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Regularity on Semigroups of Transformations Preserving an Equivalence Relation and a Cross-Section with Fixed Sets

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Abstract

For a nonempty set X, let ρ be an equivalence relation on X and R a cross-section of the partition X/ρ induced by ρ. The semigroup $T(X,\rho,R)$ contains all transformations on the set X which preserve both ρ and R. For a nonempty subset Y of X, the subsemigroup $T_Y(X,\rho,R)$ of $T(X,\rho,R)$ is defined as: $T_Y(X,\rho,R) =$

$$\{\alpha \in T(X,\rho,R) : x\alpha = x \text{ for all } x \in X \text{ in which } x\rho \cap Y \neq \emptyset\}.$$

In this paper, we present characterizations of the regularity for $T_Y(X,\rho,R)$. Moreover, we enumerate the number of elements in $T_Y(X,\rho,R)$ corresponding to such regularity.

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1 Introduction

Let X be a nonempty set. The semigroup $T(X)$ of full transformations on the set X consists of all functions from X into X itself together with composition as its operation. It is well known that a transformation semigroup plays a crucial role in the study of algebra and semigroup theory. An analogue of Cayley's theorem shows that any semigroup can be realized as a transformation semigroup of some appropriate set. It has a tautological semigroup action on such a set. Many authors studied several subsemigroups of $T(X)$ and important results were discovered in the past few decades. One of interesting subsemigroups of $T(X)$ we will focus in this paper is concerned with an equivalence relation and a cross-section.

Let ρ be an equivalence relation on a given nonempty set X and R a cross-section of the partition X/ρ (i.e., each ρ-class contains exactly one element of R). The subsemigroup $T(X, \rho, R)$ of $T(X)$ is defined as follows:

$$T(X, \rho, R) = \{\alpha \in T(X) : R\alpha \subseteq R \text{ and } (x, y) \in \rho \Rightarrow (x\alpha, y\alpha) \in \rho\}.$$

In 2003, Araújo and Konieczny [1] stated that $T(X, \rho, R)$ is the centralizer of the idempotent transformation with kernel ρ and image R. Moreover, they studied automorphism groups of $T(X, \rho, R)$. Later, in 2004, they [2] also determined the structure of $T(X, \rho, R)$ in terms of Green's relations and described the regularity of $T(X, \rho, R)$. In 2011, Sun et al. [3] considered the natural partial order on $T(X, \rho, R)$ and observed the maximal elements, minimal elements and covering elements in $T(X, \rho, R)$. More recently, in 2019, Sawatraksa et al. [4] investigated the regularity of $T(X, \rho, R)$ consisting of left regular, right regular and completely regular.

From now on, we consider the special subset of $T(X, \rho, R)$ defined by, for a fixed nonempty subset Y of X,

$$T_Y(X, \rho, R) = \{\alpha \in T(X, \rho, R) : x\alpha = x \text{ for all } x \in X \text{ in which } x\rho \cap Y \neq \emptyset\}.$$

It is not hard to verify that $T_Y(X, \rho, R)$ is a subsemigroup of $T(X, \rho, R)$. In fact, it contains an identity function id_X. Moreover, if we take $\rho = X \times X$, then $T_Y(X, \rho, R) = \{id_X\}$. Furthermore, if $\rho = \{(x, x) : x \in X\}$, then $T_Y(X, \rho, R) = Fix(X, Y)$ which is another subsemigroup of $T(X)$ defined by

$$Fix(X, Y) = \{\alpha \in T(X) : y\alpha = y \text{ for all } y \in Y\}.$$

The semigroup $Fix(X, Y)$ was first studied in 2013 by Honyam and Sanwong [5]. They proved that $Fix(X, Y)$ is a regular subsemigroup of $T(X)$ and determined Green's relations and ideals of $Fix(X, Y)$. Consequently, we may consider our semigroup $T_Y(X,\rho,R)$ as a generalization of $Fix(X,Y)$, as well.

In this paper, we investigate the regularity of $T_Y(X,\rho,R)$ consisting of left regular, right regular and completely regular. Further, we provide necessary and sufficient conditions for $T_Y(X,\rho,R)$ to be a regular semigroup. In addition, we enumerate the number of elements in $T_Y(X,\rho,R)$ corresponding to such the regularity we mentioned above.

2 Preliminaries and Notations

In this section, we describe some basic preliminaries and relevant notations used in what follows about algebra and semigroup theory. The reader is referred to [6] and [7] for additional information.

Let S be a semigroup. An element $a \in S$ is said to be *regular* if there exists $x \in S$ such that $a = axa$; *left regular* if there exists $x \in S$ such that $a = xa^2$; *right regular* if there exists $x \in S$ such that $a = a^2x$; and *completely regular* if there exists $x \in S$ in which $a = axa$ and $ax = xa$. Clearly, every completely regular element is regular, left regular and right regular. If all elements in S are regular, then S will be called a *regular semigroup*. Moreover, if all elements in S are left (right, completely) regular, then S is called a *left* (*right, completely*) *regular semigroup*. Furthermore, necessary and sufficient conditions for elements to be completely regular were proved by Petrich and Reilly in 1999 as stated in the following theorem.

Theorem 2.1. [8] Let S be a semigroup and $a \in S$. Then a is completely regular if and only if a is left and right regular.

The regularity of semigroups is a popular topic in the study of semigroup theory as it plays an important role in characterizing Green's relations of semigroups. Many researches dealt with the regularity of several semigroups [9], [10] and [11].

We now introduce some useful notations which are indispensable for this paper.

Let $\mathscr{A}$ and $\mathscr{B}$ be families of sets. If for each set $A \in \mathscr{A}$ there is a set $B \in \mathscr{B}$ such that $A \subseteq B$, then we say that $\mathscr{A}$ *refines* $\mathscr{B}$ and is denoted by $\mathscr{A} \hookrightarrow \mathscr{B}$. For each $\alpha \in T_Y(X,\rho,R)$, denote by $\ker(\alpha)$ the *kernel* of α which is the equivalence relation

$$\ker(\alpha) = \{(x,y) \in X \times X : x\alpha = y\alpha\}.$$

Moreover,

$$\blacktriangledown\alpha = \{(r\rho)\alpha : r \in R\} \text{ and } \blacktriangledown_Y\alpha = \{(r\rho)\alpha : r \in R \text{ and } r\rho \cap Y = \emptyset\}.$$

The following example illustrates the above notation.

Example 2.1. Let $X = \{1, 2, \ldots, 10\}, Y = \{3, 4\}, X/\rho = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}, \{9, 10\}\}$ and $R = \{1, 4, 6, 9\}$. Consider $\alpha \in T_Y(X, \rho, R)$ defined by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 5 & 1 & 2 & 2 & 6 & 8 \end{pmatrix}.$$

It follows that $\ker(\alpha) = \{(1, 6), (6, 1), (2, 7), (7, 2), (2, 8), (8, 2), (7, 8), (8, 7)\} \cup \{(x, x) : x \in X\}$ and

$$\begin{aligned} \blacktriangledown\alpha &= \{(r\rho)\alpha : r \in R\} \\ &= \{(1\rho)\alpha, (4\rho)\alpha, (6\rho)\alpha, (9\rho)\alpha\} \\ &= \{\{1, 2, 3\}, \{4, 5\}, \{1, 2\}, \{6, 8\}\} \end{aligned} \qquad \begin{aligned} \blacktriangledown_Y\alpha &= \{(r\rho)\alpha : r \in R \text{ and } r\rho \cap Y = \emptyset\} \\ &= \{(6\rho)\alpha, (9\rho)\alpha\} \\ &= \{\{1, 2\}, \{6, 8\}\}. \end{aligned}$$

Moreover, we have $\blacktriangledown_Y\alpha \hookrightarrow \blacktriangledown\alpha$.

3 Left, Right and Completely Regular $T_Y(X, \rho, R)$

This section provides the characterizations of left, right and completely regularity for $T_Y(X, \rho, R)$. In general, $T_Y(X, \rho, R)$ is not a left, right and completely regular semigroup as the following example shows.

Example 3.1. Let $X = \{1, 2, \ldots, 10\}, Y = \{1, 2\}, X/\rho = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8\}, \{9, 10\}\}$ and $R = \{1, 4, 7, 9\}$. Consider $\alpha \in T_Y(X, \rho, R)$ defined by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 1 & 3 & 3 & 4 & 5 & 4 & 6 \end{pmatrix}.$$

(i) α is not left regular: To show this, suppose to get a contradiction that α is left regular. Then there is $\beta \in T_Y(X, \rho, R)$ such that $\alpha = \beta\alpha^2$ which leads to $4 = 7\alpha = 7\beta\alpha^2 \in X\alpha^2$ which is the desired contradiction.

(ii) α is not right regular: To show this, suppose to get a contradiction that α is right regular. Then there is $\beta \in T_Y(X, \rho, R)$ such that $\alpha = \alpha^2\beta$ which leads to $4 = 7\alpha = 7\alpha^2\beta = (7\alpha)\alpha\beta = 4\alpha\beta = 1\beta = 1$ which is impossible.

(iii) α is not completely regular: This follows easily from Theorem 2.1.

The following lemma is useful for characterizing the left regularity of elements in $T_Y(X, \rho, R)$. Note that if every equivalence class contains a certain element of Y, then $T_Y(X, \rho, R)$ is a singleton semigroup which is also left regular. To characterize the left regularity, we then consider the case that some ρ-class exists which is disjoint from Y.

Lemma 3.1. Let $\alpha, \beta \in T_Y(X, \rho, R)$. Then $\alpha = \gamma\beta$ for some $\gamma \in T_Y(X, \rho, R)$ if and only if $\blacktriangledown_Y \alpha \hookrightarrow \blacktriangledown \beta$.

Proof. Assume that $\alpha = \gamma\beta$ for some $\gamma \in T_Y(X, \rho, R)$. Let $r \in R$ be such that $r\rho \cap Y = \emptyset$. Then $(r\rho)\alpha = (r\rho)\gamma\beta = ((r\rho)\gamma)\beta$. Since $\gamma \in T_Y(X, \rho, R)$, $(r\rho)\gamma \subseteq s\rho$ for some $s \in R$ and so $(r\rho)\alpha \subseteq (s\rho)\beta \in \blacktriangledown\beta$. Thus $\blacktriangledown_Y \alpha \hookrightarrow \blacktriangledown \beta$.

Conversely, assume that $\blacktriangledown_Y \alpha \hookrightarrow \blacktriangledown \beta$. Define $\gamma \in T_Y(X, \rho, R)$ on each ρ-class as follows. Let $r \in R$. If $r\rho \cap Y \neq \emptyset$, then we define $x\gamma = x$ for all $x \in r\rho$. If $r\rho \cap Y = \emptyset$, then $(r\rho)\alpha \subseteq (s\rho)\beta \subseteq t\rho$ for some $s, t \in R$. For each $x \in r\rho$, choose $y_x \in s\rho$ such that $y_x\beta = x\alpha$ (if $x = r$, we choose $y_x = s$ since $r\alpha = t = s\beta$) and define $x\gamma = y_x$. By the definition of γ, we have $\gamma \in T_Y(X, \rho, R)$. To prove that $\alpha = \gamma\beta$, let $x \in X$. If $x\rho \cap Y \neq \emptyset$, then $x\gamma\beta = (x\gamma)\beta = x\beta = x = x\alpha$. If $x\rho \cap Y = \emptyset$, then $x\gamma\beta = (x\gamma)\beta = y_x\beta = x\alpha$. Therefore, $\alpha = \gamma\beta$. $\square$

As an immediate consequence of Lemma 3.1, we have the following theorem.

Theorem 3.2. Let $\alpha \in T_Y(X, \rho, R)$. Then α is left regular if and only if $\blacktriangledown_Y \alpha \hookrightarrow \blacktriangledown \alpha^2$.

From the characterization for left regularity of elements in $T_Y(X, \rho, R)$ as presented in Theorem 3.2, we obtain necessary and sufficient conditions for which the semigroup $T_Y(X, \rho, R)$ is left regular as follows.

Theorem 3.3. $T_Y(X, \rho, R)$ is a left regular semigroup if and only if one of the following statements hold.

(i) $r\rho \cap Y \neq \emptyset$ for all $r \in R$.

(ii) There is exactly one element $r \in R$ such that $r\rho \cap Y = \emptyset$ and $|r\rho| \leq 2$.

Proof. Assume that $T_Y(X, \rho, R)$ is a left regular semigroup. We consider two cases.

Case 1: There exists $r \in R$ such that $r\rho \cap Y = \emptyset$ and $|r\rho| > 2$. We write $r\rho = \{r, a, b, \ldots\}$ and define $\alpha \in T_Y(X, \rho, R)$ by

$$\alpha = \begin{pmatrix} x & r & a & b \\ x & r & r & a \end{pmatrix}_{x \in X \setminus \{r,a,b\}}.$$

It follows that $r\rho \setminus \{b\} \in \blacktriangledown_Y \alpha$ but $r\rho \setminus \{b\} \nsubseteq A$ for all $A \in \blacktriangledown \alpha^2$ and so $\blacktriangledown_Y \alpha \not\twoheadrightarrow \blacktriangledown \alpha^2$.

Case 2: There are $r, s \in R$ such that $r \neq s$ and $r\rho \cap Y = \emptyset = s\rho \cap Y$. Let $t \in R$ be such that $t\rho \cap Y \neq \emptyset$. Define $\alpha \in T_Y(X, \rho, R)$ by

$$\alpha = \begin{pmatrix} x & r\rho & s\rho \\ x & t & r \end{pmatrix}_{x \in X \setminus (r\rho \cup s\rho)}.$$

It follows that $\{r\} \in \blacktriangledown_Y \alpha$ but $\{r\} \nsubseteq A$ for all $A \in \blacktriangledown \alpha^2$ and so $\blacktriangledown_Y \alpha \not\twoheadrightarrow \blacktriangledown \alpha^2$.

From the two cases, α is not left regular by Theorem 3.2. So condition (i) or (ii) holds.

Conversely, assume that the conditions hold. If $r\rho \cap Y \neq \emptyset$ for all $r \in R$, then $T_Y(X, \rho, R)$ has only one element and thus $T_Y(X, \rho, R)$ is a left regular semigroup. If there is exactly one element $r \in R$ such that $r\rho \cap Y = \emptyset$ and $|r\rho| \leq 2$, then $r\rho = \{r\}$ or $r\rho = \{r, a\}$. For the case $r\rho = \{r\}$, we have that every element in

$$T_Y(X, \rho, R) = \left\{ \begin{pmatrix} x & r \\ x & s \end{pmatrix}_{x \in X \setminus \{r\}} : s \in R \right\}$$

is an idempotent. For the case $r\rho = \{r, a\}$, we obtain that every element in

$$T_Y(X, \rho, R) = \left\{ \begin{pmatrix} x & r & a \\ x & s & b \end{pmatrix}_{x \in X \setminus \{r,a\}} : s \in R \text{ and } b \in s\rho \right\}$$

is an idempotent. Therefore, $T_Y(X, \rho, R)$ is a left regular semigroup. □

We now enumerate the number of elements in a left regular semigroup $T_Y(X, \rho, R)$ as the following theorem shows.

Theorem 3.4. Let $|X| = n$ and $|R| = m$. If $T_Y(X, \rho, R)$ is a left regular semigroup, then

$$|T_Y(X, \rho, R)| = \begin{cases} 1 & \text{if } \forall r \in R, r\rho \cap Y \neq \emptyset; \\ m & \text{if } \exists!\, r \in R, r\rho \cap Y = \emptyset \text{ and } |r\rho| = 1; \\ n & \text{if } \exists!\, r \in R, r\rho \cap Y = \emptyset \text{ and } |r\rho| = 2. \end{cases}$$

Proof. Assume that $T_Y(X,\rho,R)$ is a left regular semigroup. By Theorem 3.3, we have three cases to consider.

Case 1: $r\rho\cap Y\neq\emptyset$ for all $r\in R$. Clearly, $|T_Y(X,\rho,R)|=1$.

Case 2: There exists a unique $r\in R$ such that $r\rho\cap Y=\emptyset$ and $|r\rho|=1$; that is, $r\rho=\{r\}$. So we can write a left regular element $\alpha\in T_Y(X,\rho,R)$ as follows.

$$\alpha=\begin{pmatrix} x & r\\ x & s\end{pmatrix}_{x\in X\setminus\{r\},s\in R}.$$

We can observe that there are m ways to place the image of r. Thus $|T_Y(X,\rho,R)|=m$.

Case 3: There exists a unique $r\in R$ such that $r\rho\cap Y=\emptyset$ and $|r\rho|=2$. For convenience, we let $r\rho=\{r,a\}$, where $r\neq a\in X$. For $\alpha\in T_Y(X,\rho,R)$, we can write

$$\alpha=\begin{pmatrix} x & r & a\\ x & s & b\end{pmatrix}_{x\in X\setminus\{r,a\},\, s\in R,\, b\in s\rho}.$$

Since r is mapped to $s\in R$, the element a has to be mapped to an element in $s\rho$. Therefore, there are $|s\rho|$ ways to place the image of a. From the fact that the image of r would be any element in a cross-section R, we obtain $|T_Y(X,\rho,R)|=\sum\limits_{s\in R}|s\rho|=|X|=n$. □

Before we characterize the right regularity of elements in $T_Y(X,\rho,R)$, we need the following lemma.

Lemma 3.5. Let $\alpha,\beta\in T_Y(X,\rho,R)$. Then $\alpha=\beta\gamma$ for some $\gamma\in T_Y(X,\rho,R)$ if and only if $\ker(\beta)\subseteq\ker(\alpha)$.

Proof. Assume that $\alpha=\beta\gamma$ for some $\gamma\in T_Y(X,\rho,R)$. Let $a,b\in X$ be such that $a\beta=b\beta$. Then $a\alpha=a\beta\gamma=b\beta\gamma=b\alpha$. Hence $\ker(\beta)\subseteq\ker(\alpha)$.

Conversely, assume that $\ker(\beta)\subseteq\ker(\alpha)$. We define $\gamma\in T_Y(X,\rho,R)$ on each ρ-class as follows. Let $r\in R$.

If $r\notin X\beta$ or $r\rho\cap Y\neq\emptyset$, then we define $x\gamma=x$ for all $x\in r\rho$.

If $r\in X\beta$ and $r\rho\cap Y=\emptyset$, then there is $s\in R$ such that $s\beta=r$ and $s\alpha=t$ for some $t\in R$. For this case, we define γ in two steps. Let $x\in r\rho$.

(i) If $x=a\beta\in X\beta$, then we define $x\gamma=a\alpha$.

(ii) If $x\notin X\beta$, then we define $x\gamma=t$.

It is clear that γ is well-defined since $\ker(\beta)\subseteq\ker(\alpha)$. To prove that $\alpha=\beta\gamma$, let $x\in X$. If $(x\beta)\rho\cap Y\neq\emptyset$, then $(x\beta)\beta=x\beta$ which leads to $x\beta=(x\beta)\alpha=x\alpha$ since $\ker(\beta)\subseteq\ker(\alpha)$. So $x\beta\gamma=(x\beta)\gamma=x\beta=x\alpha$. If

$(x\beta)\rho \cap Y = \emptyset$, then there exists $r \in R \cap x\rho$. Thus $r\beta \in R \cap (x\beta)\rho$. It follows that $x\beta\gamma = (x\beta)\gamma = x\alpha$.

Finally, we prove that $\gamma \in T_Y(X, \rho, R)$. If $r \notin X\beta$ or $r\rho \cap Y \neq \emptyset$, then $r\gamma = r$ and $(r\rho)\gamma = r\rho$. If $r = s\beta \in X\beta$ and $r\rho \cap Y = \emptyset$, then $r\gamma = s\alpha = t$. Let $x \in r\rho$. For each $x \notin X\beta$, we have $x\gamma = t \in t\rho$. For each $x = b\beta \in X\beta$, we have $b \in u\rho$ for some $u \in R$. Since $b\beta = x \in r\rho$, $u\beta = r = s\beta$ and so $u\alpha = s\alpha = t$. This implies $x\gamma = b\alpha \in (u\rho)\alpha \subseteq t\rho$. It follows that $\alpha \in T(X, \rho, R)$ and by the definition of γ, we have $\gamma \in T_Y(X, \rho, R)$. □

As an immediate consequence of Lemma 3.5, we have the following theorem.

Theorem 3.6. Let $\alpha \in T_Y(X, \rho, R)$. Then α is right regular if and only if $\ker(\alpha^2) \subseteq \ker(\alpha)$.

The following theorem provides the characterization of the right regularity for the semigroup $T_Y(X, \rho, R)$.

Theorem 3.7. $T_Y(X, \rho, R)$ is a right regular semigroup if and only if one of the following statements hold.

(i) $r\rho \cap Y \neq \emptyset$ for all $r \in R$.

(ii) There is exactly one element $r \in R$ such that $r\rho \cap Y = \emptyset$ and $|r\rho| \leq 2$.

Proof. The concept of the proof is imitated from the proof of Theorem 3.3. In order to verify the conditions (i) and (ii), the functions $\alpha \in T_Y(X, \rho, R)$ constructed in Case 1 and Case 2 in the proof of Theorem 3.3 can be duplicately defined here. Moreover, we can show that $\ker(\alpha^2) \nsubseteq \ker(\alpha)$ and apply Theorem 3.6 to complete the proof of this part. For the converse, as in the proof of Theorem 3.3, we see that all elements in $T_Y(X, \rho, R)$ are precisely idempotents which are also right regular. Hence the right regularity of $T_Y(X, \rho, R)$ is proved. □

By applying Theorems 2.1, 3.2 and 3.6, we obtain the following corollary.

Corollary 3.8. Let $\alpha \in T_Y(X, \rho, R)$. Then α is completely regular if and only if $\blacktriangledown_Y\alpha \hookrightarrow \blacktriangledown\alpha^2$ and $\ker(\alpha^2) \subseteq \ker(\alpha)$.

As direct consequences of Theorems 2.1, 3.3 and 3.7, we have the following corollary.

Corollary 3.9. The following statements are equivalent.

(i) $T_Y(X,\rho,R)$ is a left regular semigroup;

(ii) $T_Y(X,\rho,R)$ is a right regular semigroup;

(iii) $T_Y(X,\rho,R)$ is a completely regular semigroup;

(iv) One of the following statements hold.

(a) $r\rho \cap Y \neq \emptyset$ for all $r \in R$.

(b) There is exactly one element $r \in R$ such that $r\rho \cap Y = \emptyset$ and $|r\rho| \leq 2$.

As in the above corollary, the conditions for being left regular, right regular and completely regular of $T_Y(X,\rho,R)$ are equivalent, we obtain that the cardinalities of left regular, right regular and completely regular $T_Y(X,\rho,R)$ coincide.

Corollary 3.10. Let $|X| = n$ and $|R| = m$. If $T_Y(X,\rho,R)$ is a right (completely) regular semigroup, then

$$|T_Y(X,\rho,R)| = \begin{cases} 1 & \text{if } \forall r \in R, r\rho \cap Y \neq \emptyset; \\ m & \text{if } \exists!\, r \in R, r\rho \cap Y = \emptyset \text{ and } |r\rho| = 1; \\ n & \text{if } \exists!\, r \in R, r\rho \cap Y = \emptyset \text{ and } |r\rho| = 2. \end{cases}$$

4 Regular $T_Y(X,\rho,R)$

In general, $T_Y(X,\rho,R)$ need not be regular. By following Example 3.1, we can construct the function $\alpha \in T_Y(X,\rho,R)$ by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 1 & 3 & 3 & 4 & 5 & 4 & 6 \end{pmatrix}$$

which is not regular. To prove this, suppose to the contrary that α is regular. Then there is $\beta \in T_Y(X,\rho,R)$ such that $\alpha = \alpha\beta\alpha$ which leads to $5 = 8\alpha = 8\alpha\beta\alpha = 5\beta\alpha = (5\beta)\alpha$ and $6 = 10\alpha = 10\alpha\beta\alpha = 6\beta\alpha = (6\beta)\alpha$. It follows that $5\beta = 8$ and $6\beta = 10$. Since $(5,6) \in \rho$, $(8,10) = (5\beta, 6\beta) \in \rho$ which is a contradiction.

In fact, if every equivalence class contains some element of Y, then $T_Y(X,\rho,R)$ has exactly one element and so $T_Y(X,\rho,R)$ is a regular semigroup. So we consider the case where there is some ρ-class $r\rho$ in which $r\rho \cap Y = \emptyset$ when we characterize a regular element of $T_Y(X,\rho,R)$.

Let A be a nonempty subset of X. An equivalence relation ρ on X induces a partition A/ρ of A as follows.

$$A/\rho = \{x\rho \cap A : x \in X \text{ and } x\rho \cap A \neq \emptyset\}.$$

Moreover, we define

$$A_Y = \{x \in X : x\rho \cap Y = \emptyset\}.$$

Then an equivalence relation ρ on X induces an equivalence relation ρ^* on A_Y by

$$\rho^* = \rho \cap (A_Y \times A_Y).$$

In addition, we define the following notation.

$$A/\rho^* = \{x\rho^* \cap A : x \in A_Y \text{ and } x\rho^* \cap A \neq \emptyset\}.$$

Indeed, A/ρ^* can be an empty set whenever $A \cap A_Y = \emptyset$. Moreover, if $A \subseteq A_Y$, an equivalence relation ρ^* on A_Y induces a partition A/ρ^* of A.

Actually, we see that the set A/ρ^* defined above can be rewritten as

$$A/\rho^* = \{r\rho \cap A : r \in R \text{ and } r\rho \cap Y = \emptyset \neq r\rho \cap A\}.$$

We now present the characterization for the regularity of elements in $T_Y(X, \rho, R)$ as follows.

Theorem 4.1. Let $\alpha \in T_Y(X, \rho, R)$. Then α is regular if and only if $X\alpha/\rho^* \subseteq \blacktriangledown\alpha$.

Proof. Assume that α is regular. Then $\alpha = \alpha\beta\alpha$ for some $\beta \in T_Y(X, \rho, R)$. If $X\alpha \cap A_Y = \emptyset$, then $X\alpha/\rho^* = \emptyset \subseteq \blacktriangledown\alpha$. If $X\alpha \cap A_Y \neq \emptyset$, then $X\alpha/\rho^* \neq \emptyset$ and let $r\rho \cap X\alpha \in X\alpha/\rho^*$. Then $r\rho \cap Y = \emptyset$ and $r\rho \cap X\alpha \neq \emptyset$ which leads to $r = s\alpha$ for some $s \in R$. From $\beta \in T_Y(X, \rho, R)$, we obtain that $(r\rho)\beta \subseteq t\rho$ for some $t \in R$. Now, we prove that $r\rho \cap X\alpha = (t\rho)\alpha$. Consider $r = s\alpha = s\alpha\beta\alpha = r\beta\alpha = (r\beta)\alpha \in (t\rho)\alpha$. We get $t\alpha = r$ and thus $(t\rho)\alpha \subseteq r\rho$ since all elements in $(t\rho)\alpha$ belong to the same class. So $(t\rho)\alpha \subseteq r\rho \cap X\alpha$. In other words, if $x\alpha \in r\rho \cap X\alpha$, then $x\alpha\beta = (x\alpha)\beta \in (r\rho)\beta \subseteq t\rho$ and $x\alpha = x\alpha\beta\alpha = (x\alpha\beta)\alpha \in (t\rho)\alpha$. That means $r\rho \cap X\alpha \subseteq (t\rho)\alpha$. It follows that $r\rho \cap X\alpha = (t\rho)\alpha \in \blacktriangledown\alpha$.

Conversely, assume that $X\alpha/\rho^* \subseteq \blacktriangledown\alpha$. We define $\beta \in T_Y(X, \rho, R)$ on each ρ-class as follows. Let $r \in R$.

If $r\rho \cap X\alpha = \emptyset$ or $r\rho \cap Y \neq \emptyset$, we define $x\beta = x$ for all $x \in r\rho$.

If $r\rho \cap X\alpha \neq \emptyset$ and $r\rho \cap Y = \emptyset$, then $r\rho \cap X\alpha \in X\alpha/\rho^* \subseteq \blacktriangledown\alpha$ and thus $r\rho \cap X\alpha = (s\rho)\alpha$ for some $s \in R$. Now, we define β in two steps. Let $x \in r\rho$.

(i) If $x \in X\alpha$, then $x \in r\rho \cap X\alpha = (s\rho)\alpha$. We choose $y_x \in s\rho$ (if $x = r$, we may choose $y_x = s$) such that $y_x\alpha = x$ and define $x\beta = y_x$.

(ii) If $x \notin X\alpha$, then we define $x\beta = s$.

By the construction of β, we have $\beta \in T_Y(X, \rho, R)$ and $x\alpha\beta\alpha = (x\alpha)\beta\alpha = (x\alpha)\alpha = x\alpha$ (if $(x\alpha)\rho \cap Y \neq \emptyset$) and $x\alpha\beta\alpha = (x\alpha)\beta\alpha = y_{x\alpha}\alpha = x\alpha$ (if $(x\alpha)\rho \cap Y = \emptyset$) for all $x \in X$. Therefore, α is regular, as required. $\square$

Note that in the case where every ρ-class is mapped to some ρ-class $s\rho$ such that $s\rho \cap Y \neq \emptyset$, we obtain that $X\alpha/\rho^* = \emptyset$; that is, α is regular by Theorem 4.1.

We now introduce some terminologies which are important for studying the regularity of $T_Y(X, \rho, R)$. An equivalence relation ρ on X is a T^*-*relation* if there is exactly one ρ-class containing more than two elements and other ρ-classes are singletons. If there is $n \geq 1$ such that each ρ-class has at most n elements, then we say that ρ is n-*bounded*.

The following theorem gives a characterization when $T_Y(X, \rho, R)$ is a regular semigroup.

Theorem 4.2. $T_Y(X, \rho, R)$ is regular if and only if one of the following statements hold.

(i) $A_Y = \emptyset$.

(ii) $A_Y \neq \emptyset$; and ρ^* is 2-bounded or a T^*-relation.

Proof. Let $T_Y(X, \rho, R)$ be a regular semigroup. Suppose that $A_Y \neq \emptyset$ and ρ^* is not 2-bounded. This implies that there is $r \in A_Y \cap R$ such that $|r\rho^*| \geq 3$. We write $r\rho^* = \{r, a_1, a_2, \ldots\}$. If $|A_Y \cap R| = 1$, then ρ^* is a T^*-relation. If $|A_Y \cap R| \geq 2$, then let $s \in A_Y \cap R$ be such that $s \neq r$. To prove that $|s\rho^*| = 1$, suppose, on the contrary, that $|s\rho^*| \geq 2$. Write $s\rho^* = \{s, b_1, \ldots\}$. We consider $\alpha \in T_Y(X, \rho, R)$ defined by

$$\alpha = \begin{pmatrix} x & r & a_1 & a_2 & s & b_1 & z \\ x & r & a_1 & a_1 & r & a_2 & r \end{pmatrix}_{x \in X \setminus A_Y,\ z \in A_Y \setminus \{r, a_1, a_2, s, b_1\}}.$$

Observe that $\{r, a_1, a_2\} \in X\alpha/\rho^*$ and either $\blacktriangledown\alpha = \{x\rho, \{r, a_1\}, \{r, a_2\} : x \in X \setminus A_Y\}$ or $\blacktriangledown\alpha = \{x\rho, \{r, a_1\}, \{r, a_2\}, \{r\} : x \in X \setminus A_Y\}$. It follows that $X\alpha/\rho^* \nsubseteq \blacktriangledown\alpha$. By Theorem 4.1, α is not regular which is a contradiction. So $|s\rho^*| = 1$ for all $s \in A_Y \cap R$ and $s \neq r$. Hence ρ^* is a T^*-relation.

Conversely, assume that the conditions hold. If $A_Y = \emptyset$, then $T_Y(X, \rho, R)$ has exactly one element and thus $T_Y(X, \rho, R)$ is a regular semigroup. For the case $A_Y \neq \emptyset$; and ρ^* is 2-bounded or a T^*-relation, let $\alpha \in T_Y(X, \rho, R)$. That $X\alpha/\rho^* \subseteq \blacktriangledown\alpha$ is clear when $X\alpha/\rho^* = \emptyset$. Let $r\rho \cap X\alpha \in X\alpha/\rho^*$. Then $r\rho \cap Y = \emptyset$ and $r\rho \cap X\alpha \neq \emptyset$. Thus $r\rho = r\rho^*$. Now, we consider two cases.

Case 1: ρ^* is 2-bounded. Then $r\rho$ has at most two elements. If $r\rho \cap X\alpha = \{r\}$, then $\emptyset \neq (s\rho)\alpha \subseteq r\rho \cap X\alpha = \{r\}$ for some $s \in R$ and so $r\rho \cap X\alpha = (s\rho)\alpha \in \blacktriangledown\alpha$. If $r\rho \cap X\alpha = \{r, a\}$ where $a \neq r$, then there exists $b \in X$ such that $b\alpha = a$ which leads to $a = b\alpha \in (t\rho)\alpha$ for some $t \in R$. Hence $r\rho \cap X\alpha = (t\rho)\alpha \in \blacktriangledown\alpha$.

Case 2: ρ^* is a T^*-relation. It is clear when $r\rho \cap X\alpha = \{r\}$ as in Case 1. If $r\rho \cap X\alpha$ has at least three elements, then $|r\rho| \geq 3$. Since every ρ^*-class except $r\rho^*$ has only one element, $r\rho \cap X\alpha = (r\rho)\alpha \in \blacktriangledown\alpha$.

From the two cases described above, we obtain α is regular by Theorem 4.1. Therefore, $T_Y(X, \rho, R)$ is a regular semigroup, as required. □

Recall that $A_Y = \{x \in X : x\rho \cap Y = \emptyset\}$. Consider

$$\begin{aligned}(X \backslash A_Y)/\rho &= \{x\rho \cap (X \backslash A_Y) : x \in X \text{ and } x\rho \cap (X \backslash A_Y) \neq \emptyset\} \\ &= \{x\rho : x \in X \text{ and } x\rho \cap Y \neq \emptyset\}.\end{aligned}$$

For a finite set X, we now present the number of elements in a regular semigroup $T_Y(X, \rho, R)$.

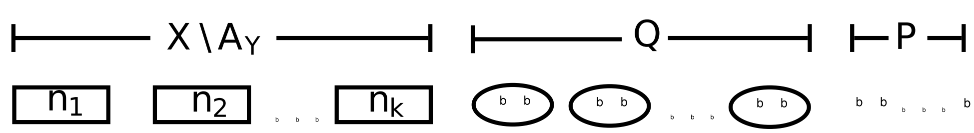
Theorem 4.3. Let $|X| = n, |R| = m$ and $|(X \backslash A_Y)/\rho| = k$ in which $|x_i\rho| = n_i$ for all $x_i \in X \backslash A_Y, i = 1, 2, \ldots, k$. Moreover, let $P = \{x \in A_Y : |x\rho| = 1\}$ with $|P| = p$ and $Q = \{x \in A_Y : |x\rho| \neq 1\}$ with $|Q| = q$. If $T_Y(X, \rho, R)$ is a regular semigroup, then

$$|T_Y(X, \rho, R)| = \begin{cases} 1 & \text{if } A_Y = \emptyset; \\ n^{\frac{q}{2}} m^p & \text{if } A_Y \neq \emptyset \text{ and } \rho^* \text{ is 2-bounded;} \\ (p + q^{q-1} + \sum_{i=1}^{k} n_i^{q-1}) m^p & \text{if } A_Y \neq \emptyset \text{ and } \rho^* \text{ is a } T^*\text{-relation.} \end{cases}$$

Proof. Assume that $T_Y(X, \rho, R)$ is a regular semigroup. By Theorem 4.2, we have three cases to consider.

Case 1: $A_Y = \emptyset$. This implies that $|T_Y(X, \rho, R)| = 1$, immediately.

Case 2: $A_Y \neq \emptyset$ and ρ^* is 2-bounded. Consider the following figure.



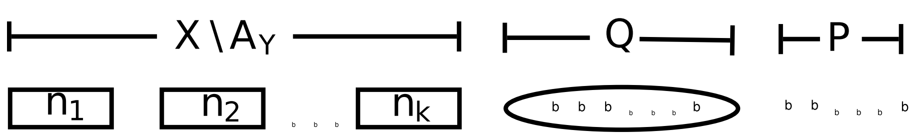
By the definition of $T_Y(X, \rho, R)$, it is obvious that each element in $x_i\rho$ is mapped to itself for all $i = 1, 2, \ldots, k$.

Now, consider each $x\rho$, where $x \in Q$. It is known that $|x\rho| = 2$. Let $r \in R \cap x\rho$. If r is mapped to $s \in R$, then another element in $x\rho$ can be mapped to any element in $s\rho$. Hence there are $\sum\limits_{s \in R} |s\rho| = |X| = n$ possibilities for considering the images of elements in $x\rho$. Since the number of such $x\rho$ where $x \in Q$ is $\frac{q}{2}$, we have $n^{\frac{q}{2}}$ possibilities to construct the images of elements in Q.

Next, consider the possibilities of images of elements in P. Clearly, each element in P has to be mapped to the element in a cross-section R. Thus we have m^p ways to construct the images of elements in P.

Therefore, $|T_Y(X, \rho, R)| = n^{\frac{q}{2}} m^p$, as required.

Case 3: $A_Y \neq \emptyset$ and ρ^* is a T^*-relation. For this case, we will consider only the possibilities of images of elements in Q. The remainder would be similar to the argument mentioned in Case 2. Consider the following figure.



Let $r \in R \cap Q$. Since ρ^* is a T^*-relation, $|r\rho| \geq 3$; that is, $q \geq 3$. If r is mapped to $s \in R$, then the number of possibilities of images of elements in $Q\backslash\{r\}$ are $|s\rho|$ ways. In fact, the class $s\rho$ would be one class in $(X\backslash A_Y)/\rho$ or $\{r\rho\}$ or $\{a\rho : a \in P\}$. Therefore, the number of ways to construct the images of elements in Q is

$$n_1^{q-1} + n_2^{q-1} + \ldots + n_k^{q-1} + q^{q-1} + p.$$

Consequently, $|T_Y(X, \rho, R)| = (p + q^{q-1} + \sum\limits_{i=1}^{k} n_i^{q-1}) m^p$. $\square$

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