International Journal of Mathematics and
Computer Science, **16**(2021), no. 1, 1–8

An Algorithm for Computing Least-squares Function-valued Padé-type Approximation

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(Received March 2, 2020, Accepted May 13, 2020)

Abstract

The computational problem of a special determinant is investigated. The determinant appears in the construction of the least-squares function-valued Padé-type approximation for computing Fredholm integral equations of the second kind. Here a common computational approach of determinant cannot be used. The Compact Recursive Projection Algorithm is mainly used in this paper. Finally, an example is given to illustrate that the method is effective and stable.

1 Introduction

Consider Fredholm integral equation of the second kind

$$f(s,\lambda) = y(s) + \lambda \int_a^b K(s,t) f(t,\lambda)dt,\ (a \le s,\ t \le b), \tag{1.1}$$

where $y(s) \in L^2[a,b]$ and $K(s,t)$ is an L^2 kernel which are defined in $[a,b]$ and $[a,b] \times [a,b]$, respectively.

The technique utilized for solving the integral equation is based on successive substitution which is an iterative procedure, yielding a sequence of approximation leading to an infinite power series solutions. So we turn to

Key words and phrases: Orthogonal polynomial, Function-valued Padé-type approximation, Fredholm integral equation, Compact recursive projection algorithm.
AMS (MOS) Subject Classifications: 35B27.
ISSN 1814-0432, 2021, http://ijmcs.future-in-tech.net

consider the generating function $f(s,\lambda)$ of this kind of series of function given by

$$f(s,\lambda) = y_0(s) + y_1(s)\lambda + \cdots + y_n(s)\lambda^n + \cdots , \tag{1.2}$$

where

$$y_i = y_i(s) = \int_a^b K^i(s,t)y(t)dt, \ (i \geq 1), \tag{1.3}$$

In (1.3), $K^i(s,t)$ is called the kernel. In this paper, suppose that $f(s,\lambda)$ is analytic as a function of λ at the $\lambda = 0$. Thus, the above series converges for values of $|\lambda|$ which are sufficiently small. At the same time, suppose that $y_i(s) = y_i \in L^2[a,b]$ and their inner product be defined as

$$(y_i, y_j) = y_{i,j} = \int_a^b y_i(s)y_j(s)ds \tag{1.4}$$

In general, the solution of Fredholm integral equation of the second kind is usually expressed as a non-linear function (or rational function) with poles. So, Padé approximation methods are used by some authors obtaining the approximate solution of integral equations [1, 2]. Recently, the method of function-valued Padé-type approximation ($FPTA$) [3, 4] was applied to solve the integral equations (1.1) or to accelerate the convergence of the power series (1.2) and to estimate the characteristic value of the integral equations. However, function-valued Padé-type approximation can be quite sensitive to perturbation on the coefficients of the power series. In order to fill up this gap, least-squares function-valued Padé-type approximation ($LSFPTA$) [5] was constructed. To improve its computation, an efficient Compact Recursive Projection Algorithm is presented by means of Sylvester identity equation [6]. An example of Fredholm integral equation is given to illustrate that the method is effective and stable.

2 Least-squares Orthogonal Polynomials

We now give the definition of $LSFPTA$ [5] which is an extension of that of $FPTA$ [3].

Let P denote the set of scalar polynomials in one real variable whose coefficients belong to the complex fields C and P_k denote the set of elements of P of degree less than or equal to k.

Let $\phi^{(l)} : P \longrightarrow C$, $l = m - n + 1$ be a linear functional on P, only acting on x, defined by $\phi^{(m-n+1)}(x^k) = y_{l+k}(s)$, $k = 0, 1, \cdots$.

Let $v_n \in P_n$ be a scalar polynomial of degree n, $v_n(\lambda) = b_0 + b_1\lambda + \cdots + b_n\lambda^n$, $\tilde{v}_n(\lambda) = \lambda^n v_n(\lambda^{-1})$ and assume that the coefficient $b_n \neq 0$. Define the function-valued coefficients polynomial W_l by,

$$W_l(s,\lambda) = \phi^{(l)} \frac{v_n(x) - v_n(\lambda)}{x - \lambda} \tag{2.5}$$

Set

$$\tilde{W}_l(s,\lambda) = \lambda^{n-1} W_l(s,\lambda^{-1}) \tag{2.6}$$

and

$$P_{mn}(s,\lambda) = \tilde{v}_n(\lambda) \sum_{i=0}^{m-n} y_i(s)\lambda^i + \lambda^l \tilde{W}_l(s,\lambda).$$

It can be proved that if $\tilde{v}_n(0) \neq 0$, then

$$P_{mn}(s,\lambda)/\tilde{v}_n(\lambda) - f(s,\lambda) = O(\lambda^{m+1}) \tag{2.7}$$

In this way, the rational function $R_{m,n}(s,\lambda) = P_{mn}(s,\lambda)/v_n(\lambda)$ is defined as a $LSFPTA$ of type (m/n) for the given series $f(s,\lambda)$ and is denoted by $(m/n)_f^{LS}(s,\lambda)$. The polynomial $v_n(\lambda)$ in (2.7) is called a least-squares orthogonal polynomial with respect to the generalized linear functional $\phi^{(m-n+1)}$. Next, we shall give a determinant expression of least-squares orthogonal polynomial $v_n(\lambda)$ in (2.6).
Let

$$A_n = \begin{bmatrix} (r_1,r_1) & (r_2,r_1) & \cdots & (r_n,r_1) \\ (r_1,r_2) & (r_2,r_2) & \cdots & (r_n,r_2) \\ \vdots & \vdots & \ddots & \vdots \\ (r_1,r_n) & (r_2,r_n) & \cdots & (r_n,r_n) \end{bmatrix}. \tag{2.8}$$

Theorem 2.1 [5]

$$v_n(\lambda) = det \begin{bmatrix} 1 & \eta^{(n)} \\ \theta^{(n)} & A_n \end{bmatrix} /detA_n \tag{2.9}$$

with

$$r_i = (y_{i,l}, y_{i+1,l}, \cdots, y_{i+h,l}),\ l = m-n+1,\ h \geq n,$$

$$(r_i, r_j) = y_{i,l}y_{j,l} + y_{i+1,l}y_{j+1,l} + \cdots + y_{i+h,l}y_{j+h,l},$$

$$\theta^{(n)} = ((r_0,r_1),(r_0,r_2),\cdots,(r_0,r_n))^T, \eta^{(n)} = (\lambda,\lambda^2,\lambda^3,\cdots,\lambda^n).$$

However, it is very difficult to compute a high order $LSFPTA$ by using determinant (2.9). The aim of the paper is to establish an efficient recursive algorithm for $v_n(\lambda)$. Therefore, we need the following well-known result.

Lemma 2.2 (Sylvester Identity Equation [6]).
Set

$$A = \begin{bmatrix} a_{0,1} & a_{0,2} & \cdots & a_{0,n} \\ a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \end{bmatrix} and\, B = \begin{bmatrix} a_{0,1} & a_{0,2} & \cdots & a_{0,n-1} \\ a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-2,1} & a_{n-2,2} & \cdots & a_{n-2,n-1} \end{bmatrix}$$

Then $detB \cdot detB^{'} = detA \cdot detC - detA^{'} \cdot detC^{'}$, where

$$A^{'} = \begin{bmatrix} u^{'} & B \\ a_{n-1,1} & w \end{bmatrix}, C = \begin{bmatrix} x_0 & x^{'} \\ u^{'} & B \end{bmatrix}, C^{'} = \begin{bmatrix} x^{'} & x_n \\ B & u^{''} \end{bmatrix}, B^{'} = \begin{bmatrix} x_0 & x \\ u & A \end{bmatrix},$$

$$x = (x_1, x_2, \cdots x_n), x^{'} = (x_1, x_2, \cdots x_{n-1}), u = (a_{0,0}, a_{1,0}, \cdots a_{n-1,0})^T,$$

$$u^{'} = (a_{0,0}, a_{1,0}, \cdots a_{n-2,0})^T, u^{''} = (a_{0,n}, a_{1,n}, \cdots a_{n-2,n})^T, w = (a_{0,0}, a_{n-1,1}, \cdots a_{n-1,n-1}).$$

3 Compact Recursive Projection Algorithm

Definition 3.1 Let $v_j(\lambda) = b_0 + b_1\lambda + b_2\lambda^2 + \cdots + b_j\lambda^j$ be an arbitrary polynomial of degree j and $r_i \in C^{n\times l}(i = 0, 1, \cdots)$ be a vector, compact projection operator is defined by $[r_i, v_j(\lambda)] = b_0(r_i, r_0) + b_1(r_i, r_1) + b_2(r_i, r_2) + \cdots + b_j(r_i, r_j)$.

Example 3.2 Let$v_3(\lambda) = b_0 + b_1\lambda + b_2\lambda^2 + b_3\lambda^3$. By definition 3.1, we have

$$[r_i, v_3(\lambda)] = b_0(r_i, r_0) + b_1(r_i, r_1) + b_2(r_i, r_2) + b_3(r_i, r_3).$$

Lemma 3.3 Let $v_k^{(i)} = M_k^{(i)}/N_k^{(i)}$ be a sequence of different orthogonal polynomials, where $N_k^{(i)}$ and $M_k^{(i)}$ be defined as follows: $i = 0, 1, \cdots, n, k = 1, 2, \cdots$.

$$N_k^(i) = det \begin{bmatrix} \lambda^i & \lambda^{i+1} & \cdots & \lambda^{i+k} \\ (r_1, \lambda^i) & (r_1, \lambda^{i+1}) & \cdots & (r_1, \lambda^{i+k}) \\ \vdots & \vdots & \ddots & \vdots \\ (r_k, \lambda^i) & (r_k, \lambda^{i+1}) & \cdots & (r_k, \lambda^{i+k}) \end{bmatrix},$$

$$M_k^{(i)} = det \begin{bmatrix} (r_1, \lambda^{i+1}) & (r_1, \lambda^{i+2}) & \cdots & (r_1, \lambda^{i+k}) \\ (r_2, \lambda^{i+1}) & (r_2, \lambda^{i+2}) & \cdots & (r_2, \lambda^{i+k}) \\ \vdots & \vdots & \ddots & \vdots \\ (r_k, \lambda^{i+1}) & (r_k, \lambda^{i+2}) & \cdots & (r_k, \lambda^{i+k}) \end{bmatrix}.$$

then $v_n^{(0)} = v_n(\lambda)$.

Proof. Use the definition of compact projection operator and theorem 2.1.

Lemma 3.4 Let $M_k^{(i-1)}$, $M_{k-1}^{(i+1)}$, $M_k^{(i-1)}$ and $M_{k-1}^{(i)}$ be defined in lemma 3.3, respectively. Then

$$\frac{[r_k, v_{k-1}^{(i)}]}{[r_k, v_{k-1}^{(i+1)}]} = \frac{M_k^{(i-1)} M_{k-1}^{(i+1)}}{M_k^{(i)} M_{k-1}^{(i)}}. \tag{3.10}$$

Proof. By using the properties of determinant, we deduce that

$$\begin{aligned} [r_k, v_{k-1}^{(i)}] &= [r_k, \frac{N_{k-1}^{(i)}}{M_{k-1}^{(i)}}] \\ &= \frac{1}{M_{k-1}^{(i)}} det \begin{bmatrix} (r_k, \lambda^i) & (r_k, \lambda^{i+1}) & \cdots & (r_k, \lambda^{i+k-1}) \\ (r_1, \lambda^i) & (r_1, \lambda^{i+1}) & \cdots & (r_1, \lambda^{i+k-1}) \\ \vdots & \vdots & \ddots & \vdots \\ (r_{k-1}, \lambda^i) & (r_{k-1}, \lambda^{i+1}) & \cdots & (r_{k-1}, \lambda^{i+k-1}) \end{bmatrix} \\ &= \frac{(-1)^{k-1}}{M_{k-1}^{(i)}} det \begin{bmatrix} (r_1, \lambda^i) & (r_1, \lambda^{i+1}) & \cdots & (r_1, \lambda^{i+k-1}) \\ (r_2, \lambda^i) & (r_2, \lambda^{i+1}) & \cdots & (r_2, \lambda^{i+k-1}) \\ \vdots & \vdots & \ddots & \vdots \\ (r_k, \lambda^i) & (r_k, \lambda^{i+1}) & \dots & (r_k, \lambda^{i+k-1}) \end{bmatrix} = \frac{(-1)^{k-1} M_k^{(i-1)}}{M_{k-1}^{(i)}}. \end{aligned} \tag{3.11}$$

Similarly,

$$[r_k, v_{k-1}^{(i+1)}] = \frac{(-1)^{k-1} M_k^{(i)}}{M_{k-1}^{(i+1)}} \tag{3.12}$$

Comparing (3.11) with (3.12), we get

$$\frac{[r_k, v_{k-1}^{(i)}]}{[r_k, v_{k-1}^{(i+1)}]} = \frac{M_k^{(i-1)} M_{k-1}^{(i+1)}}{M_k^{(i)} M_{k-1}^{(i)}}$$

Theorem 3.5 If $M_k^{(i)}M_{k-1}^{(i)} \neq 0$ $(k = 1, 2, \cdots, i = 0, 1, \cdots, n)$ in (3.10), then

$$v_k^{(i)} = v_{k-1}^{(i)} - \frac{[r_k, v_{k-1}^{(i)}]}{[r_k, v_{k-1}^{(i+1)}]} v_{k-1}^{(i+1)}, \tag{3.13}$$

where $v_0^{(i)} = \lambda^{(i)}$.

Proof. By using lemma 2.2, we deduce that

$$N_k^{(i)} M_{k-1}^{(i)} = N_{k-1}^{(i)} M_k^{(i)} - N_{k-1}^{(i+1)} M_k^{(i-1)} \tag{3.14}$$

according to $M_k^{(i)}M_{k-1}^{(i)} \neq 0$, we obtain

$$\frac{N_k^{(i)}}{M_k^{(i)}} = \frac{N_{k-1}^{(i)}}{M_{k-1}^{(i)}} - \frac{M_k^{(i-1)} N_{k-1}^{(i+1)}}{M_{k-1}^{(i)} M_k^{(i)}}$$

By using the definition of lemma 3.3, we have

$$\frac{N_k^{(i)}}{M_k^{(i)}} = \frac{N_{k-1}^{(i)}}{M_{k-1}^{(i)}} - \frac{M_k^{(i-1)} M_{k-1}^{(i+1)} v_{k-1}^{(i+1)}}{M_{k-1}^{(i)} M_k^{(i)}}.$$

Thus, we derive from lemma 3.3 and lemma 3.4 that

$$v_k^{(i)} = v_{k-1}^{(i)} - \frac{M_k^{(i-1)} M_{k-1}^{(i+1)}}{M_{k-1}^{(i)} M_k^{(i)}} v_{k-1}^{(i+1)}, \textit{and} \;\; v_k^{(i)} = v_{k-1}^{(i)} - \frac{[r_k, v_{k-1}^{(i)}]}{[r_k, v_{k-1}^{(i+1)}]} v_{k-1}^{(i+1)} \tag{3.15}$$

Note : Compute structure charts for $v_k^{(i)}$:

$$\begin{array}{ccccccc}
1 & \lambda & \lambda^2 & \lambda^3 & \cdots & \lambda^{n-1} & \lambda^n \\
v_1^{(0)} & v_1^{(1)} & v_1^{(2)} & v_1^{(3)} & \cdots & v_1^{(n-1)} & \\
v_2^{(0)} & v_2^{(1)} & v_2^{(2)} & \cdots & v_2^{(n-1)} & & \\
\vdots & \vdots & \vdots & \vdots & & & \\
v_{n-2}^{(0)} & v_{n-2}^{(1)} & v_{n-2}^{(2)} & & & & \\
v_{n-1}^{(0)} & v_{n-1}^{(1)} & & & & & \\
v_n^{(0)} & & & & & &
\end{array}$$

We now give the complete algorithm to compute least-squares function-valued Padé-type approximants $(m/n)_f^{LS}(s, \lambda)$ for the power series $f(s, \lambda)$.

Algorithm 3.6 :
Step 1: Compute (r_i, r_j) by (2.9), $i, j = 0, 1, 2, \cdots, n$.
Step 2: Compute $v_n^{(0)}(\lambda) = v_n(\lambda)$ by process of structure charts.
Step 3: Compute the denominator polynomial $\tilde{v}_n(\lambda) = \lambda^n v_n(\lambda^{-1})$ of $LSFPA$ for $f(s, \lambda)$.
Step 4: Compute numerator polynomial $P_{m,n}(s, \lambda)$ of $LSFPA$ by (5,6).
Step 5: Obtain $(m/n)_f^{LS} f(s, \lambda)$ for $f(s, \lambda)$.

$$(m/n)_f^{LS} f(s, \lambda) = \frac{P_{m,n}(s, \lambda)}{\tilde{v}_n(\lambda)}.$$

Example 3.7 Consider the following Fredholm integral equation of the second kind

$$\phi(s) = 1 + \lambda \int_0^1 [1 + |s - y|]\phi(y)dy. \tag{3.16}$$

Its kernel is $K(s, y) = 1 + |s - y|, 0 \leq s, y \leq 1$.
The exact solution of the equation is given by

$$\phi(s) = \frac{2cosh(s - \frac{1}{2})}{2cosh\frac{1}{2}v - 3vsinh\frac{1}{2}v} \tag{3.17}$$

where $v = \sqrt{2\lambda}$.

Solution: The first few terms of the power series for $f(s, \lambda)$ are

$$f(s, \lambda) = \phi(s) = 1 + [\frac{5}{4} + (s - \frac{1}{2})^2]\lambda + [\frac{161}{96} + \frac{5}{4}(s - \frac{1}{2})^2 + \frac{1}{6}(s - \frac{1}{4})^4]\lambda^2 + \cdots.$$

By Algorithm 3.6, we get

$$(2/2)_f^{LS}(s, \lambda) = \frac{P_{2,2}(s, \lambda)}{\tilde{v}_2(\lambda)} = \frac{1 + a(s) + b(s)\lambda^2}{1 - 2.675\lambda + 1.788\lambda^2},$$

with $a(s) = [-1.425 + (s - 0.5)^2]$, $b(s) = [0.122 - 1.425(s - 0.5)^2 + 0.167(s - 0.5)^4]$.

It is easily verified that $f(s, \lambda) - (2/2)_f^{LS}(s, \lambda) = O(\lambda^5)$.

Acknowledgement. This work was supported by Changzhou Key Laboratory of High Technology (CM20183004) and Scientific research platform of Changzhou College of Information Technology (KYPT201801G).

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