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A remark on sampling Nymann-Beurling criterion for the Riemann hypothesis

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Abstract

We explain the asymptotic behavior of Nymann-Beurling criterion for the Riemann hypothesis by the method of probability theory.

1 Introduction and result

The Riemann zeta function is a function of a complex number $s = \sigma + ti$, defined on $\sigma > 1$ by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The Riemann zeta function plays an important role in Analytic Number Theory, for instance the prime number theorem. The distribution of prime numbers requires information about the location of non-trivial zeroes of the Riemann zeta-function, which is still an open question in mathematics. The Riemann hypothesis asserts that all non-trivial zeroes of the function ζ lie on the critical line $\{s \in \mathbb{C} | \Re(s) = 1/2\}$ and the Lindelöf hypothesis asserts that for any $\epsilon > 0$, $\zeta\left(1/2 + it\right) = O(t^{\epsilon})$, as t tends to infinity. In 2007, Lifshits and

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Weber [3] investigated the behavior of the Riemann zeta function $\zeta(\frac{1}{2}+it)$, when t is sampled by the Cauchy random walk. They showed that, for any real $b>2$,

$$\sum_{k=1}^{n}\zeta(1/2+iS_k) \overset{\text{(a.s.)}}{=} n+O(n^{1/2}(\log n)^b),$$

with S_k denoting the Cauchy random variables. This indicates that the value of the Riemann zeta-function in the critical line are small on average, which supports the Lindelöf hypothesis. In 2015, Srichan [4] extended the results of Lifshits and Weber to the Dirichlet L-functions and the Hurwitz zeta-function. He showed that the value of that class of zeta-function in the critical line are also small on average, which supports the generalized Lindelöf hypothesis. The work of Nymann and Beurling, et al. stated a criterion of the Riemann hypothesis that the Riemann hypothesis is true if and only if

$$\lim_{N\to\infty}\inf_{A_N}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|1-\zeta\left(1/2+it\right)A_N\left(1/2+it\right)\right|^2\frac{dt}{\frac{1}{4}+t^2}=0, \tag{1.1}$$

where the infimum is taken over all Dirichlet polynomial $A_N:=\sum_{n\leq N}a_n n^{-s}$ of length N with complex coefficients. In 2012, Bettin, Conrey, and Farmer [1] gave a conditional result on the constant in the Báez-Duarte reformulation of the Nymann-Beurling criterion for the Riemann hypothesis. By assuming the Riemann hypothesis and

$$\sum_{0<\Im(\rho)\leq T}\frac{1}{|\zeta'(\rho)|^2}\ll T^{\frac{3}{2}-\delta}$$

for some $\delta>0$, they obtained the result; that is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|1-\zeta(1/2+it)A_N(1/2+it)|^2\frac{dt}{\frac{1}{4}+t^2}\sim\frac{2+\gamma-\log 4\pi}{\log N},$$

by using the Dirichlet polynomial $A_N(s):=\sum_{n=1}^{N}\left(1-\frac{\log n}{\log N}\right)\frac{\mu(n)}{n^s}$. Moreover, using the criterion of Balazard and Saias, Bettin, Conrey, and Farmer deduced that the Riemann hypothesis follows from

$$\lim_{N\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\zeta(1/2+it)A_N(1/2+it)}{\frac{1}{2}+it}\right|^2 dt=1. \tag{1.2}$$

Substituting $t=\frac{x}{2}$, we rewrite (1.2) as

$$\lim_{N\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+\frac{i}{2}x\right)A_N\left(\frac{1}{2}+\frac{i}{2}x\right)\right|^2\frac{dx}{1+x^2}=1. \tag{1.3}$$

Since the distribution density of S_k is $\frac{k}{\pi(k^2+x^2)}$, it would be interesting to investigate the limit (1.3) in sense of the expectation

$$\mathbb{E}\Big|\zeta\left(\frac{1}{2}+i\frac{S_1}{2}\right)A_N\left(\frac{1}{2}+i\frac{S_1}{2}\right)\Big|^2, \tag{1.4}$$

with S_k denoting the Cauchy random variables. From the Nymann-Beurling criterion for the Riemann hypothesis, the expectation in (1.4) is equal to one if and only if the Riemann hypothesis true. However the calculation the expectation in (1.4) need some conditional result as in [1]. Without conditions, we shall study this problem in an easier situation that is similar to (1.4). We study the first moment

$$\mathbb{E}\Big[\zeta\left(\frac{1}{2}+iS_k\right)A_N\left(\frac{1}{2}+iS_k\right)\Big] \tag{1.5}$$

as $k\to\infty$. We expect that the value in (1.5) should be one as $N\to\infty$.

Let $X_1, X_2, ...$ denote an infinite sequence of independent Cauchy distributed random variables. Then the time t is modelled by the sequence of partial sums $S_k = X_1 + X_2 + ... + X_k$.

Here, we investigate the almost sure asymptotic behavior of the system

$$(\zeta A_N)_k := \zeta\left(\frac{1}{2}+iS_k\right)A_N\left(\frac{1}{2}+iS_k\right), \qquad k=1,2,...$$

Our method is based on the classical approximation for the Riemann zeta-function ([5], Theorem 4.11 p. 77) which will be shown in the next section. Thus, the asymptotic behavior of $(\zeta A_N)_k$ follows from an investigation of auxiliary system, for $n=1,2,...,x>0$,

$$Z_x\left(\frac{1}{2}+iS_k\right)A_N\left(\frac{1}{2}+iS_k\right) = \left(\sum_{n\le x}\frac{1}{n^{\left(\frac{1}{2}+iS_k\right)}} - \frac{x^{1-\left(\frac{1}{2}+iS_k\right)}}{1-\left(\frac{1}{2}+iS_k\right)}\right)A_N\left(\frac{1}{2}+iS_k\right).$$

We obtain the following theorem.

Theorem 1.1. *We have*

$$\lim_{k\to\infty}\lim_{N\to\infty}\mathbb{E}\left[\zeta\left(\frac{1}{2}+iS_k\right)A_N\left(\frac{1}{2}+iS_k\right)\right]=1.$$

2 Lemmas

This section consists of lemmas that are useful for the proof of the theorem.

Lemma 2.1. *[5] We have*

$$\zeta(s)=\sum_{n\le x}\frac{1}{n^s}-\frac{x^{1-s}}{1-s}+O(x^{-\sigma})$$

uniformly for $\sigma \geq \sigma_0 > 0$, $|t| < 2\pi x/C$, *when* $C > 1$.

Lemma 2.2. *[Landau's Handbuch [2]] We have*

$$\sum_{n\le x}\frac{\mu(n)}{n}=o(1),$$

as $x \to \infty$.

Lemma 2.3. *[Landau's Handbuch [2]] We have*

$$\sum_{n\le x}\frac{\mu(n)\log n}{n}=-1+o(1),$$

as $x \to \infty$.

In [3], we note that

$$\lim_{x\to\infty}\mathbb{E}\left|Z_x\left(\frac{1}{2}+iS_k\right)-\zeta\left(\frac{1}{2}+iS_k\right)\right|^2=0,$$

Then, $Z_x\left(\frac{1}{2}+iS_k\right)A_N\left(\frac{1}{2}+iS_k\right)$ provides a good approximation to $\zeta\left(\frac{1}{2}+iS_k\right)A_N(\frac{1}{2}+iS_k)$. Then we have

Lemma 2.4. *For each positive integer* k,

$$\lim_{x\to\infty}\mathbb{E}\left|Z_x\left(\frac{1}{2}+iS_k\right)A_N\left(\frac{1}{2}+iS_k\right)-\zeta\left(\frac{1}{2}+iS_k\right)A_N\left(\frac{1}{2}+iS_k\right)\right|=0.$$

3 Proof of Theorem 1.1

Proof. Let $s=\sigma+iS_k$. Write

$$Z_x(\sigma+iS_k)A_N(\sigma+iS_k)=B_1(\sigma+iS_k)-B_2(\sigma+iS_k),$$

where

$$B_1(\sigma+iS_k)=\sum_{n\le x}\sum_{m\le N}a(m)(nm)^{-\sigma-iS_k}=\sum_{n\le x}n^{-\sigma}\sum_{m\le N}\frac{a(m)}{m^\sigma}(nm)^{-iS_k}$$

and

$$B_2(\sigma+iS_k)=\frac{x^{1-\sigma}}{1-(\sigma+iS_k)}\sum_{m\le N}\frac{a(m)}{m^\sigma}(xm)^{-iS_k}=x^{1-\sigma}\sum_{m\le N}a(m)m^{-\sigma}(xm)^{-iS_k}\frac{1}{1-(\sigma+iS_k)}.$$

By the characteristic function of the standard Cauchy distribution

$$\mathbb{E}(nm)^{-iS_k}=\mathbb{E}e^{-iS_k(\log nm)}=(nm)^{-k}.$$

Then, we have the expected value

$$\mathbb{E}B_1(\sigma+iS_k)=\sum_{n\le x}n^{-\sigma}\sum_{m\le N}a(m)m^{-\sigma}\mathbb{E}(nm)^{-iS_k}=\sum_{n\le x}\frac{1}{n^{\sigma+k}}\sum_{m\le N}\frac{a(m)}{m^{\sigma+k}}.$$

By the integral representation $\frac{1}{1-s}=\int_0^1u^{-s}du$, $\Re s<1$, we have

$$\begin{aligned}
\mathbb{E}B_2(\sigma+iS_k)&=x^{1-\sigma}\sum_{m\le N}\frac{a(m)}{m^\sigma}\mathbb{E}\left[(xm)^{-iS_k}\int_0^1u^{-(\sigma+iS_k)}du\right]\\
&=x^{1-\sigma}\sum_{m\le N}\frac{a(m)}{m^\sigma}\int_0^1u^{-\sigma}\mathbb{E}(xmu)^{-iS_k}du\\
&=x^{1-\sigma}\sum_{m\le N}\frac{a(m)}{m^\sigma}\int_0^1u^{-\sigma}e^{-k|\log(xmu)|}du\\
&=x^{1-\sigma}\sum_{m\le N}\frac{a(m)}{m^\sigma}\left[(xm)^k\int_0^{1/xm}u^{k-\sigma}du+(xm)^{-k}\int_{1/xm}^1u^{-k-\sigma}du\right]\\
&=x^{1-\sigma}\sum_{m\le N}\frac{a(m)}{m^\sigma}\left[\frac{(xm)^{\sigma-1}}{k-(\sigma-1)}+\frac{(xm)^{\sigma-1}-(xm)^{-k}}{k+(\sigma-1)}\right]\\
&=\sum_{m\le N}\frac{a(m)}{m^\sigma}\left[\frac{2km^{\sigma-1}}{k^2-(\sigma-1)^2}-\frac{m^{-k}x^{1-\sigma-k}}{k+(\sigma-1)}\right]\\
&=\frac{2k}{k^2-(\sigma-1)^2}\sum_{m\le N}\frac{a(m)}{m}-\frac{x^{1-\sigma-k}}{k+(\sigma-1)}\sum_{m\le N}\frac{a(m)}{m^{\sigma+k}}.
\end{aligned}$$

Then, we have the expected value

$$\mathbb{E}[Z_x(\sigma+iS_k)A_N(\sigma+iS_k)] = \sum_{n\le x}\frac{1}{n^{\sigma+k}}\sum_{p\le N}\frac{a(p)}{p^{\sigma+k}} - \frac{2k}{k^2-(\sigma-1)^2}\sum_{p\le N}\frac{a(p)}{p} + \frac{x^{1-\sigma-k}}{k+(\sigma-1)}\sum_{p\le N}\frac{a(p)}{p^{\sigma+k}}.$$

Taking $\sigma=\frac{1}{2}$, we have

$$\mathbb{E}\left[Z_x\left(\frac{1}{2}+iS_k\right)A_N\left(\frac{1}{2}+iS_k\right)\right] = \sum_{n\le x}\frac{1}{n^{\frac{1}{2}+k}}\sum_{m\le N}\frac{a(m)}{m^{\frac{1}{2}+k}} - \frac{2k}{k^2-\frac{1}{4}}\sum_{m\le N}\frac{a(m)}{m} + \frac{x^{\frac{1}{2}-k}}{k-\frac{1}{2}}\sum_{m\le N}\frac{a(m)}{m^{\frac{1}{2}+k}}.$$

In view of Lemma 2.4, we have

$$\begin{aligned}\mathbb{E}\left[\zeta\left(\frac{1}{2}+iS_k\right)A_N\left(\frac{1}{2}+iS_k\right)\right] &= \lim_{x\to\infty}\mathbb{E}\left[Z_x\left(\frac{1}{2}+iS_k\right)A_N\left(\frac{1}{2}+iS_k\right)\right]\\ &= \zeta\left(\frac{1}{2}+k\right)\sum_{m\le N}\frac{a(m)}{m^{\frac{1}{2}+k}} - \frac{2k}{k^2-\frac{1}{4}}\sum_{m\le N}\frac{a(m)}{m}\\ &= \zeta\left(\frac{1}{2}+k\right)\sum_{m\le N}\frac{\mu(m)}{m^{\frac{1}{2}+k}} - \frac{\zeta\left(\frac{1}{2}+k\right)}{\log N}\sum_{m\le N}\frac{\mu(m)\log m}{m^{\frac{1}{2}+k}}\\ &\quad - \frac{2k}{k^2-\frac{1}{4}}\sum_{m\le N}\frac{\mu(m)}{m} - \frac{2k}{(k^2-\frac{1}{4})\log N}\sum_{m\le N}\frac{\mu(m)\log m}{m}.\end{aligned}$$

In view of

$$\sum_{m\le N}\frac{\mu(m)}{m^{\frac{1}{2}+k}} = \frac{1}{\zeta\left(\frac{1}{2}+k\right)} + O(N^{-k+\frac{1}{2}}), \qquad \sum_{m\le N}\frac{\mu(m)\log m}{m^{\frac{1}{2}+k}} = O(N^{-k+\frac{1}{2}}\log N),$$

by Lemmas 2.2 and 2.3, we have

$$\mathbb{E}\left[\zeta\left(\frac{1}{2}+iS_k\right)A_N\left(\frac{1}{2}+iS_k\right)\right] = 1 + O(N^{-k+\frac{1}{2}}) - \frac{2k}{k^2-\frac{1}{4}}\Big(o(1)+\frac{1+o(1)}{\log N}\Big).$$

Taking $N\to\infty$, we have

$$\lim_{N\to\infty}\mathbb{E}\left[\zeta\left(\frac{1}{2}+iS_k\right)A_N\left(\frac{1}{2}+iS_k\right)\right] = 1 - o(1)\frac{2k}{k^2-\frac{1}{4}}.$$

Thus, as $k \to \infty$,

$$\lim_{k\to\infty} \lim_{N\to\infty} \mathbb{E}\left[\zeta\left(\frac{1}{2}+iS_k\right) A_N\left(\frac{1}{2}+iS_k\right)\right] = 1.$$

This completes the proof of the Theorem. □

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