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An adaptive RBF Controller for a class of Nonaffine Nonlinear systems

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Abstract

In this paper, an RBF neural adaptive control scheme is developed for a class of uncertain nonaffine nonlinear system. RBF neural network and Nussbaum function have been used to find the unknown part of control system. The Lyapunov theory is used to prove the stability of proposed control law and the update law. A Simulation example is provided to check the effectiveness of controller.

1 Introduction

In the last three decades, several adaptive neural and fuzzy controllers have been intensively conducted for uncertain nonaffine nonlinear systems and some developments have been achieved in the literature [1] - [2] including direct and indirect schemes. Adaptive control for nonaffine nonlinear systems with uncertain parameters are studied by many [3]– [8].

In [3]– [5], a class of SISO nonlinear control systems have been considered in which the unknown function depends on the response and its derivatives. In practice, there are numerous systems which can be represented by this kind of control systems [3]– [4]. Another type of unknown control systems, where the states are input, the response (output) and its derivatives have been considered [5].

In [5], such a controller is designed to track the output of reference signals, where all the

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signals and their derivative are bounded up to the n^{th} order. In a later paper, [6], a similar system was considered with the uncertainties in the unknown part of nonlinear functions. Still later, [7], an adaptive scheme was developed for a control system, where the output was dependent on states, input and derivative, of inputs up to n^{th} order.
In [8], a controller was designed for a class of nonlinear system represented by the input–output model with unmodeled dynamics. The stability of the nonaffine nonlinear system is much more challenging than than that of the linear one. In spite of that, several control systems fall into such category where the structure is nonaffine nonlinear, like chemical system, wind turbine, aircraft flight control and mechanical systems [9]– [11].
In this paper, we have designed a RBFNN based controller for a class of nonaffine nonlinear systems. First, the nonaffine system is transformed into an affine one by using the Mean Value Theorem. Then, an RBF neural network is used to approximate the uncertain affine part with a Nussbaum type function to construct the control and update laws. Using Lyapunov Stability Theorem, we prove that all the signals in the closed loop system are uniformly bounded. Finally, the theoretical results are given through numerical simulation studies.
The rest of the paper is as follows. In section 2, the problem formulation and preliminaries with the description of RBF Neural Network and Nussbaum function are introduced. In section 3, the control law and stability analysis are presented. An example is presented in Section 4 to demonstrate the effectiveness of the proposed controller. Finally, we conclude our paper in section 5.

2 Problem Formulation

Let us take the following single input–single output (SISO) nonlinear nonaffine system

$$y^n \;=\; f\left(Y, U\right), \tag{2.1}$$

where $Y = [y, \dot{y}, \ddot{y}, \cdots, y^{n-1}]$ and $U = [u, \dot{u}, \ddot{u}, \cdots, u^m]$; y is the output and $\dot{y}, \ddot{y}, \cdots$ are the respective order derivatives of y. u is the control input and $\dot{u}, \ddot{u}, \cdots$ are the respective order derivatives of u , f is a nonlinear function that satisfies the following conditions.
Let

$$x_1 = y, x_2 = \dot{y}, \cdots, x_n = y^{(n-1)}$$

$$z_1 = u, z_2 = \dot{u}, \cdots, z_m = u^{(m-1)},$$

then system (2.1) can be written as

$$\begin{aligned} \dot{x_i} &= x_{i+1}, (i = 1, 2, 3, \cdots, n-1) \\ \dot{x_n} &= f(x, z, v) \\ \dot{z}_j &= z_{j+1}, (j = 1, 2, 3, \cdots, m-1) \\ \dot{z}_m &= v, \end{aligned} \tag{2.2}$$

where $v = u^m$ is the control input for the augmented system (2.2) and $x = [x_1, x_2, \cdots, x_n]^T$, $z = [z_1, z_2, \cdots, z_m]^T$ are the states and available for the system. The initial states of the system are chosen such that $z(0) \in Z_0$, $x(0) \in X_0$, where Z_0 & X_0 are the compact subsets of R^m & R^n respectively

By using the Mean Value Theorem, the function f in (2.2) can be transformed to:

$$f(x, z, v) = f(x, z, v^*) + g_\lambda (v - v^*), \tag{2.3}$$

where $g_\lambda = g(x, z, \lambda v + (1-\lambda) v^*), 0 < \lambda < 1$ and $g = \dfrac{\partial f(x, z, v)}{\partial v}, v^* \in V_0 \subset R$. By choosing $v^* = 0$ [12], the function f in (2.3) can be expressed as:

$$f(x, z, v) = f(x, z, 0) + g_\lambda v.$$

Then system (2.2) can be transformed into the form:

$$\begin{aligned} \dot{x}_i &= x_{i+1}, (i = 1, 2, 3, \cdots, n-1) \\ \dot{x}_n &= f(x, z) + g_\lambda v \\ \dot{z}_j &= z_{j+1}, (j = 1, 2, 3, \cdots, m-1) \\ \dot{z}_m &= v. \end{aligned} \tag{2.4}$$

Let $Y_d = \left[y_d, y_d^{(1)}, y_d^{(2)}, \cdots, y_d^{(n-1)} \right]$.

Regarding the construction of the control law, the following assumptions are made:

Assumption 1: The desired signal $y_d(t)$ and its derivatives $y_d^{(n)}(t)$ are smooth and bounded.

Assumption 2: The function $g_\lambda = \dfrac{\partial f}{\partial v}$ is a nonzero function with unknown sign of direction but satisfies $0 < g_{min} \le |g_\lambda| \le g_{max}$, where g_{min} and g_{max} are positive constants.

Let $e = [e_1, e_2, \cdots, e_n]^T$, where $e_1 = x_1 - y_d, e_2 = x_2 - y_d^{(1)}, \cdots, e_n = x_n - y_d^{(n-1)}$. Define a new variable e_s as follows:

$$e_s = \left(\frac{d}{dt} + \delta \right)^{n-1} e_1 = [\Lambda \ \ 1] \, e, \tag{2.5}$$

where δ is positive constant, Λ is a vector and $\Lambda = \left[\delta^{n-1}, (n-1)\delta^{n-2}, \frac{(n-1)(n-2)}{2!}\delta^{n-3}, \cdots, (n-1)\delta\right]^T$.
It has been shown in [13] that $e_s \longrightarrow 0$ as $t \longrightarrow \infty$. Therefore, we can conclude that e and consequently all the derivatives of e up to $(n-1)^{th}$ order converge to zero.
Using (2.4), the time derivative of (2.5) can be written as

$$\dot{e}_s = f(x,z) + \mu + g_\lambda v - y_d^{(n)}, \tag{2.6}$$

where $\mu = [0 \ \ \Lambda]^T e$.

Before starting the controller design, we first introduce the basic approximation property of RBF neural network and Nussbaum type function.

2.1 Radial Basis Function Neural Network

In the field of control system, an RBFNN has been proved to be a universal function approximator, which can uniformly approximate any unknown continuous function defined on compact set up to a small error of tolerance [1]. A neural network with a radial basis function (RBF) can be considered as a two-level network in which a hidden layer performs a fixed nonlinear transformation without configurable parameters to map the input space to the intermediate space, and then the output layer combines the output data of middle layer linearly as the output of entire networks.
Mathematically, let $f : S \longrightarrow R^m$ be a smooth continuous function, where S is a compact set. If the activation function (RBF) is suitably selected, then, for a sufficiently large number of hidden-layer neurons, there exist weights and thresholds such that

$$f(x) = w^T \Psi(x) + \epsilon(x), \tag{2.7}$$

where w, Ψ and ϵ are the weights, Activation function and NN approximation error respectively

2.2 Nussbaum function

To compensate the influence of an unknown direction of controller, we consider the following continuous function (known as Nussbaum function) which satisfies the following two conditions [15]:

$$\lim_{\varsigma \longrightarrow \infty} \sup \frac{1}{\varsigma} \int_0^\varsigma N(\zeta) d\zeta = \infty, \tag{2.8}$$

$$\lim_{\varsigma \longrightarrow \infty} \inf \frac{1}{\varsigma} \int_0^\varsigma N(\zeta) d\zeta = -\infty. \tag{2.9}$$

The continuous functions $t^2 \cos t, e^{t^2} \cos(\frac{\pi}{2}t)$ satisfy the above two conditions and hence they are Nussnaum functions. The function $t^2 \cos t$ has been used to design the controller here.

The following lemma is used to design the controller of the system defined in (2.2).

Lemma 1: [14] Let $V(t)$ and $\zeta(t)$ be the smooth functions defined on $[0, t_f)$ with $V(t) \geq 0, \forall t \in [0, t_f)$ and let $N(\zeta)$ be an Nussbaum function. If the following inequality holds:

$$\begin{aligned} 0 &\leq V(t) \leq c_0 \\ &+ e^{-c_1 t} \int_0^t (H(x(\sigma))N(\zeta) + 1)\, \dot{\zeta} e^{c_1 \sigma} d\sigma, \end{aligned} \tag{2.10}$$

where $c_1 \geq 0$ and $H(x(\sigma))$ is a piecewise continuous time varying function which takes the values in the unknown closed intervals $I = [t^-, t^+]$ with $0 \notin I$ and c_0 is a constant, then $V(t), \zeta(t)$ and $\int_0^t H(x(\sigma))N(\zeta)\dot{\zeta} d\sigma$ are bounded on $[0, t_f)$. If the result of the closed loop system is bounded, then $t_f = \infty$.

3 Controller Design and Stability Analysis

In this section, our main task is to design the controller and the update law of the system (2.2) using Lyapunov Theory.
Consider the following Lyapunov–like function:

$$V(t) = \frac{1}{2} e_s^2 + \frac{1}{2} \tilde{w}^T \Gamma^{-1} \tilde{w}, \tag{3.11}$$

where $\tilde{w} = w^* - \hat{w}$ and $\hat{w}$ is the estimate of w^*, $\Gamma = \Gamma^T > 0$ is any constant matrix. Then the time derivative of V can be written as:

$$\dot{V}(t) = e_s \dot{e}_s - \tilde{w}^T \Gamma^{-1} \dot{\hat{w}}. \tag{3.12}$$

Using (2.6), the equation (3.12) can be written as:

$$\dot{V} = e_s f(x, z) + e_s \mu + e_s g_\lambda v - e_s y_d^{(n)} - \tilde{w}^T \Gamma^{-1} \dot{\hat{w}} \tag{3.13}$$

Since $f(x, z)$ is a smooth function on the compact set $\Omega_{(e+Y_d, z)} \in R^{m+n}$, it can be approximated by RBF neural network as follows:

$$f(x, z) = w^{*T} \Psi(x, z) + \epsilon^*, \tag{3.14}$$

where w^* is the optimal parameter and ϵ^* is the optimal approximation error such that $\forall\epsilon > 0, |\epsilon^*| \leq \epsilon$
Noting $e_s\epsilon \leq \frac{e_s^2}{4} + \epsilon^2$, we have

$$\begin{aligned} \dot{V} &\leq e_s w^{*T}\Psi(x,z) + \frac{e_s^2}{4} + \epsilon^2 + e_s\mu + e_s g_\lambda v \\ &\quad -e_s y_d^{(n)} - \tilde{w}^T\Gamma^{-1}\dot{\hat{w}} \end{aligned} \tag{3.15}$$

Let us define the controller:

$$v = N(\zeta)\dot{\zeta}, \tag{3.16}$$

where $\dot{\zeta} = e_s\hat{w}^T\Psi + ke_s^2 - e_s y_d^{(n)} + e_s\mu$ and k is a positive constant.
From the Lyapunov stability theory, the weight update law for $\hat{w}$ is determined as:

$$\dot{\hat{w}} = \Gamma\left(\Psi e_s - \gamma\hat{w}\right) \tag{3.17}$$

where $\hat{w}$ is the estimate of w^*.

Theorem 1: Consider the closed loop system (2.2) consisting of the system dynamics, the control law (3.16), the update law (3.17) with assumptions 1–2. Given a compact set $\Omega_{(x,z)} \in R^{m+n}$, for any $(x(0), z(0)) \in \Omega_{(x,z)}$, there exist constants k and γ such that all the signals in the closed loop system are bounded.
Proof: Consider the positive definite Lyapunov function V defined in (3.11):

$$V(t) = \frac{1}{2}e_s^2 + \frac{1}{2}\tilde{w}^T\Gamma^{-1}\tilde{w}$$

The time derivative of V is given by (3.15)

$$\begin{aligned} \dot{V} &\leq e_s w^{*T}\Psi(x,z) + \frac{e_s^2}{4} + \epsilon^2 + e_s\mu + e_s g_\lambda v \\ &\quad -e_s y_d^{(n)} - \tilde{w}^T\Gamma^{-1}\dot{\hat{w}}. \end{aligned} \tag{3.18}$$

Substituting (3.16), $\dot{\zeta}$ and (3.17) in the above inequality, then after simplification, we obtain:

$$\begin{aligned} \dot{V} &\leq -(k - \frac{1}{4})e_s^2 + e_s\tilde{w}^T\Psi(x,z) + \epsilon^2 \\ &\quad +(e_s g_\lambda N(\zeta) + 1)\dot{\zeta} - \tilde{w}^T\Gamma^{-1}\dot{\hat{w}} \\ &\leq -(k - \frac{1}{4})e_s^2 + \gamma\tilde{w}^T\hat{w} + \epsilon^2 + (e_s g_\lambda N(\zeta) + 1)\dot{\zeta}. \end{aligned} \tag{3.19}$$

Using $2\tilde{w}^T\hat{w} \leq -\|\tilde{w}\|^2 + \|w^*\|^2$, we have

$$\begin{aligned}\dot{V}(t) &\leq -(k-\frac{1}{4})e_s^2 - \frac{\gamma}{2}(\|\tilde{w}\|^2 - \|w^*\|^2) + \epsilon^2 \\ &\quad +(HN(\zeta)+1)\dot{\zeta} \\ &\leq -(k-\frac{1}{4})e_s^2 - \frac{\gamma}{2\lambda_{max}(\Gamma^{-1})}\tilde{w}^T\Gamma^{-1}\tilde{w} \\ &\quad +\frac{\gamma}{2}\|w^*\|^2 + (HN(\zeta)+1)\dot{\zeta} + \epsilon^2, \qquad (3.20)\end{aligned}$$

where $\lambda_{max}(\Gamma^{-1})$ represents the largest eigenvalue of matrix Γ^{-1} and $H = e_s g_\lambda$. Now, let us define the new constants:

$$\alpha = min\left\{k-\frac{1}{4}, \frac{\gamma}{\lambda_{max}(\Gamma^{-1})}\right\}, \qquad (3.21)$$

$$\beta = \epsilon^2 + \frac{\gamma}{2}\|w^*\|^2. \qquad (3.22)$$

Thus, we have:

$$\dot{V}(t) \leq -\alpha V + \beta + (HN(\zeta)+1)\dot{\zeta}. \qquad (3.23)$$

Multiplying Eq. (3.23) by $e^{\alpha t}$, gives

$$(V(t)e^{\alpha t})' \leq \left[\beta + (HN(\zeta)+1)\dot{\zeta}\right]e^{\alpha t}. \qquad (3.24)$$

Integrating Eq. (3.24) on $[0,t]$, gives

$$V(t) \leq \frac{\beta}{\alpha} + V(0) + e^{-\alpha t}\int_0^t \left[HN(\zeta)+1)\dot{\zeta}\right]e^{\alpha\sigma}d\sigma. \qquad (3.25)$$

By Lemma 1, $\int_0^t \left[HN(\zeta)+1)\dot{\zeta}\right]e^{\alpha\sigma}d\sigma, V(t)$ and $\dot{\zeta}(t)$ must be bounded on $[0,t_f)$. Let A_0 be the upper bound of $\int_0^t \left[HN(\zeta)+1)\dot{\zeta}\right]e^{\alpha\sigma}d\sigma$ over $[0,t_f)$ for all $t_f > 0$. Substituting this bound into (3.25), we obtain

$$\frac{1}{2}e_s^2 \leq V(t) \leq \frac{\beta}{\alpha} + V(0) + A_0. \qquad (3.26)$$

So, we have

$$|e_s| \leq \sqrt{2\left(\frac{\beta}{\alpha} + V(0) + A_0\right)} \qquad (3.27)$$

from Eqs. (3.26) and (3.27), it can be shown that $V(t)$ and e_s are bounded by $\frac{\beta}{\alpha}+V(0)+A_0$ and $\sqrt{2\left(\frac{\beta}{\alpha}+V(0)+A_0\right)}$ respectively. This proves that all the signals involved in the closed loop system (2.2);i.e., x, z and w are bounded. Furthermore, the error e_1 is also bounded. This completes the proof.

4 Simulation Results

This section presents the simulation results to validate the effectiveness of the proposed controller. To illustrate that, consider the following nonlinear nonaffine system:

$$\ddot{y} = (1+y)\,\dot{u} - \dot{y} + y^2 + \sin(\dot{u})u. \tag{4.28}$$

The control objective is to design the controller such that system output $y = x_1(t)$ tracks the desired trajectory $y_d = .1\sin t$.
Let $x_1 = y, x_2 = \dot{y}, z_1 = u, v = \dot{u}$.
Then we have the following equivalent system to (4.28)

$$\begin{aligned} \dot{x_1} &= x_2, & (4.29)\\ \dot{x_2} &= (1+x_1)v - x_2 + x_1^2 + \sin(v)z_1, & (4.30)\\ \dot{z_1} &= v. & (4.31) \end{aligned}$$

In this simulation, we have used one RBF neural network unit to approximate the unknown function $f(x,z)$. Let the input variables of RBF neural network be x_1, x_2 and $x_3 = z_1$. Five Gaussian functions are defined for each input variables with variance 1. The design parameters are selected as: $\delta = 1.3$,$\Gamma = 25I, \gamma = .0001, k = .5$. The initial conditions are $(x(0), z(0)) = [.01, .01, .01]$ and weight vectors are initiated by $w(0) = 0.1$. The initial value of $\zeta = 0$.

The simulation results are shown in Figure 1–2. Fig. 1(a) gives the output of the closed loop system. Fig. 1(b) shows that the output of the closed loop system tracks the reference signal fairly well. Fig. 2(a) is the tracking error and shows that the controller is working fine. Fig. 2(b) shows that the parameter ζ is bounded.

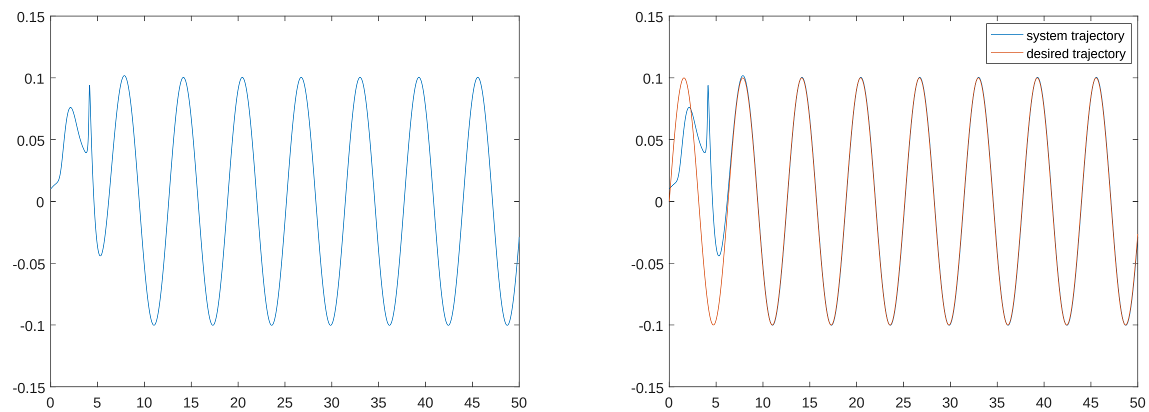

Figure 1: (a) System Trajectory (b) Desired Trajectory vs system trajectory

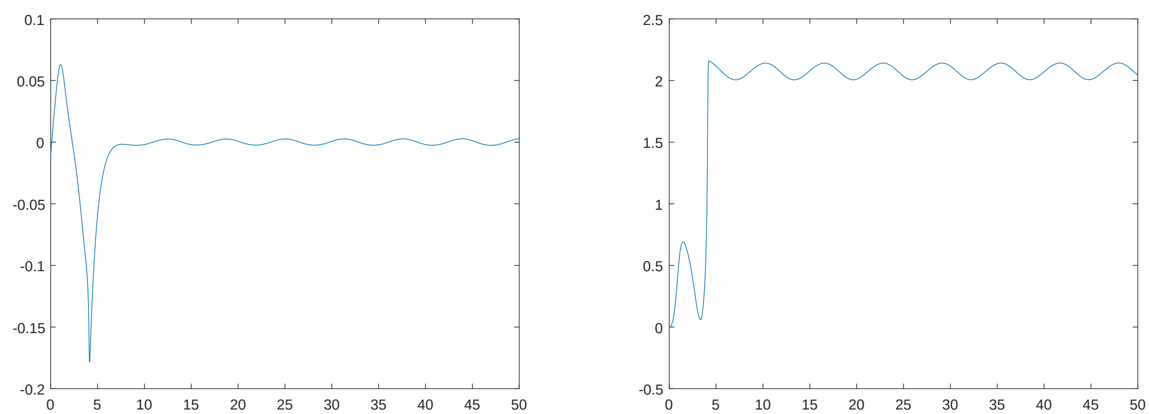

Figure 2: (a) Tracking error (b) The curve of ζ

5 Conclusion

We have constructed an adaptive feedback scheme for a class of SISO nonaffine nonlinear system. The RBF Neural Network and Nussbaum function were used to approximate the unknown nonlinear function. Using Lyapunov Stability Theory, we showed that all the signals in a closed loop system are bounded. Computer simulation studies verify that the designed controller achieves the desired performance fairly well.

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