International Journal of Mathematics and
Computer Science, **16**(2021), no. 1, 449–457

Symmetric bi-additive maps of rings with generalized skew-centralizing traces

Sitanan Khun-in, Utsanee Leerawat

Department of Mathematics
Faculty of Science
Kasetsart University
Bangkok 10900, Thailand

email: sit_anan@hotmail.com, fsciutl@ku.ac.th

(Received September 5, 2020, Accepted October 1, 2020)

Abstract

In this paper, we characterize symmetric bi-additive mappings on rings with left identity under some conditions involving skew-centralizing. Moreover, we describe the structure of the trace of symmetric bi-additive mappings on rings.

1 Introduction

Throughout this paper, let R denote a ring with center $Z(R)$. Recall that a ring R is prime if $aRb = \{0\}$ with $a, b \in R$ implies $a = 0$ or $b = 0$ and semiprime in case $aRa = \{0\}$ with $a \in R$ implies $a = 0$. For an integer $n > 1$, an element $x \in R$ is called $n-$torsion free if $nx = 0$ implies $x = 0$. A ring R is called an $n-$torsion free ring if every element in R is $n-$torsion free. Moreover, a ring R is called an $n!-$torsion free if it is $d-$torsion free for any divisor d of $n!$. For any $x, y \in R$ the symbol $[x, y]$ will represent the commutator $xy-yx$ and the symbol $< x, y >$ stands for the skew-commutator $xy + yx$.

For a subset S of R, a mapping $f : S \longrightarrow R$ is called commuting (centralizing) on S if $[f(x), x] = 0$ (resp. $[f(x), x] \in Z(R)$) for all $x \in S$.

Key words and phrases: Bi-additives, symmetric, traces, prime rings, skew-commutings, skew-centralizings.
AMS (MOS) Subject Classifications: 16N60, 16E99, 16W20, 08A35.
Corresponding author email: fsciutl@ku.ac.th
ISSN 1814-0432, 2021, http://ijmcs.future-in-tech.net

A mapping $f : S \longrightarrow R$ is called skew-commuting (skew-centralizing) on S if $< f(x), x >= 0$ (resp. $< f(x), x >\in Z(R)$) holds for all $x \in S$.

Over the last sixty years, several authors $[1, 2, 3, 4, 9]$ have proved commutativity theorems for prime rings admitting automorphisms which are centralizing on appropriate subsets of a ring R.

In [8], Bresar proved that there are no nonzero skew-commuting additive mappings on a $2-$torsion free semiprime rings. In other words, if R is a $2-$torsion free semiprime ring and $f : R \longrightarrow R$ is an additive mapping such that $< f(x), x >= 0$ for all $x \in R$, then $f(x) = 0$ for all $x \in R$. Park and Jung [7] introduced the concept of $n-$skew commuting (resp. $n-$skew centralizing) on R if $< f(x), x^n >= 0$ (resp. $< f(x), x^n >\in Z(R)$) for all $x \in R$.

Recently, Leerawat and Lapuangkham [11] introduced and investigated some results on the notion of generalizations of commuting/centralizing and skew-commuting/skew-centralizing in suitable rings. Later, Leerawat and Lapuangkham [12] obtained a characterization of generalizations of skew-commuting in prime rings. More precisely, let R be a $2-$torsion free prime ring with $g : R \longrightarrow R$ non-zero isomorphism or anti-isomorphism of R and $f : R \longrightarrow R$ an additive mapping. If f and g are generalized skew-commuting on R, then $f(x) = 0$ for all x in R.

A mapping $G : R \times R \longrightarrow R$ is said to be symmetric if $G(x, y) = G(y, x)$ for all $x, y \in R$. A mapping $g : R \longrightarrow R$ defined by $g(x) = G(x, x)$,where $G : R \times R \longrightarrow R$ is symmetric mappings, is called a trace of G. Vukman $[5, 6]$ investigated symmetric bi-derivations in prime and semiprime rings. In Muthana [10] obtained a similar type of results on Lie ideals of R. Jung [13] studied bi-additive maps with trace which is skew-commuting or skew-centralizing on rings with left identity and proved the following result:
Let R be a $2-$torsion-free ring with left identity e. Let $G : R \times R \longrightarrow R$ be a symmetric bi-additive mapping and let g be the trace of G. If g is skew-commuting on R, then $G = 0$. The aim of this paper is to investigate some results on the notion of generalized skew-commuting/skew-centralizing of traces in rings which partially extend some results of Jung [13].

2 Preliminaries

In this section, we recall some known definitions.

Definition 2.1. *A mapping* $G : R \times R \longrightarrow R$ *is called symmetric if* $G(x, y) = G(y, x)$ *holds for all* $x, y \in R$*. A mapping* $g : R \longrightarrow R$ *defined by* $g(x) = G(x, x)$ *for all* $x \in R$*, where* $G : R \times R \longrightarrow R$ *is a symmetric mapping, is called a trace of* G*. A mapping* $G : R \times R \longrightarrow R$ R *is called bi-additive if for all* $x, y, z \in R$*, the following conditions hold:*

(i) $G(x + y, z) = G(x, z) + G(y, z)$*,*

(ii) $G(x, y + z) = G(x, y) + G(x, z)$*.*

The following are some basic properties of a symmetric bi-additive mapping $G : R \times R \longrightarrow R$ *with the trace* g *of* G*.*
The proofs of these properties are straightforward and hence omitted.

(1) $g(x + y) = g(x) + g(y) + 2G(x, y)$ *for all* $x, y \in R$*,*

(2) $G(x, 0) = G(0, x) = 0$ *for all* $x \in R$*,*

(3) $G(-x, y) = -G(x, y) = G(x, -y)$ *for all* $x, y \in R$*,*

(4) $g(-x) = g(x)$ *for all* $x \in R$*,*

(5) $g(x + y) + g(x - y) = 2g(x) + 2g(y)$ *for all* $x, y \in R$*.*

Definition 2.2. ([11])
Let f *and* g *be mapping from* R *into itself.*

(1) We say that f *and* g *are generalized centralizing on* R *if* $[f(x), g(x)] \in Z(R)$ *for all* $x \in R$*, and we say that* f *and* g *are generalized commuting on* R *if* $[f(x), g(x)] = 0$ *for all* $x \in R$*.*

(2) We say that f *and* g *are genneralized skew-centralizing on* R *if* $< f(x), g(x) > \in Z(R)$ *for all* $x \in R$ *, and we say that* f *and* g *are generalized skew-commuting on* R *if* $< f(x), g(x) >= 0$ *for all* $x \in R$*.*

3 Main results

In this section, we extend a part of [13] to the generalized skew-commuting/centralizing traces.

Theorem 3.1. *Let* R *be a* $2-$*torsion free ring with left identity* e*.*
Let $G : R \times R \longrightarrow R$ *be a symmetric bi-additive mapping with trace* g*.*
Let $f : R \longrightarrow R$ *be an additive mapping such that* $f(e) = e$*. If* f *and* g *are generalized skew-commuting on* R*, then* $G(x, y) = 0$ *for all* $x, y \in R$*.*

Proof. Let f and g be generalized skew-commuting on R. Then

$$f(x)g(x) + g(x)f(x) = 0 \text{ for all } x \in R. \tag{3.1.1}$$

Replacing x by e in (3.1.1), we obtain

$$g(e) + g(e)e = 0, \tag{3.1.2}$$

and right multiplication by e gives $2g(e)e = 0$. Since R is a $2-$torsion free ring, $g(e)e = 0$. Hence, by(3.1.2), $g(e) = 0$.
Replacing x by $x + e$ in (3.1.1) and using $g(x + e) = g(x) + 2G(x, e)$, we get

$$2f(x)G(x,e)+g(x)+2G(x,e)+g(x)e+2G(x,e)f(x)+2G(x,e)e = 0 \text{ for all } x \in R. \tag{3.1.3}$$

Take $-x$ for x in (3.1.3) and compare the relation, so obtained, with (3.1.3). Since R is a $2-$ torsion free ring, it follows that

$$G(x, e) + G(x, e)e = 0, \text{ for all } x \in R. \tag{3.1.4}$$

Right multiplication of (3.1.4) by e gives

$$2G(x, e)e = 0, \text{ for all } x \in R. \tag{3.1.5}$$

Since R is a $2-$torsion free ring, $G(x, e)e = 0$ for all $x \in R$. Then, by (3.1.4), $G(x, e) = 0$ for all $x \in R$. Therefore, $g(x + e) = g(x) + 2G(x, e) = g(x)$ for all $x \in R$. Replacing x by $x + e$ in (3.1.1) and using $g(x + e) = g(x)$ for all $x \in R$, it follows that

$$g(x) + g(x)e = 0, \text{ for all } x \in R. \tag{3.1.6}$$

Right multiplication of (3.1.6) by e yields $2g(x)e = 0$. Since R is a $2-$torsion free ring, it follows that $g(x)e = 0$ and hence the relation (3.1.6) implies $g(x) = 0$ for all $x \in R$. Since g is the trace of G, $G(x, x) = 0$ for all $x \in R$.
By linearization, we get

$$G(y, x) + G(x, y) = 0, \text{ for all } x, y \in R. \tag{3.1.7}$$

Since G is symmetric and R is a $2-$ torsion free ring, it follows that
$G(x, y) = 0$ for all $x, y \in R$. The proof is complete. □

Theorem 3.2. *Let R be a $2-$torsion free ring with left identity e. Let $G : R \times R \longrightarrow R$ be a symmetric bi-additive mapping with the trace g.*
Let $f : R \longrightarrow R$ be an additive mapping such that $f(e) = e$. If f and g are generalized skew-centralizing on R then f and g are generalized commuting on R. Moreover, there exist an additive map $\lambda : R \longrightarrow R$ and a symmetric map $H : R \times R \longrightarrow R$ with trace $h : R \longrightarrow R$ such that

$$g(x) = \lambda(x)f(x) + h(x) \text{ for all } x \in R.$$

Proof. Assume that f and g are generalized skew-centralizing on R then

$$f(x)g(x) + g(x)f(x) \in Z(R) \text{ for all } x \in R. \tag{3.2.1}$$

Replacing x by e in (3.2.1), we have $g(e) + g(e)e \in Z(R)$.
Commuting with e gives $g(e) = g(e)e$. Since R is $2-$ torsion free, $g(e) \in Z(R)$. Replacing x by $x + e$ in (3.2.1), we have

$$\begin{aligned} &2f(x)G(x,e) + g(x) + 2G(x,e) + 2f(x)g(e) \\ &+2G(x,e)f(x) + g(x)e + 2G(x,e)e \in Z(R) \text{ for all } x \in R. \end{aligned} \tag{3.2.2}$$

Substituting $-x$ for x in (3.2.2) and comparing (3.2.2) with the result, we obtain

$$4G(x,e) + 4G(x,e)e + 4f(x)g(e) \in Z(R) \text{ for all } x \in R. \tag{3.2.3}$$

Commuting with e gives

$$4(G(x,e) - G(x,e)e) = 0 \text{ for all } x \in R.$$

Since R is $2-$ torsion free, $G(x,e) = G(x,e)e$ for all $x \in R$.
By (3.2.3), and using R is $2-$ torsion free, we have

$$2G(x,e) + f(x)g(e) \in Z(R) \text{ for all } x \in R. \tag{3.2.4}$$

Commuting with $f(x)$ gives $f(x)G(x,e) = G(x,e)f(x)$ for all $x \in R$, and by (3.2.2) and (3.2.4) we have

$$4f(x)G(x,e) + g(x) + g(x)e \in Z(R) \text{ for all } x \in R, \tag{3.2.5}$$

and commuting with e gives $g(x) = g(x)e$ for all $x \in R$. Applying this relation to (3.2.5) and R is $2-$ torsion free, we obtain

$$2f(x)G(x,e) + g(x) \in Z(R) \text{ for all } x \in R. \tag{3.2.6}$$

Commuting with $f(x)$, we have $[f(x), g(x)] = 0$ for all $x \in R$. This means that f and g are generalized commuting on R.
Next, let $\lambda(x) = -2G(x, e)$ for all $x \in R$. Then $\lambda : R \longrightarrow R$ is an additive map.
Define $H : R \times R \longrightarrow R$ by

$$H(x, y) = G(x, y) + G(x, e)f(x) + G(y, e)f(y) \text{ for all } x, y \in R.$$

It is obvious that H is symmetric. Let h be the trace of H. Then

$$\begin{aligned} h(x) = H(x, x) &= G(x, x) + 2G(x, e)f(x) \\ &= g(x) - \lambda(x)f(x) \text{ for all } x \in R. \end{aligned}$$

Hence $g(x) = \lambda(x)f(x) + h(x)$ for all $x \in R$. □

Theorem 3.3. *Let R be a $2-$torsion free ring with left identity e. Let f and h be additive mappings such that $f(e) = e$. If f and h are generalized skew-centralizing on R, then f and h are generalized commuting on R.*

Proof. By assumption, we have $< f(x), h(x) > \in Z(R)$ for all $x \in R$. Then

$$f(x) < f(x), h(x) >=< f(x), h(x) > f(x) \text{ for all } x \in R. \tag{3.3.1}$$

This implies that

$$< [h(x), f(x)], f(x) >= 0 \text{ for all } x \in R. \tag{3.3.2}$$

Let $G : R \times R \longrightarrow R$ be a mapping defined by

$$G(x, y) = [h(x), f(y)] + [h(y), f(x)] \text{ for all } x, y \in R.$$

Then it is clear that G is symmetric and bi-additive, and that the mapping $g : R \longrightarrow R$ defined by $g(x) = G(x, x)$ for all $x \in R$ is the trace of G.
Note that for each $x \in R$,

$$\begin{aligned} < g(x), f(x) > &= g(x)f(x) + f(x)g(x) \\ &= G(x, x)f(x) + f(x)G(x, x) \\ &= 2([h(x), f(x)]f(x) + f(x)[h(x), f(x)]) \\ &= 2 < [h(x), f(x)], f(x) > . \end{aligned}$$

Applying (3.3.2), we obtain

$$< g(x), f(x) >= 0 \text{ for all } x \in R.$$

Therefore, f and g are generalized skew-commuting on R. Using Theorem 3.1, we have $G(x,y) = 0$ for all $x, y \in R$. In particular, $G(x,x) = 0$ for all $x \in R$. Using the fact that R is a $2-$ torsion free ring, we have $[h(x), f(x)] = 0$ for all $x \in R$. Hence f and h are generalized commuting on R. □

Theorem 3.4. *Let $n \geq 2$ be a fixed positive integer and let R be an $(n+1)!-$torsion free ring with left identity e. Let $G : R \times R \longrightarrow R$ be a symmetric bi-additive mapping with the trace g. If $f : R \longrightarrow R$ is an endomorphism of R such that $f(e) = e$ and $< f(x^n), g(x) >= 0$ for all $x \in R$, then $G(x,y) = 0$ for all $x, y \in R$.*

Proof. Suppose that

$$< f(x^n), g(x) >= 0 \text{ for all } x \in R. \tag{3.4.1}$$

For $x = e$ in (3.4.1), we have

$$g(e) + g(e)e = 0. \tag{3.4.2}$$

Using a similar approach as in the proof of Theorem 3.1, we have $g(e) = 0$. Let t be any positive integer. Replacing x by $x + te$ in (3.4.1) and using $g(x+te) = g(x) + 2tG(x,e)$ for all $x \in R$, we get

$$tP_1(x,e) + t^2P_2(x,e) + ... + t^nP_n(x,e) + t^{n+1}P_{n+1}(x,e) = 0, \tag{3.4.3}$$

for all $x \in R$, where $P_k(x,e)$ is the sum of terms involving x and e such that $P_k(x,te) = t^kP_k(x,e)$, for $k = 1, 2, ..., n, n+1$.
Next, replacing t by $1, 2, ..., n+1$ in turn and consider the resulting system of $n+1$ homogeneous equations, we get

$$AX = [0],$$

where $A = \begin{bmatrix} 1 & 1^2 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ n+1 & (n+1)^2 & \cdots & (n+1)^{n+1} \end{bmatrix}$,
$X = \begin{bmatrix} P_1(x,e) \\ P_2(x,e) \\ \vdots \\ P_{n+1}(x,e) \end{bmatrix}$,

and [0] is $(n+1)\times 1$ zero matrix.
Since $\det A = \prod_{i=0}^{n}(n+1-i)!$ and R is $(n+1)!-$ torsion free, it follows that the system has only the trivial solution.
In particular, $P_{n+1}(x,e)=0$ implies that $2e^nG(x,e)+2G(x,e)e^n=0$ for all $x\in R$. Since R is $(n+1)!-$ torsion free and $e^n=e$, it follows that

$$G(x,e)+G(x,e)e=0 \text{ for all } x\in R.$$

Right multiplication of this equation by e and using R is $(n+1)!-$ torsion free, we have $G(x,e)e=0$ for all $x\in R$. Therefore, $G(x,e)=0$ for all $x\in R$. Next, $P_n(x,e)=0$ implies that

$$e^ng(x)+g(x)e^n=0 \text{ for all } x\in R.$$

Then
$$g(x)+g(x)e=0 \text{ for all } x\in R.$$

As in the proof of Theorem 3.1, we get $g(x)=0$ for all $x\in R$ and $G(x,y)=0$ for all $x,y\in R$. The proof of the theorem is complete. □

4 Acknowledgement

This research was supported by the International SciKU Branding (ISB), Faculty of Science, Kasetsart University, Bangkok, Thailand.

References

[1] J. Luh, A note on commuting automorphisms of rings, Amer. Math. Monthly, **77,** (1970), 61–62.

[2] J. Mayne, Centralizing automorphisms of prime rings, Canad. Math. Bull., **19,** (1976), 113–115.

[3] J. Mayne, Ideals and centralizing mappings in prime rings, Proc. Amer. Math. Soc., **86,** (1982), 211–212, Erratum, **89,** (1983), 187.

[4] J. Mayne, Centralizing mappings of prime rings, Canad. Math. Bull., **27,** (1984), 122–126.

[5] J. Vukman, Symmetric bi-derivation on prime and semiprime rings, Aequations Math., **38,** (1989), 245–254.

[6] J. Vukman, Two results concerning symmetric bi-derivation on prime rings, Aequations Math., **40,** (1990), 181–189.

[7] K. H. Park, Y. S. Jung, Skew-commuting and commuting mappings in rings, Aequationes Math., **64,** (2002), 136–144 ; Erratum, **70,** (2005), 199–200.

[8] M. Bresar, On skew commuting mappings of rings, Bull. Austral. Math. Soc. 47(1993), 291-296.

[9] M. F. Smiley, Remarks on the commutativity of rings, Proc. Amer. Math. Soc. 10(1959), 466-470.

[10] N. M. Muthana, Left cetralizer traces, generalized bi-derivations left bimultipliers and generalized Jordan biderivations, The aligarh Bulletin of Mathematics, **26,** no. 2, (2007), 33–45.

[11] U. Leerawat, S. Lapuangkham, Generalized centralizing and skew-centralizing mappings on rings, International Journal of Mathematics and Computer Science, **15,** no. 2, (2020), 577–585.

[12] U. Leerawat, S. Lapuangkham, On generalized skew–commuting mappings of prime rings, JP Journal of Algebra, Number Theory and Applications, **46,** no. 2,(2020), 133–144.

[13] Y. S. Jung, Bi-additive maps and commutativity in rings, Kyungpook Math. J., **42,** (2002), 381–389.