n)\right]\right) \\
&= S^{n_i}\left(\sigma(f_i), S^n\left(\hat{\sigma}[t_1], \hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_n]\right), \ldots, S^n\left(\hat{\sigma}[t_{n_i}], \hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_n]\right)\right) \\
&= S^n\left(S^{n_i}\left(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]\right), \hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_n]\right) \\
&= S^n\left(\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})], \hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_n]\right) \\
&= S^n\left(\hat{\sigma}[t], \hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_n]\right).
\end{aligned}$$

□

3 Variable Fixed Strong Hyperidentities in $clone_{gfv}(\tau)$

We begin this section with the following definition.

Definition 3.1. Let V be a variety of type τ. An identity $s \approx t$ in IdV is said to be a *gfv-identity* of V if $s, t \in W_\tau^{fv}(X)$.

Let $Id^{gfv}V$ be the set of all gfv-identities of V. That is,

$$Id^{gfv}V := \{s \approx t \mid s \approx t \in IdV, s, t \in W_\tau^{fv}(X)\}.$$

The purpose of next theorem is to show that $Id^{gfv}V$ is a congruence on $clone_{gfv}(\tau)$.

Theorem 3.2. Let V be a variety of type τ. Then $Id^{gfv}V$ is a congruence on $clone_{gfv}(\tau)$.

Proof. We will prove that from $r \approx t$, $r_k \approx t_k \in Id^{gfv}V, k = 1, 2, \ldots, n$, there follows $S^n(r, r_1, \ldots, r_n) \approx S^n(t, t_1, \ldots, t_n) \in Id^{gfv}V$. Firstly, we prove by induction on the complexity of a term $t \in W_\tau^{gfv}(X)$ that for every $n \in \mathbb{N}$ from $t_k \approx r_k \in Id^{gfv}V, k = 1, 2, \ldots, n$, there follows $S^n(t, t_1, \ldots, t_n) \approx S^n(t, r_1, \ldots, r_n) \in Id^{gfv}V$.

If $t = x_i$ for $1 \leq i \leq n$, then

$$\begin{aligned} S^n(x_i, t_1, \ldots, t_n) &= t_i \\ &\approx r_i \\ &= S^n(x_i, r_1, \ldots, r_n) \in Id^{gfv}V. \end{aligned}$$

If $t = x_j$ for $j > n$, then

$$\begin{aligned} S^n(x_j, t_1, \ldots, t_n) &= x_j \\ &\approx x_j \\ &= S^n(x_j, r_1, \ldots, r_n) \in Id^{gfv}V. \end{aligned}$$

Assume that $t = f_i(s_1, \ldots, s_{n_i})$ where $s_1, \ldots, s_{n_i} \in W_\tau^{fv}(X)$ such that

$$S^n(s_k, t_1, \ldots, t_n) \approx S^n(s_k, r_1, \ldots, r_n) \in Id^{gfv}V$$

for all $1 \leq k \leq n_i$. Then

$$\begin{aligned} S^n(t, t_1, \ldots, t_n) &= S^n(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_n) \\ &= f_i\left(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_{n_i}, t_1, \ldots, t_n)\right) \\ &\approx f_i\left(S^n(s_1, r_1, \ldots, r_n), \ldots, S^n(s_{n_i}, r_1, \ldots, r_n)\right) \\ &= S^n(f_i(s_1, \ldots, s_{n_i}), r_1, \ldots, r_n) \in Id^{gfv}V. \end{aligned}$$

Consequently, if $t \approx s \in Id^{gfv}V$, then $S^n(r, r_1, \ldots, r_n) \approx S^n(t, t_1, \ldots, t_n) \in Id^{gfv}V$. □

Using Theorem 3.2, for a variety V of type τ we have $clone_{gfv}(V) = clone_{gfv}(\tau)/Id^{gfv}V$.

Definition 3.3. Let V be a variety of type τ. Then $s \approx t \in Id^{gfv}V$ is called a *gfv-strong hyperidentity in V* if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id^{gfv}V$ for every $\sigma \in Hyp_G^{fv}(\tau)$. If every identity in $Id^{gfv}V$ is a gfv-strong hyperidentity, then the variety V is called a *gfv-strongly solid*.

Theorem 3.4. Let V be a variety of type τ. If $Id^{gfv}V$ is a fully invariant congruence relation on $clone_{gfv}(\tau)$, then the variety V is a gfv-strongly solid.

Proof. Let $s \approx t \in Id^{gfv}V$ and let $\sigma \in Hyp_G^{fv}(\tau)$ be a fv-generalized hypersubstitution of type τ. Since by Theorem 2.3 the extension $\hat{\sigma}$ of σ is an endomorphism of $clone_{gfv}(\tau)$, we have that $\hat{\sigma}[t] \approx \hat{\sigma}[s] \in Id^{gfv}V$. Therefore, $s \approx t$ is a gfv-strong hyperidentity in V. So V is a gfv-strongly solid.

□

Recall that, for a variety V of type τ, $Id^{gfv}V$ is a congruence on $clone_{gfv}(\tau)$ by Theorem 3.2. We then form the quotient algebra

$$clone_{gfv}(V) = clone_{gfv}(\tau)/Id^{gfv}V.$$

Note that we have a natural homomorphism

$$nat_{Id^{gfv}V} : clone_{gfv}(\tau) \longrightarrow clone_{gfv}(V)$$

such that

$$nat_{Id^{gfv}V}(t) = [t]_{Id^{gfv}V}.$$

Then we obtain the following theorems.

Theorem 3.5. Let V be a variety of type τ. If $s \approx t \in Id^{gfv}V$ is an identity in $clone_{gfv}(\tau)$, then $s \approx t$ is a gfv-strong hyperidentity in V.

Proof. Let $s \approx t \in Id^{gfv}V$ be an identity in $clone_{gfv}(V)$ and let $\sigma \in Hyp_G^{fv}(\tau)$. Then $\hat{\sigma} : clone_{gfv}(\tau) \longrightarrow clone_{gfv}(\tau)$ is an endomorphism by Theorem 2.3. Thus the natural mapping $nat_{Id^{gfv}V}$

$$nat_{Id^{gfv}V} \circ \hat{\sigma} : clone_{gfv}(\tau) \longrightarrow clone_{gfv}(V)$$

is a homomorphism. Then

$$\begin{aligned} s \approx t \in Id^{gfv}V &\implies nat_{Id^{gfv}V} \circ \hat{\sigma}(s) = nat_{Id^{gfv}V} \circ \hat{\sigma}(t) \\ &\implies nat_{Id^{gfv}V}(\hat{\sigma}[s]) = nat_{Id^{gfv}V}(\hat{\sigma}[t]) \\ &\implies [\hat{\sigma}[s]]_{Id^{gfv}V} = [\hat{\sigma}[t]]_{Id^{gfv}V} \\ &\implies \hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id^{gfv}V. \end{aligned}$$

This means that $s \approx t$ is satisfied as a gfv-strong hyperidentity in V. □

Let V be a variety of type τ. Define

$$HId^{gfv}V := \left\{ s \approx t \mid s, t \in W_\tau^{fv}(X), \hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV \text{ for all } \sigma \in Hyp_G^{fv}(\tau) \right\}.$$

Theorem 3.6. Let V be a variety of type τ. Then $HId^{gfv}V$ is a congruence on $clone_{gfv}(\tau)$.

Proof. Let $s \approx t \in HId^{gfv}V$ and let $s_1 \approx t_1, \ldots, s_n \approx t_n \in HId^{gfv}V$. Then $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id^{gfv}V$ and $\hat{\sigma}[s_1] \approx \hat{\sigma}[t_1], \ldots, \hat{\sigma}[s_n] \approx \hat{\sigma}[t_n] \in Id^{gfv}V$ for all $\sigma \in Hyp_G^{fv}(\tau)$. We will prove that

$$\hat{\sigma}\left[S^n(s, s_1, \ldots, s_n)\right] \approx \hat{\sigma}\left[S^n(t, t_1, \ldots, t_n)\right] \in Id^{gfv}V$$

for all $\sigma \in Hyp_G^{fv}(\tau)$. Since $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id^{gfv}V$, $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$ and $\hat{\sigma}[s], \hat{\sigma}[t] \in W_\tau^{fv}(X)$. Similarly, since $\hat{\sigma}[s_1] \approx \hat{\sigma}[t_1], \ldots, \hat{\sigma}[s_n] \approx \hat{\sigma}[t_n] \in Id^{gfv}V$, we have

$$\hat{\sigma}[s_1] \approx \hat{\sigma}[t_1], \ldots, \hat{\sigma}[s_n] \approx \hat{\sigma}[t_n] \in IdV$$

and

$$\hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_n], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n] \in W_\tau^{fv}(X).$$

From $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$ and $\hat{\sigma}[s_1] \approx \hat{\sigma}[t_1], \ldots, \hat{\sigma}[s_n] \approx \hat{\sigma}[t_n] \in IdV$, we have

$$S^n\left(\hat{\sigma}[s], \hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_n]\right) \approx S^n\left(\hat{\sigma}[t], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]\right) \in IdV.$$

By $\hat{\sigma}[s], \hat{\sigma}[t] \in W_\tau^{fv}(X)$ and $\hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_n], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n] \in W_\tau^{fv}(X)$, it follows that

$$S^n\left(\hat{\sigma}[s], \hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_n]\right), S^n\left(\hat{\sigma}[t], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]\right) \in W_\tau^{fv}(X).$$

Therefore,

$$S^n\left(\hat{\sigma}[s], \hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_n]\right) \approx S^n\left(\hat{\sigma}[t], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]\right) \in Id^{gfv}V.$$

Since $\hat{\sigma}$ is an endomorphism, we have

$$\hat{\sigma}\left[S^n(s, s_1, \ldots, s_n)\right] \approx \hat{\sigma}\left[S^n(t, t_1, \ldots, t_n)\right] \in Id^{gfv}V.$$ □

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