International Journal of Mathematics and
Computer Science, **16**(2021), no. 1, 459–469

The Algebraic Structures of Quantifier Free Formulas Induced by Terms of a Fixed Variable

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(Received August 31, 2020, October 1, 2020)

Abstract

Quantifier free formulas are significant tools for describing properties of algebraic systems. Based on the concept of terms of a fixed variable, we introduce a new idea of quantifier free formulas defined by such terms, called quantifier free formulas of a fixed variable. The generalized formula-term clone of a fixed variable is constructed by using the generalized superposition. Finally, the set of all mapping which map an operation symbol to a term of fixed variable and a relation symbol to quantifier free formulas of a fixed variable together with one associative binary operation form a semigroup.

1 Introduction and Preliminaries

Let $X := \{x_1, x_2, \ldots\}$ be a countably infinite set of symbols called *variables.* We often refer to these variables as letters to X as an alphabet, and also refer to the set $X_n := \{x_1, x_2, \ldots, x_n\}$ as an n-element alphabet. Let $(f_i)_{i\in I}$ be an indexed set which is disjoint from X. Each f_i is called an n_i-ary operation symbol, where $n_i \geq 1$ is a natural number. Let τ be a function which assigns

Key words and phrases: Term, clone, Menger algebra, hypersubstitution, hyperidentities,Quantifier free Formula.
AMS (MOS) Subject Classifications: 08A62, 08B15, 20M10.
ISSN 1814-0432, 2021, http://ijmcs.future-in-tech.net

to every f_i the number n_i as its arity. The sequence of the values of function τ, written as $(n_i)_{i\in I}$, is called a *type.* An *n-ary term of type* τ is defined inductively as follows :

(i) Every variable $x_j \in X_n$ is an n-ary term of type τ.

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms of type τ and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of type τ.

The set of all n-ary terms of type τ, closed under finite number of applications of (ii), is denoted by $W_\tau(X_n)$. We call the set $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ is the set of all terms of type τ.

The generalized superposition of terms is defined by Leeratanavalee in [5] as follows

$$S^n : W_\tau(X)^{n+1} \longrightarrow W_\tau(X)$$

is defined by the following steps: for any term $t \in W_\tau(X)$,

(i) if $t = x_j, 1 \leq j \leq n$, then $S^n(x_j, t_1, \ldots, t_n) := t_j$,

(ii) if $t = x_j, n < j \in \mathbb{N}$, then $S^n(x_j, t_1, \ldots, t_n) := x_j$,

(iii) if $t = f_i(s_1, \ldots, s_{n_i})$, then
$S^n(t, t_1, \ldots, t_n) := f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_{n_i}, t_1, \ldots, t_n))$.

It turns out that the algebra $clone_g(\tau) := (W_\tau(X); S^n, (x_i)_{i\in\mathbb{N}})$ consisting the set $W_\tau(X)$, with the $(n+1)$-ary generalized superposition operation S^n and variables act as infinitely many nullary operations satisfies the following four identities.

(Cg1) $\widetilde{S}^n(X_0, \widetilde{S}^n(Y_1, X_1, \ldots, X_n), \ldots, \widetilde{S}^n(Y_n, X_1, ..., X_n)) \approx$
$\widetilde{S}^n(\widetilde{S}^n(X_0, Y_1, \ldots, Y_n), X_1, \ldots, X_n)$.

(Cg2) $\widetilde{S}^n(\lambda_i, X_1, \ldots, X_n) \approx X_i$, for $1 \leq i \leq n$.

(Cg3) $\widetilde{S}^n(\lambda_i, X_1, \ldots, X_n) \approx \lambda_i$, for $i > n$.

(Cg4) $\widetilde{S}^n(X_1, \lambda_1, \ldots, \lambda_n) \approx X_1$, where $\widetilde{S}^n$ is operation symbols corresponding to the operations S^n of $clone_g(\tau)$ where $\lambda_1, ..., \lambda_n$ are nullary operation symbols and $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are variables.

In [5], a generalized hypersubstitution of type τ is defined to be a mapping σ taking an operation symbol to a term which does not necessarily preserve arities. We denote the set of all generalized hypersubstitutions of type τ by $Hyp_G(\tau)$.

Then the generalized hypersubstitution σ can be extended to a mapping

$$\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$$

by the following steps:

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i where $\hat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

Together with one binary operation $\circ_G$ on $Hyp_G(\tau)$ which defined by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings and σ_{id} which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$, $(Hyp_G(\tau); \circ_G, \sigma_{id})$ is a monoid. See [4] for details.

Now we recall the definition of algebraic systems. Let J be a nonempty indexed set and let $(\gamma_j)_j \in J$ be a sequence of relation symbols.Let $\tau' := (n_j)_j \in J$ where n_j is the arity of the γ_j for every $j \in J$. The pair (τ, τ') is called *the type* of algebraic system.

Definition 1.1. *An algebraic system type (τ, τ') is the triple consisting a nonemty set A together with a sequence $\left(f_i^A\right)_{i\in I}$ of operation on A where f_i^A is $n_i - ary$ for $i \in I$ and sequence $(\gamma_j^A)_{j\in J}$ of relations on A where γ_j^A is n_j-ary for $j \in J$, i.e., $\mathcal{A} := (A, (f_i^A)_{i\in I}, (\gamma_j^A)_{j\in J})$.*

We now present a concrete example of algebraic systems. The ordered semigroup $(S, \cdot, \leq)$ is an algebraic system of type $((2), (2))$.

To explain the properties of algebraic systems, the concept of *quantifier free formulas* which is first introduced by Mal'cev in 1973 is needed. For more details see also [6]. Next, we recall the formal definition of n-ary quantifier free formulas which is defined by Denecke and Phusanga [1].

Definition 1.2. *An n-ary quantifier free formula of type (τ, τ') is defined in the following inductive way:*

(i) *If* t_1, t_2 *are* n*-ary terms of type* τ*, then the equation* $t_1 \approx t_2$ *is an* n*-ary quantifier free formula of type* $(\tau, \tau^{'})$*.*

(ii) *If* $j \in J$ *and* $t_1, \ldots, t_{n_j}$ *are* n*-ary term of of type* τ*, and* γ_j *is an* n_j*-ary relation symbol, then* $\gamma(t_1, \ldots, t_{n_j})$ *is an* n*-ary quantifier free formula of type* $(\tau, \tau^{'})$*.*

(iii) *If* F *is an* n*-ary quantifier free formula of type* $(\tau, \tau\prime)$*, then* $\neg F$ *is an* n*-ary quantifier free formula of type* $(\tau, \tau^{'})$*.*

(iv) *If* F_1, F_2 *is an* n*-ary quantifier free formula of type* $(\tau, \tau^{'})$*, then* $F_1 \vee F_2$ *is an* n*-ary quantifier free formula of type* $(\tau, \tau^{'})$*.*

Let $\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ be the set of all n-ary quantifier free formula of type $(\tau, \tau^{'})$ and let $\mathcal{F}_{(\tau,\tau')}(W_\tau(X)) := \bigcup_{n=1}^{\infty} \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ be the set of all quantifier free formula of type $(\tau, \tau^{'})$.

The definition of generalized superposition operation have already defined in [8] by the following.

Definition 1.3. *Let* $t_1, \ldots, t_n \in W_\tau(X)$ *and* S^n *be the generalized superposition of terms. The operation* $R^n : \mathcal{F}_{(\tau,\tau')}(W_\tau(X)) \times W_\tau(X)^n \longrightarrow \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$ *where* $n \in \mathbb{N}^+$ *be defined by the following inductive step:*

(i) *If* $s_1, s_2 \in W_\tau(X)$*, then*

$$R^n(s_1 \approx s_2, t_1, \ldots, t_n) := S^n(s_1, t_1, \ldots, t_n) \approx S^n(s_2, t_1, \ldots, t_n).$$

(ii) *If* $j \in J$ *and* $s_1, \ldots, s_{n_j} \in W_\tau(X)$*, then*

$$R^n(\gamma_j(s_1, \ldots, s_{n_j}), t_1, \ldots, t_n) := \gamma_j(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_{n_j}, t_1, \ldots, t_n)).$$

(iii) *If* $F \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$ *is already defined, then*

$$R^n(\neg F, t_1, \ldots, t_n) := \neg R^n(F, t_1, \ldots, t_n).$$

(iv) *If* $F_1, F_2 \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$ *and suppose that* $R^n(F_1, t_1, \ldots, t_n), R^n(F_2, t_1, \ldots, t_n)$ *are already defined, then*

$$R^n(F_1 \vee F_2, t_1, \ldots, t_n) := R^n(F_1, t_1, \ldots, t_n) \vee R^n(F_2, t_1, \ldots, t_n).$$

Then the algebra

$$Formclone_g(\tau,\tau^{'}) := (\mathcal{F}_{(\tau,\tau')}(W_\tau(X)), W_\tau(X), R^n, S^n, (x_i)_{1\leq i\leq n, n\in\mathbb{N}}),$$

which is called the *generalized formula-term clone* of type $(\tau,\tau^{'})$ is constructed.

Similar to the generalized hypersubstitutions of type τ, the generalized hypersubstitutions for algebraic systems of type $(\tau,\tau^{'})$ are given by D. Phusanga [8], that is a mapping $\sigma : \{f_i \mid i \in I\} \bigcup \{\gamma_j \mid j \in J\} \longrightarrow W_\tau(X) \bigcup \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$ which maps operation symbols to terms and relation symbols to quantifier free formulas. Let $Hyp_G(\tau,\tau^{'})$ be the set of all generalized hypersubstitutions for algebraic systems of type $(\tau,\tau^{'})$.

To define a binary operation on $Hyp_G(\tau,\tau^{'})$, the following definition is needed.

Definition 1.4. *Let $\sigma \in Hyp(\tau,\tau^{'})$. Then σ induces a mapping*

$$\widehat{\sigma} : W_\tau(X) \bigcup \mathcal{F}_{(\tau,\tau')}(W_\tau(X)) \longrightarrow W_\tau(X) \bigcup \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$$

by setting

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_1,\ldots,t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_{n_i}])$*, for any n-ary operation symbol f_i where $\hat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined,*

(iii) $\hat{\sigma}[t_1 \approx t_2] := \hat{\sigma}[t_1] \approx \hat{\sigma}[t_2]$,

(iv) $\hat{\sigma}[\gamma_j(t_1,\ldots,t_{n_j})] := R^{n_j}(\sigma(\gamma_j), \hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_{n_j}])$ *for* $j \in J$,

(v) $\hat{\sigma}[\neg F] := \neg\hat{\sigma}[F]$*, for every* $F \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$,

(vi) $\hat{\sigma}[F_1 \vee F_2] := \hat{\sigma}[F_1] \vee \hat{\sigma}[F_2]$*, for every* $F_1, F_2 \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$.

We recall that $\circ$ denotes the usual composition of binary operations and it is easy to verify that $\widehat{\sigma}_1 \circ \sigma_2 \in Hyp_G(\tau,\tau^{'})$, whenever $\sigma_1,\sigma_2 \in Hyp_G(\tau,\tau^{'})$. Then the definition of binary operation $\circ_G$ on $Hyp_G(\tau,\tau')$ is defined by $\sigma_1 \circ_G \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$ for all $\sigma_1,\sigma_2 \in Hyp_G(\tau,\tau')$. Let σ_{id} be the hypersubstitutions for algebraic systems which maps each n_i-ary operation symbol f_i to the term $f_i(x_1,\ldots,x_{n_i})$ for all $i \in I$ and maps each n_j-ary relation symbol γ_j to quantifier free formulas $\gamma_j(x_1,\ldots,x_{n_j})$ for all $j \in J$. In particular, the algebra $(Hyp_G(\tau,\tau^{'}), \circ_G, \sigma_{id})$ is a monoid where σ_{id} is an identity element with respect to $\circ_G$. For research in this way, see [3].

In this paper, we use a special class of terms to define a new concept of quantifier free formulas. Our first aim is to construct the algebra consisting the set of such quantifier free formulas with a suitable generalized superposition operation. Based on hypersubstitution theory, we complete the paper with introducing the notion of generalized hypersubstitution sending each operation symbol and each relation symbol to an interesting term and an interesting quantifier free formulas, respectively.

2 Quantifier Free Formulas Induced by Terms of a Fixed Variable

We start the main result by recalling the definition of terms of a fixed variable which is introduced by K. Wattanatripop and T. Changphas [9]. Let $var(t)$ be the set of all variables occurring in the term t.

Definition 2.1. *An n-ary term of a fixed variable of type τ is inductively defined by:*

(i) *Every $x_i \in X_n$ is an n-ary term of a fixed variable of type τ.*

(ii) *If $t_1, \ldots, t_{n_i}$ are n-ary terms of a fixed variable of type τ and $var(t_j) = var(t_k)$ for all $1 \leq j < k \leq n_i$, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of a fixed variable of type τ.*

Let $W_\tau^{fv}(X_n)$ be the set of all n-ary terms of a fixed variable of type τ. The set of all terms of a fixed variable of type τ will be denoted by $W_\tau^{fv}(X)$, i.e.,

$$W_\tau^{fv}(X) := \bigcup_{n \in \mathbb{N}^+} W_\tau^{fv}(X_n)$$

Applying the generalized hypersubstitutions, then we have

Definition 2.2. A generalized hypersubstitution $\sigma \in Hyp_G(\tau)$ is called an *fv-generalized hypersubstitution* of type τ if for all $i \in I$, $\sigma(f_i) \in W_\tau^{fv}(X)$. *Let $Hyp_G^{fv}(\tau)$ be the set of all fv-generalized hypersubstitutions of type τ.*

Lemma 2.3. *Let $\sigma \in Hyp_G^{fv}(\tau)$. Then the extension of σ maps a term of a fixed variable to a term of a fixed variable.*

Proof. Let $\sigma \in Hyp_G^{fv}(\tau)$ and let $t \in W_\tau^{fv}(X)$. We want to show that $\hat{\sigma}[t] \in W_\tau^{fv}(X)$. It is clear that for $t \in X$ because $\hat{\sigma}[t] = \hat{\sigma}[x] = x \in W_\tau^{fv}(X)$. Assume that $t = f_i(t_1, \ldots, t_{n_i})$ where $t_1, \ldots, t_{n_i} \in W_\tau^{fv}(X)$ such that $\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}] \in W_\tau^{fv}(X)$. Since $var(t_1) = \cdots = var(t_{n_i})$, we have $var\left(\hat{\sigma}[t_1]\right) = \cdots = var\left(\hat{\sigma}[t_{n_i}]\right)$. From $\sigma(f_i) \in W_\tau^{fv}(X)$, we have

$$\hat{\sigma}[t] = \hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] = S^{n_i}\left(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]\right)$$

is a term of fixed variable. □

Now, we give a definition of the quantifier free formula of a fixed variable of type $(\tau, \tau^{'})$ as follows. Let $var(F)$ be the set of all variables occurring in the quantifier free formula F.

Definition 2.4. *An n-ary quantifier free formula of a fixed variable of type $(\tau, \tau^{'})$ is defined in the following inductive way:*

(i) *If t_1, t_2 are n-ary terms of a fixed variable of type τ, then the equation $t_1 \approx t_2$ is an n-ary quantifier free fomula of a fixed variable of type $(\tau, \tau^{'})$.*

(ii) *If $j \in J$ and $t_1, \ldots, t_{n_j}$ are n-ary terms of a fixed variable of type τ, and if $var(t_l) = var(t_k)$ for all $1 \leq l < k \leq n_i$, then $\gamma_j(t_1, \ldots, t_{n_j})$ is an n-ary quantifier free formula of a fixed variable of type $(\tau, \tau^{'})$.*

(iii) *If F is an n-ary quantifier free formula of a fixed variable of type $(\tau, \tau^{'})$, then $\neg F$ is an n-ary quantifier free formula of a fixed variable of type $(\tau, \tau^{'})$.*

(iv) *If F_1, F_2 are n-ary quantifier free formulas of a fixed variable of type $(\tau, \tau^{'})$, and $var(F_1) = var(F_2)$, then $F_1 \vee F_2$ is an n-ary quantifier free formula of a fixed variable of type $(\tau, \tau^{'})$.*

Let $\mathcal{F}_{(\tau,\tau')}^{fv}(W_\tau^{fv}(X_n))$ be the set of all n-ary quantifier free formulas of a fixed variable of type $(\tau, \tau^{'})$ and let $\mathcal{F}_{(\tau,\tau')}^{fv}(W_\tau^{fv}(X)) := \bigcup_{n=1}^{\infty} \mathcal{F}_{(\tau,\tau')}^{fv}(W_\tau^{fv}(X_n))$ be the set of all quantifier free formulas of a fixed variable of type $(\tau, \tau^{'})$.

From Definiton 2.4 if we set the condition in (i) by $var(t_1) = var(t_2)$, then the quantifier free formula of a fixed variable of type $(\tau, \tau^{'})$ is called the *strong quantifier free formula of a fixed variable* of type $(\tau, \tau^{'})$. Note that every strong quantifier free formula of a fixed variable of type $(\tau, \tau^{'})$ is a quantifier free formula of a fixed variable of type $(\tau, \tau^{'})$.

We present some interesting examples of quantifier free formulas of a fixed variable. Let $(\tau, \tau\prime) = ((2),(3))$ be a type with a binary operation symbol f and a binary relation symbol γ. Consider the variables from X_4, then the example of quaternary quantifier free formulas of a fixed variable of type $((2),(3))$ are: $x_1 \approx x_1, x_1 \approx x_3,\ f(x_2,x_2) \approx f(x_3,x_3), \gamma(x_1,x_1,x_1)$, $\gamma(x_2, f(x_2,x_2), f(f(x_2,x_2),x_2)),\ \neg(f(x_1,f(x_1,x_1)) \approx f(f(x_2,x_2),x_2))$, $\neg(\gamma(x_3,x_3,x_3)), (x_1 \approx x_1) \vee \neg(\gamma(x_1,x_1,x_1))$, $\gamma(x_2,x_2,x_2) \vee \neg(f(x_2,f(x_2,x_2)) \approx f(f(x_2,x_2),x_2))$.

To ensure that after we apply the usual generalized superposition R^n to quantifier free formulas of a fixed variable, the results are also quantifier free formulas of a fixed variable, we need the next lemma.

Lemma 2.5. *Let $n \in \mathbb{N}^+$ and $F \in \mathcal{F}^{fv}_{(\tau,\tau')}(W^{fv}_\tau(X))$. If $s_1, \ldots, s_n \in W^{fv}_\tau(X)$, then we have $R^n(F, s_1, \ldots, s_n) \in F^{fv}_{(\tau,\tau')}(W^{fv}_\tau(X))$.*

Proof. Let $F \in \mathcal{F}^{fv}_{(\tau,\tau')}(W^{fv}_\tau(X))$ and $s_1, \ldots, s_n \in W^{fv}_\tau(X)$. We will give a proof by induction on the complexity of formula F.

(i) If F has the form $t_1 \approx t_2$, whenever $t_1, t_2 \in W^{fv}_\tau(X)$

$$R^n(t_1 \approx t_2, s_1, \ldots, s_n) = S^n(t_1, s_1, \ldots, s_n) \approx S^n(t_2, s_1, \ldots, s_n).$$

By [9], we have $S^n(t_1, s_1, \ldots, s_n), S^n(t_2, s_1, \ldots, s_n)$ are an n-ary term of a fixed variable. So we complete the proof in this case.

If F has the form $\gamma_j(t_1, \ldots, t_{n_j})$ and $var(t_l) = var(t_k)$ for all $1 \leq l < k \leq n_j$, then

$$R^n(\gamma_j(t_1, \ldots, t_{n_j}), s_1, \ldots, s_n) = \gamma_j(R^n(t_1, s_1, \ldots, s_n), \ldots, R^n(t_{n_j}, s_1, \ldots, s_n).$$

Since $R^n(t_1, s_1, \ldots, s_n), \ldots, R^n(t_{n_j}, s_1, \ldots, s_n) \in W^{fv}_\tau(X)$ and $var(t_1) = \ldots = var(t_{n_j})$, we obtain that $var(R^n(t_1, s_1, \ldots, s_n)) = \ldots = var(R^n(t_{n_j}, s_1, \ldots, s_n))$ and so

$$R^n(\gamma_j(t_1, \ldots, t_{n_j}), s_1, \ldots, s_n) \in \mathcal{F}^{fv}_{(\tau,\tau')}(W^{fv}_\tau(X)).$$

(iii) If has the form $\neg F$, and asume that $F \in \mathcal{F}^{fv}_{(\tau,\tau')}(W^{fv}_\tau(X))$ is a quantifier free formula of a fixed variable. So $R^n(\neg F, s_1, \ldots, s_n) = \neg(R^n(F, s_1, \ldots, s_n)) \in \mathcal{F}^{fv}_{(\tau,\tau')}(W^{fv}_\tau(X))$.

(iv) If $F_1, F_2 \in \mathcal{F}^{fv}_{(\tau,\tau')}(W^{fv}_\tau(X))$ and $var(F_1) = var(F_2)$, then

$$R^n(F_1 \vee F_2, s_1, \ldots, s_n) = S^n(F_1, s_1, \ldots, s_n) \vee S^n(F_2, s_1, \ldots, s_n).$$

Since $var(F_1) = var(F_2)$, we obtain that

$$var(R^n(F_1, s_1, \ldots, s_n)) = var(R^n(F_2, s_1, \ldots, s_n)). \qquad \square$$

The conclusion of Lemma 2.5 show that the set $\mathcal{F}^{fv}_{(\tau,\tau')}(W^{fv}_\tau(X))$ is closed under the genralized superposition operation. Then we can from the following algebra

$fv - Formclone_g(\tau, \tau') := (\mathcal{F}^{fv}_{(\tau,\tau')}(W^{fv}_\tau(X))_{n\geq 1}, (R^n)_{n\geq 1}, (x_i)_{1\leq i\leq n, n\in\mathbb{N}})$, which is called the $fv - formulas$ *clone of type* (τ, τ'). Moreover, we have

Theorem 2.6. $fv - Formclone_g(\tau, \tau')$ *is a subalgebra of* $Formclone_g(\tau, \tau')$.

3 The Semigroup of Fixed Variable Generalized Hypersubstitutions for Algebric Systems

In this section, we present the definition of mapping which takes $n_i - ary$ operation symbols to terms of a fixed variable and $n_j - ary$ relation symbols to quantifier free formulas of a fixed variable. This mapping does not necessary preserve the arity.

Definition 3.1. A generalized hypersubstitution $\sigma \in Hyp_G(\tau)$ is called an fv-*generalized hypersubstitution of type* τ if for all $i \in I, j \in J, \sigma(f_i) \in W^{fv}_\tau(X)$ and $\sigma(\gamma_j) \in \mathcal{F}^{fv}_{(\tau,\tau')}(W^{fv}_\tau(X))$. Let $Hyp^{fv}_G(\tau, \tau')$ be the set of all fv-generalized hypersubstitution of type (τ, τ').

Then we obtain

Theorem 3.2. *If* $\sigma \in Hyp^{fv}_G(\tau, \tau')$, *then we have*

$$\hat{\sigma} : W^{fv}_\tau(X) \cup \mathcal{F}^{fv}_{(\tau,\tau')}(W^{fv}_\tau(X)) \longrightarrow W^{fv}_\tau(X) \cup \mathcal{F}^{fv}_{(\tau,\tau')}(W^{fv}_\tau(X)).$$

Proof. Let $\sigma \in Hyp^{fv}_G(\tau, \tau')$ and $\beta \in W^{fv}_\tau(X) \cup \mathcal{F}^{fv}_{(\tau,\tau')}(W^{fv}_\tau(X))$. We want to show that $\hat{\sigma}[\beta] \in W^{fv}_\tau(X) \cup \mathcal{F}^{fv}_{(\tau,\tau')}(W^{fv}_\tau(X))$.

Let β be an n-ary term of a fixed variable. It is clear when $\beta = x_i \in X$ by the definition. Now, assume that $\beta = f_i(t_1, \ldots, t_{n_i})$ where $t_1, \ldots, t_{n_i} \in W^{fv}_\tau(X)$ such that $\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}] \in W^{fv}_\tau(X)$. Since $var(t_1) = \ldots = var(t_{n_i})$, we have $var(\hat{\sigma}[t_1]) = \ldots = var(\hat{\sigma}[t_{n_i}])$. From $\sigma(f_i) \in W^{fv}_\tau(X)$, then $\hat{\sigma}[\beta] = \hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] = S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ is term of a fixed variable.

Now, let $\beta = F$ is a formula we will give a proof by induction on the complexity of formula F.

(i) If F has the form $t_1 \approx t_2$, then we have $\hat{\sigma}[t_1], \hat{\sigma}[t_2] \in W_\tau^{fv}(X)$ by Lemma 2.3, so $\hat{\sigma}[F] = \hat{\sigma}[t_1 \approx t_2] = \hat{\sigma}[t_1] \approx \hat{\sigma}[t_2] \in \mathcal{F}_{(\tau,\tau')}^{fv}(W_\tau^{fv}(X))$.

(ii) If F has the form $\gamma_j(t_1, \ldots, t_{n_j})$ where $t_1, \ldots, t_{n_j} \in W_\tau^{fv}(X)$ and $var(t_1) = \ldots = var(t_{n_j})$. Since $\sigma(\gamma_j) \in \mathcal{F}_{(\tau,\tau')}^{fv}(W_\tau^{fv}(X))$ and $\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_j}] \in W_\tau^{fv}(X)$. It is easy to see that $var(\hat{\sigma}[t_1]) = \ldots = var(\hat{\sigma}[t_{n_j}])$ by assumption. Then $\hat{\sigma}[F] = \hat{\sigma}[\gamma_j(t_1, \ldots, t_{n_j})] = R^{n_j}(\sigma(\gamma_j), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_j}]) \in W_\tau^{fv}(X) \cup \mathcal{F}_{(\tau,\tau')}^{fv}(X)$

(iii) If F has the form $\neg F$ and assume that $\hat{\sigma}[F] \in \mathcal{F}_{(\tau,\tau')}^{fv}(X)$, then $\hat{\sigma}[\neg F] = \neg\hat{\sigma}[F] \in \mathcal{F}_{(\tau,\tau')}^{fv}(W_\tau^{fv}(X))$.

(iv) If $F = F_1 \vee F_2$ whenever $var(F_1) = var(F_2)$ and assume that $\hat{\sigma}[F_1], \hat{\sigma}[F_2] \in \mathcal{F}_{(\tau,\tau')}^{fv}(W_\tau^{fv}(X))$, then $\hat{\sigma}[F_1 \vee F_2] = \hat{\sigma}[F_1] \vee \hat{\sigma}[F_2]$. Since $var(F_1) = var(F_2)$, we have $var(\hat{\sigma}[F_1]) = var(\hat{\sigma}[F_2])$. Thus $\hat{\sigma}[F_1 \vee F_2] \in \mathcal{F}_{(\tau,\tau')}^{fv}(W_\tau^{fv}(X))$. □

The fact that the extension $\hat{\sigma}$ is an endomorphism of $fv{-}Formclone(\tau, \tau')$ can be shown in the next lemma.

Lemma 3.3. *Let $\sigma \in Hyp_G^{fv}(\tau, \tau')$. If $s_1, \ldots, s_n \in W_\tau^{fv}(X)$ where $n \in \mathbb{N}^+$, then*

$$\hat{\sigma}[R^n(\beta, s_1, \ldots, s_n)] = R^n(\hat{\sigma}[\beta], \hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_n])$$

for any $\beta \in \mathcal{F}_{(\tau,\tau')}^{fv}(W_\tau^{fv}(X))$.

Proof. It is not difficult to prove that by induction on the definition of a quantifier free formula of a fixed variable β. □

Theorem 3.4. *The algebra $(Hyp_G^{fv}(\tau, \tau'), \circ_G)$ is a subsemigroup of $(Hyp_G(\tau, \tau'), \circ_G)$.*

Proof. Let σ_1, σ_2 be arbitrary elements in $Hyp_G^{fv}(\tau, \tau')$. We prove that $\sigma_1 \circ_G \sigma_2 \in Hyp_G^{fv}(\tau, \tau')$. By Lemma 3.3, we have that $\hat{\sigma_1}$ takes the elements in $W_\tau^{fv}(X) \cup \mathcal{F}_{(\tau,\tau')}^{fv}(W_\tau^{fv}(X))$ to itself. Then so is a composition $\hat{\sigma_1} \circ \sigma_2$. This shows that $Hyp_G^{fv}(\tau, \tau')$ is a subsemigroup of $Hyp_G(\tau, \tau')$ with respect to $\circ_G$. □

Acknowledgment. This research was supported by Rajamangala University of Technology Lanna.

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