Some relations among Pythagorean triples

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Abstract

Some relations among Pythagorean triples are established. The main tool is a fundamental characterization of the Pythagorean triples through a cathetus that allows to determine the relationships between two Pythagorean triples with an assigned cathetus $a$ and $b$ and the Pythagorean triple with cathetus $a \cdot b$.

1 Introduction

Let $x$, $y$ and $z$ be positive integers satisfying

$$x^2 + y^2 = z^2.$$

Such a triple $(x, y, z)$ is called a Pythagorean triple and if, in addition, $x$, $y$ and $z$ are co-prime, it is called a primitive Pythagorean triple. First, let us recall a recent novel formula that allows to obtain all Pythagorean triples as follows.

Theorem 1.1. ([1]) $(x, y, z)$ is a Pythagorean triple if and only if there exists $d \in C(x)$ such that

$$x = x, \quad y = \frac{x^2 - d}{2}, \quad z = \frac{x^2 + d}{2}, \quad (1.1)$$

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with $x$ positive integer, $x \geq 1$, and where

$$C(x) = \begin{cases} D(x), & \text{if } x \text{ is odd,} \\ D(x) \cap P(x), & \text{if } x \text{ is even,} \end{cases}$$

with

$$D(x) = \{d \in \mathbb{N} \text{ such that } d \leq x \text{ and } d \text{ divisor of } x^2\} ,$$

and if $x$ is even with $x = 2^n k$, $n \in \mathbb{N}$ and $k \geq 1$ is a fixed odd number, with

$$P(x) = \{d \in \mathbb{N} \text{ such that } d = 2^s l, \text{ with } l \text{ divisor of } x^2 \text{ and } s \in \{1, 2, \ldots, n - 1\}\} .$$

In [2] we found relations between the primitive Pythagorean triple $(x, y, z)$ generated by any predeterminated positive odd integer $x$ using (1.1) and the primitive Pythagorean triple generated by $x^m$ with $m \in \mathbb{N}$ and $m \geq 2$. In [2] we took care of relations only for the case in which the primitive triple $(x, y, z)$ is generated with $d \in C(x)$ only with $d = 1$ and the primitive triple $(x^m, y', z')$ is generated with $d_m \in C(x^m)$ only with $d_m = 1$ obtaining formulas that give us $y'$ and $z'$ directly from $x, y, z$.

**Theorem 1.2.** ([2]) Let $(x, y, z)$ be the primitive Pythagorean triple generated by any predeterminated positive odd integer $x \geq 1$ using (1.1) with $z - y = d = 1$ and let $(x^m, y', z')$ be the primitive Pythagorean triple generated by $x^m$, $m \in \mathbb{N}$, $m \geq 2$, using (1.1) with $z' - y' = d_m = 1$, we have the following formulas

$$y' = y \left[1 + \sum_{p=1}^{m-1} x^{2p}\right],$$

$$z' = y \left[1 + \sum_{p=1}^{m-1} x^{2p}\right] + 1,$$

for every $m \in \mathbb{N}$ and $m \geq 2$.

Moreover, we have

$$z \left[(-1)^{m-1} + \sum_{p=1}^{m-1} (-1)^{m-1-p} x^{2p}\right] = \begin{cases} y' & \text{if } m \text{ is even,} \\ z' & \text{if } m \text{ is odd,} \end{cases}$$

(1.3)

and

$$z \left[(-1)^{m-1} + \sum_{p=1}^{m-1} (-1)^{m-1-p} x^{2p}\right] + (-1)^{m-2} = \begin{cases} z' & \text{if } m \text{ is even,} \\ y' & \text{if } m \text{ is odd.} \end{cases}$$

(1.4)
This was the first step to investigate on other relations between Pythagorean triples.

We want to find relations between the Pythagorean triple \((a, a_1, a_2)\), \((b, b_1, b_2)\), \((a \cdot b, y, z)\) generated by \(a, b, a \cdot b\) respectively using (1.1) with \(a_2 - a_1 = d_1 \in C(a)\), \(b_2 - b_1 = d_2 \in C(b)\), \(z - y = d_3 \in C(a \cdot b)\) to obtain formulas that give us \(y, z\) and \(d_3\) directly from \(a_1, a_2, b_1, b_2, d_1, d_2\).

### 2 Results

The following theorem holds.

**Theorem 2.1.** Let \((a, a_1, a_2)\), \((b, b_1, b_2)\), \((a \cdot b, y, z)\) be the Pythagorean triples generated by \(a, b, a \cdot b\) respectively using (1.1) with \(a_2 - a_1 = d_1 \in C(a)\), \(b_2 - b_1 = d_2 \in C(b)\), \(z - y = d_3 \in C(a \cdot b)\). Then

\[
y = a_1 b_2 + a_2 b_1, \quad z = a_1 b_2 + a_2 b_1 + d_1 d_2,
\]

and moreover,

\[
y = a_1 b_1 + a_2 b_2 - d_1 d_2, \quad z = a_1 b_1 + a_2 b_2,
\]

with \(d_3 = d_1 \cdot d_2 \in C(a \cdot b)\).

**Proof.** To prove (2.1) we verify that

\[
z^2 - y^2 = (a \cdot b)^2,
\]

with \(y\) and \(z\) given in (2.1).

To do this, consider the Pythagorean triples generated by \(a\) and \(b\) respectively using (1.1)

\[
a, \quad a_1 = \frac{a^2 - d_1^2}{2d_1}, \quad a_2 = \frac{a^2 + d_1^2}{2d_1}, \quad d_1 \in C(a),
\]

\[
b, \quad b_1 = \frac{b^2 - d_2^2}{2d_2}, \quad b_2 = \frac{b^2 + d_2^2}{2d_2}, \quad d_2 \in C(b).
\]

Writing (2.3) with \(y\) and \(z\) given from (2.1), we have

\[
(a_1 b_2 + a_2 b_1 + d_1 d_2)^2 - (a_1 b_2 + a_2 b_1)^2 = (a \cdot b)^2;
\]

that is,
\[ d_1^2d_2^2 + 2a_1b_2d_1d_2 + 2a_2b_1d_1d_2 = a^2b^2, \]

and using (2.4) we obtain

\[ d_1^2d_2^2 + 2\frac{a^2 - d_1^2 b^2 + d_1^2}{2d_1}d_1d_2 + 2\frac{a^2 + d_1^2}{2d_1}d_1d_2 = a^2b^2, \]

\[ 2d_1^2d_2^2 + (a^2 - d_1^2)(b^2 + d_2^2) + (a^2 + d_1^2)(b^2 - d_2^2) = 2a^2b^2, \]

from which it is easy to see that

\[ 2a^2b^2 = 2a^2b^2. \]

As a result, (2.3) is an identity with \( y \) and \( z \) given from (2.1) and that the triple

\[ a \cdot b \quad a_1b_2 + a_2b_1 \quad a_1b_2 + a_2b_1 + d_1d_2 \]

is a Pythagorean triple with \( d_3 = (d_1 \cdot d_2) \in C(a \cdot b) \).

Therefore, (2.1) holds.

To prove (2.2), using (2.1) and (1.1) we consider

\[ y = a_1b_2 + a_2b_1 = \frac{(a \cdot b)^2 - (d_1 \cdot d_2)^2}{2d_1d_2} = \frac{a^2b^2 - d_1^2d_2^2}{2d_1d_2} = \frac{(a_2^2 - a_1^2)(b_2^2 - b_1^2) - d_1^2d_2^2}{2d_1d_2} \]

\[ = \frac{d_1(a_2 + a_1)d_2(b_2 + b_1) - d_1^2d_2^2}{2d_1d_2} = \frac{(a_2 + a_1)(b_2 + b_1) - d_1d_2}{2}; \]

that is,

\[ 2a_1b_2 + 2a_2b_1 = a_1b_1 + a_2b_2 + a_1b_2 + a_2b_1 - d_1d_2. \]

Then

\[ y = a_1b_2 + a_2b_1 = a_1b_1 + a_2b_2 - d_1d_2. \]

Since \( z - y = d_1d_2 \),

\[ z = a_1b_1 + a_2b_2. \]

So the triple

\[ a \cdot b \quad y = a_1b_1 + a_2b_2 - d_1d_2 \quad z = a_1b_1 + a_2b_2 \]

is a Pythagorean triple with \( d_3 = (d_1 \cdot d_2) \in C(a \cdot b) \).
Therefore, (2.2) holds as well. Consequently, formulas (2.1) and (2.2) have thus been proved with $d_3 = (d_1 \cdot d_2) \in C(a \cdot b)$.

We note that from (2.1) it is possible to find easily the formulas (2.1) in which $d = d_m = 1$ almost in the case $m = 2$ and $m = 3$ while for $m > 3$ it is more difficult for calculus. In fact, if we consider the Pythagorean triple $(a, a_1, a_1 + 1)$ and having by (1.1) $a_1 = \frac{a^2 - 1}{2}$ from which $2a_1 + 1 = a^2$, then using (2.1) we obtain for $m = 2$

$$y' = a_1b_2 + a_2b_1 = 2a_1(a_1 + 1) = a_1(1 + 1 + 2a_1) = a_1(1 + a^2),$$

$$z' = a_1(1 + a^2) + 1,$$

for $m = 3$

$$y' = a_1b_2 + a_2b_1 = 2a_1^2(1 + a^2) + a_1(1 + a^2) + a_1 = a_1[2a_1(1 + a^2) + (1 + a^2) + 1]$$

$$= a_1[1 + (1 + a^2)(1 + 2a_1)] = a_1[1 + (1 + a^2)a^2] = a_1[1 + a^2 + a^4]$$

$$z' = a_1[1 + a^2 + a^4] + 1,$$

which is formulas (1.2) in the cases $m = 2$ and $m = 3$.

For future work, it may be interesting to study the relationships between Eisenstein Triples after the result found in [3] which gives a characterization of Eisenstein Triples through a side of the triangle.

References


