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Composition of Happy Functions and Digit Maps

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Abstract

Chase [1] introduced the concept of digit maps generalizing that of happy functions. We extend the investigation further by considering compositions of various digit maps. We prove that if F is such a composition and x is any positive integer, then the sequence $(F^{(n)}(x))_{n\geq 0}$ either converges or eventually becomes a cycle. Furthermore, we show that the number of all possible limits and cycles is finite.

1 Introduction

For integers $e, b \geq 2$, let $S_{e,b} : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ be the function that takes a nonnegative integer x to the sum of the e-th powers of its digits in base b, that is,

$$S_{e,b}(x) = a_k^e + a_{k-1}^e + \cdots + a_0^e,$$

if $x = (a_k a_{k-1} \cdots a_0)_b = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_0$ is the b-adic expansion of x with $a_k \neq 0$ and $a_i \in \{0, 1, \ldots, b-1\}$ for all $i = 0, 1, \ldots, k$. We call $S_{e,b}$ an (e, b)-happy function and if there exists $n \in \mathbb{N}$ such that $S_{e,b}^{(n)}(x) = 1$,

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then we call x an (e, b)-happy number. Here and throughout this article, $f^{(0)}$ is the identity function mapping x to x and $f^{(n)} = f^{(n-1)} \circ f$ is the n-fold composition of f. In addition, if we write a number without specifying a base, then it is always written in base 10.

It is well-known that [7, Section E34] for any $x \in \mathbb{N}$, the sequence $(S_{2,10}^{(n)}(x))_{n\geq 0}$ either converges to 1 or eventually becomes the cycle

$$(4, 16, 37, 58, 89, 145, 42, 20).$$

For example, the sequence $(S_{2,10}^{(n)}(13))_{n\geq 0}$ is $(13, 10, 1, 1, \ldots)$ and $(S_{2,10}^{(n)}(2))_{n\geq 0}$ is $(2, 4, 16, \ldots, 20, 4, 16, \ldots)$, so 13 is $(2, 10)$-happy but 2 is not. As usual, $(a_1, a_2, \ldots, a_k)$ and any of its cyclic permutation are considered the same cycle.

El-Sedy and Siksek [3] were the first to prove that there exist arbitrarily long strings of consecutive integers which are $(2, 10)$-happy. That is, for each $m \geq 1$, there exists an integer ℓ_0 such that every element of the finite sequence $\ell_0 + 1, \ell_0 + 2, \ldots, \ell_0 + m$ is a happy number. Pan [11] obtained in 2009 that if $e-1$ is not divisible by $p-1$ for any prime divisor p of $b-1$, then there exist arbitrarily long sequences of consecutive (e, b)-happy numbers.

Let P be the product of all prime divisors p of $b-1$ such that $p-1$ divides $e-1$. It is not difficult to verify that $S_{e,b}(n) \equiv n \pmod{P}$ for every n, and so if $P > 1$, then (e, b)-happy numbers do not contain consecutive integers. Zhou and Cai [17] extended Pan's result by proving that if $P > 1$, then the (e, b)-happy numbers contain arbitrarily long arithmetic progressions with common difference P.

About 9 years later, Chase [1] introduced a concept of digit maps generalizing that of happy functions and obtained a theorem extending those by Pan [11] and El-Sedy and Siksek [3]. Noppakeaw, Phoopha, and Pongsriiam [10] consider compositions of various (e, b)-happy functions. For each $\underline{e} = (e_1, e_2, \ldots, e_k) \in \mathbb{N}^k$ and $\underline{b} = (b_1, b_2, \ldots, b_k) \in \mathbb{N}^k$ with $e_i \geq 1$ and $b_i \geq 2$ for all $i = 1, 2, \ldots, k$, they [10] defined an $(\underline{e}, \underline{b})$-happy function $S_{\underline{e},\underline{b}} : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ by

$$S_{\underline{e},\underline{b}}(x) = (S_{e_1,b_1} \circ S_{e_2,b_2} \circ \cdots \circ S_{e_k,b_k})(x) \quad \text{for all } x \in \mathbb{N} \cup \{0\}.$$

and showed that for each $x \in \mathbb{N}$, the iteration sequence $\left(S_{\underline{e},\underline{b}}^{(n)}(x)\right)_{n\geq 0}$ either converges to a fixed point or eventually enters into a cycle. Moreover, they [10] proved that the number of all such fixed points and cycles is finite. This implies the possibility of obtaining similar results on $(\underline{e}, \underline{b})$-happy numbers.

For other results on happy numbers and happy functions, we refer the reader to [4, 9, 13, 14]. For results on long arithmetic progressions in other integer sequences, see [2, 5, 6, 8, 12, 15, 16] for example.

In this article, we combine the ideas from Chase [1] and Noppakeaw, Phoopha, and Pongsriiam [10] and study the composition of various digit maps. We show that such a composition also has the same property as $S_{\underline{e},\underline{b}}$.

2 Results

We first recall the definition of digit maps and u-integers from [1].

Definition 2.1. *Let $b \geq 2$ be an integer. A digit map with respect to b is a function $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ satisfying $\gcd(b, f(b-1)) = 1$, $f(0) = 0$, $f(1) = 1$, and*

$$f(x) = f(a_k) + f(a_{k-1}) + \cdots + f(a_0)$$

if $x = (a_k a_{k-1} \cdots a_0)_b = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_0$ is the b-adic expansion of x where $a_i \in \{0, 1, \ldots, b-1\}$ for all $i = 0, 1, \ldots, k$ and $a_k \neq 0$.

If f is a digit map with respect to a base $b \geq 2$ and $x, u \in \mathbb{N}$, then x is called a u-integer if $f^{(n)}(x) = u$ for some $n \geq 0$. When f is an (e, b)-happy function and $u = 1$, the u-integers are the same as (e, b)-happy numbers. So the following theorem extends those of Pan [11] and El-Sedy and Siksek [3].

Theorem 2.2. (Chase [1])*Let $b \geq 2$ be an integer. Suppose f is a digit map with respect to b and there is an $m \in \{0, 1, \ldots, b-1\}$ such that $\gcd(f(m) - m, f(b-1)) = 1$. If $u, n \in \mathbb{N}$ and u is a member of a cycle, then there exists $\ell \in \mathbb{N}$ such that $\ell, \ell+1, \ell+2, \ldots, \ell+n-1$ are u-integers.*

To extend Theorem 2.2 in the future, it may be useful to have a function g such that, for each $x \in \mathbb{N}$, the iteration sequence $(g^{(n)}(x))_{n \geq 0}$ converges to a fixed point or eventually enters into a cycle. Noppakeaw, Phoopha, and Pongsriiam [10] obtained such a function g by considering compositions of happy functions. Our purpose is to extend their result [10, Theorem 1.4] further to the compositions of various digit maps. To do this, consider the following two conditions for a function $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$.

(A) There exists $N_f \in \mathbb{N}$ such that $f(x) < x$ for all $x \geq N_f$.

(B) For each $x \in \mathbb{N} \cup \{0\}$, the sequence $(f^{(n)}(x))_{n \geq 0}$ converges to a fixed point or eventually enters into a cycle. In addition, the number of all such fixed points and cycles is finite.

We first show that a digit map satisfies the condition (A) and if $f_1, f_2, \ldots, f_k$ satisfy (A), then so does $f_1 \circ f_2 \circ \cdots \circ f_k$. A proof of a similar result was already done in [10, Theorem 1.3] but it was for $f : \mathbb{N} \to \mathbb{N}$. So we need to adjust it for $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$. Recall also that, for $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the largest integer less than or equal to x and $\lceil x \rceil$ is the smallest integer larger than or equal to x.

Theorem 2.3. *Let f be a digit map with respect to $b \geq 2$. Then there exists $M \in \mathbb{N}$ such that*

$$f(x) < x \quad \text{for all } x \geq M. \tag{2.1}$$

Proof. Let $M' = \max\{f(i) \mid i = 0, 1, \ldots, b-1\}$. Then $M' \geq f(1) = 1$. Since $e^x/x \to \infty$ as $x \to \infty$, there exists $c > 1$ such that $e^c/c > bM'/\log b$. This implies

$$c - \log c > \log b + \log M' - \log\log b. \tag{2.2}$$

Let $M = \lceil \frac{cM'}{\log b} \rceil$ and $x \geq M$. Next, we show that $f(x) < x$. We write $x = (a_k a_{k-1} \cdots a_1 a_0)_b$ where $a_k \neq 0$ and $0 \leq a_i < b$ for all $i = 0, 1, 2, \ldots, k$. Then $b^k \leq a_k b^k \leq x$. So $k \leq \frac{\log x}{\log b}$ and

$$f(x) = f(a_k) + f(a_{k-1}) + \cdots + f(a_0) \leq M'(k+1) \leq M'\big(\frac{\log x}{\log b} + 1\big). \tag{2.3}$$

Let $h(y) = \frac{y}{M'} - \frac{\log y}{\log b} - 1$ for all $y > 0$. Then $h'(y) = \frac{1}{M'} - \frac{1}{y \log b} > 0$ for all $y > \frac{M'}{\log b}$. Since $M \geq cM'/\log b > M'/\log b$ and h is increasing on $[M'/\log b, \infty)$, we obtain that if $y \geq M$, then

$$h(y) \geq h\big(M\big) \geq h\big(cM'/\log b\big) = \frac{c - \log c - \log M' + \log\log b - \log b}{\log b} > 0,$$

where the last inequality is obtained from (2.2). This shows that $h(y) > 0$ for all $y \geq M$. In particular, $h(x) > 0$, and so $1 + \log x/\log b < x/M'$. By (2.3), we obtain $f(x) < x$, as required. □

Theorem 2.4. *If $f_1, f_2, \ldots, f_k : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ satisfy the condition* (A), then $f_1 \circ f_2 \circ \cdots \circ f_k$ also satisfies (A).

Proof. We can prove this by induction on k and it is actually the same as that given by Noppakeaw et al. [10, Theorem 1.3], but for completeness, we give the proof again here. When $k = 1$, the result is obvious. Assume that $k \in \mathbb{N}$ and the result holds for k. Suppose $f_1, f_2, \ldots, f_{k+1} : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$

satisfy (A). Let $f = f_1 \circ f_2 \circ \cdots \circ f_{k+1}$ and $g = f_1 \circ f_2 \circ \cdots \circ f_k$. Then there are $m_1, m_2 \in \mathbb{N}$ such that

$$g(x) < x \quad \text{for all } x \geq m_1, \quad \text{and} \quad f_{k+1}(x) < x \quad \text{for all } x \geq m_2. \tag{2.4}$$

Let $m_3 = \max\{g(x) \mid 1 \leq x < m_1\}$ and $m = \max\{m_1, m_2, m_3\} + 1$. Let $x \geq m$. We will show that $f(x) < x$. If $f_{k+1}(x) \geq m_1$, then we obtain by (2.4) that

$$f(x) = g(f_{k+1}(x)) < f_{k+1}(x) < x.$$

On the other hand, if $f_{k+1}(x) < m_1$, then $f(x) = g(f_{k+1}(x)) \leq m_3 < m \leq x$. This completes the proof. □

Corollary 2.5. *A composition of digit maps satisfies the condition* (A).

Proof. This follows immediately from Theorems 2.3 and 2.4. □

Theorem 2.6. *If* $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ *satisfies* (A), then f satisfies (B).

Proof. This is given in [10, Theorem 1.2] for a function $f : \mathbb{N} \to \mathbb{N}$, and we can use the same method in our proof too. However, directly applying [10, Theorem 1.2] does not lead to our desired result for $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$, so we still used to give the proof here. For convenience, we write N instead of N_f and we assert that

$$\text{for every } y \in \mathbb{N} \cup \{0\}, \text{ there exists } n \in \mathbb{N} \cup \{0\} \text{ such that } f^{(n)}(y) < N. \tag{2.5}$$

If $y < N$, then we can choose $n = 0$. If $y \geq N$, then by (A), $f(y) < y$. If $f(y) < N$, then we can choose $n = 1$; otherwise, we obtain by (A) that $f^{(2)}(y) < f(y)$. We can repeat this process and obtain a strictly decreasing sequence of positive integers $f(y), f^{(2)}(y), f^{(3)}(y), \ldots$, and eventually $f^{(n)}(y) < N$ for some n. Hence (2.5) is proved.

Now let $x \in \mathbb{N} \cup \{0\}$ and suppose that $(f^{(n)}(x))_{n \geq 0}$ does not converge to a fixed point of f. By (2.5), there exists $n_1 \in \mathbb{N}$ such that $f^{(n_1)}(x) < N$. Again by (2.5), there exists $n_2 \in \mathbb{N}$ such that $f^{(n_2)}(f^{(n_1)}(x)) < N$. Repeating this process $N + 1$ times, we obtain the set of nonnegative integers

$$f^{(n_1)}(x), f^{(n_1+n_2)}(x), \ldots, f^{(n_1+n_2+\cdots+n_{N+1})}(x),$$

which are less than N. By the pigeonhole principle, some of them are the same, say

$$f^{(n_1+n_2+\cdots+n_j)}(x) = f^{(n_1+n_2+\cdots+n_j+\cdots+n_\ell)}(x) \quad \text{for some } \ell > j \geq 1.$$

Let $y = f^{(n_1+n_2+\cdots+n_j)}(x)$. Then the tail of the sequence $(f^{(n)}(x))_{n\geq 0}$ eventually becomes

$$(y, f(y), f^{(2)}(y), \ldots, f^{(n_{j+1}+n_{j+2}+\cdots+n_\ell-1)}(y), y, \ldots),$$

which is a cycle. This proves the first part of (B). Next we show that the set U_f of fixed points and cycles is finite. More precisely, we will show that

$$U_f := \{x \in \mathbb{N} \cup \{0\} \mid \exists n \in \mathbb{N}, f^{(n)}(x) = x\} \subseteq [0, M], \tag{2.6}$$

where $M = \max\{N, f(0), f(1), f(2), \ldots, f(N)\}$. First of all, by (A), if x is a fixed point of f, then $x < N$ and so $x \in [0, M]$. Suppose that x is an element in a cycle arising from the iteration $(f^{(n)}(y))_{n\geq 0}$ for some $y \in \mathbb{N} \cup \{0\}$. If $x < N$, then $x \in [0, M]$ and we are done. So suppose $x \geq N$. By (2.5), there exists $n \in \mathbb{N}$ such that $f^{(n)}(x) < N$. Since x is in a cycle, after some iterations, it must come back to x. That is, there exists $k \in \mathbb{N}$ such that $f^{(k)}(f^{(n)}(x)) = x$. If $k = 1$ or $f^{(n+k-1)}(x) \leq N$, then $x = f(f^{(n+k-1)}(x)) \leq M$ and we are done. So suppose $k \geq 2$ and $f^{(n+k-1)}(x) > N$. Let ℓ be the smallest positive integer such that $f^{(n+k-\ell)}(x) < N$. Then $1 < \ell \leq k$ and for each $1 \leq i < \ell$, $f^{(n+k-i)}(x) \geq N$. So

$$f^{(n+k-\ell+1)}(x) > f^{(n+k-\ell+2)}(x) > \cdots > f^{(n+k-1)}(x) > f^{(n+k)}(x) = x.$$

So $x < f^{(n+k-\ell+1)}(x) = f(f^{(n+k-\ell)}(x)) \leq M$. Therefore (2.6) is verified and the proof is complete. □

Corollary 2.7. *Let $f_1, f_2, \ldots, f_k$ be digit maps with respect to $b_1, b_2, \ldots, b_k$ respectively, where $b_i \geq 2$ for every i. Let $F : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ be given by $F = f_1 \circ f_2 \circ \cdots \circ f_k$. Then F satisfies* (B).

Proof. This follows immediately from Corollary 2.5 and Theorem 2.6. □

Suppose that $f_1, f_2, \ldots, f_k$ are digit maps with respect to bases b_1, b_2, ..., b_k, respectively, and $F = f_1 \circ f_2 \circ \cdots \circ f_k$. By Corollary 2.5, there is $N \in \mathbb{N}$ such that $F(x) < x$ for all $x \geq N$. Then all fixed points and cycles can be found by considering the sequence $(F^{(n)}(x))_{n\geq 0}$ where $0 \leq x < N$. We show some explicit calculations for such N in the following example.

Example 2.8. *Consider $F = f_1 \circ f_2 \circ f_3$, where f_1, f_2, f_3 are digit maps with respect to $b_1 = 6$, $b_2 = 5$, $b_3 = 4$, respectively, and for $0 \leq x < b_i$, they are defined by $f_1(x) = x^4$, $f_2(x) = x^2$, $f_3(x) = x^3$.*

First, we show that there exists an integer m such that $(f_1 \circ f_2)(x) < x$ for all $x > m$. By following the proof of Theorem 2.3, we consider $M_1' =$

$\max\{f_1(i) \mid i = 0, 1, \ldots, 5\} = 5^4$ *and* $M_2' = \max\{f_2(i) \mid i = 0, 1, 2, 3, 4\} = 4^2$. *Since* $\frac{e^{10}}{10} > \frac{6(5)^4}{\log 6}$, $\frac{e^6}{6} > \frac{5(4)^2}{\log 5}$, *we let* $c_1 = 10$ *and* $c_2 = 6$. *The corresponding* M_1 *for* f_1 *and* M_2 *for* f_2 *are* $M_1 = 3489$ *and* $M_2 = 60$. *Therefore*

$$f_1(x) < x \quad \text{for all } x \geq 3489 \quad \text{and} \quad f_2(x) < x \quad \text{for all } x \geq 60.$$

Let $m_1 = 3489$, $m_2 = 60$, *and* $m_3 = \max\{f_1(x) \mid 1 \leq x < 3489\}$. *Since* $3489 = (24053)_6$, *we see that* $m_3 = f_1((15555)_6) = 2501$.

Let $m = \max\{m_1, m_2, m_3\} + 1 = 3490$. *By the proof of Theorem 2.4, we have that*

$$\big(f_1 \circ f_2\big)(x) < x \quad \text{for all } x \geq 3490.$$

Next, we consider $f_1 \circ f_2 \circ f_3$. *Similarly,* $M_3' = 3^3$ *and* $\frac{e^7}{7} > \frac{4(3)^3}{\log 4}$, *so we let* $c_3 = 7$, *and* $M_3 = \left\lceil \frac{7(3)^3}{\log 4} \right\rceil = 137$ *and obtain*

$$f_3(x) < x \quad \text{for all } x \geq 137.$$

We let $m_1 = 3490$, $m_2 = 137$, $m_3 = \max\{(f_1 \circ f_2)(x) \mid 1 \leq x < 3490\}$. *Then*

$$\begin{aligned} m_3 &= \max\{f_1(f_2(x)) \mid 1 \leq x < (102430)_5\} = \max\{f_1(x) \mid 1 \leq x < 80\} \\ &= \max\{f_1(x) \mid 1 \leq x < (212)_6\} = 1251. \end{aligned}$$

Let $m = \max\{m_1, m_2, m_3\} + 1 = 3491$. *So*

$$\big(f_1 \circ f_2 \circ f_3\big)(x) < x \quad \text{for all } x \geq 3491.$$

The lower bound 3491 *may not be best possible but it is not difficult to search for the best one by using a computer. We can check whether* $(f_1 \circ f_2 \circ f_3)(x) < x$ *for* $x = 1, 2, 3, \ldots, 3490$. *If* $(f_1 \circ f_2 \circ f_3)(x) < x$ *for* $x = N, N+1, \ldots, 3490$ *and* $(f_1 \circ f_2 \circ f_3)(x) \geq x$ *for* $x = N-1$, *then such the integer* N *is the optimal lower bound. In fact, by using a computer, we obtain* $N = 831$.

We give two more examples to illustrate alternative calculations.

Example 2.9. *Let* $b_1 = b_2 = 10$ *and let* f_1, f_2 *be digit maps with respect to* b_1, b_2 *such that* $f_1(x) = 2x^2 - x$ *and* $f_2(x) = 3x^3 - x^2 - x$ *for* $0 \leq x < 10$. *Let* $F = f_1 \circ f_2$. *Then, for each* $x \in \mathbb{N}$, *the sequence* $(F^{(n)}(x))_{n \geq 0}$ *either converges to* 1 *or eventually becomes the cycle* $(6, 132, 240, 154, 166, 23, 211)$.

Proof. We first show that

$$F(x) = (f_1 \circ f_2)(x) < x \quad \text{for all } x \geq 10930. \tag{2.7}$$

f_1	f_2	f_3	$\underline{b}$	Fixed points of F or cycles in $(F^{(n)}(x))_{n\geq 0}$
$3x^3-x^2-x$	$2x^2-x$		(10,10)	1, (606,88,190,518,1213,87,20)
$2x^2-x$	$3x^3-x^2-x$		(10,10)	1,(6,132,240,154,166,23,211)
$2x^2-x$	$3x^3-x^2-x$		(7,5)	1,6,43,(56,16,82,112),(61,111,35,15)
$2x^2-x$	$3x^3-x^2-x$	$3x^4+2x^2-4x$	(4,5,7)	1,7,53
$2x^2-x$	$3x^3-x^2-x$	$3x^4+2x^2-4x$	(5,4,7)	1
$\lfloor e^x \rfloor - 1$			8	1,(1114,32,53,549,201,21,153,26,25, 20,59),1103,(462,1498,1126)
$\lfloor e^x \rfloor - 1$	x^2		(8,10)	1,(59,154,153,72,549,1102,402)

Table 1: Fixed points of F (except zero) or cycles in $(F^{(n)}(x))_{n\geq 0}$

If $x \in [10930, 99999]$, then $x = (a_4a_3a_2a_1a_0)_{10}$ where $0 \leq a_i \leq 9$, and so

$$f_2(x) = f_2(a_4) + f_2(a_3) + \cdots + f_2(a_0) \leq 5f_2(9) = 10485,$$

and thus

$$F(x) \leq \max\{f_1(x) \mid 1 \leq x \leq 10485\} = f_1(9999) = 4(153) = 612 < x.$$

Next, suppose that $x \geq 10^5$ and write $x = (a_ka_{k-1}\cdots a_1a_0)_{10}$ where $k \geq 5$ and $a_k \neq 0$.

It is easy to prove by induction on k that $2097(k+1) < 10^k$ for all $k \geq 5$. Then,

$$f_2(a_k) + f_2(a_{k-1}) + \cdots + f_2(a_0) \leq (k+1)f_2(9) = 2097(k+1) < 10^k.$$

Then, $F(x) \leq \max\{f_1(x) \mid 0 \leq x \leq 10^k\} = f_1(\underbrace{99\cdots 9}_{k \text{ digits}}) = 153k < 10^k \leq a_k10^k \leq x$. So (2.7) is verified. It only remains to check that, for each $x < 10930$, whether the sequence $(F^{(n)}(x))_{n\geq 0}$ converges to a fixed point or becomes a cycle. This can be done using a computer. We find that for each positive integer $x < 10930$, the sequence $(F^{(n)}(x))_{n\geq 0}$ converges to 1 or becomes the cycle $(6, 132, 240, 154, 166, 23, 211)$. □

The next example is slightly different from Example 2.9 because b_1 and b_2 are different.

Example 2.10. *Let $b_1 = 7$, $b_2 = 5$, f_1 and f_2 digit maps with respect to b_1 and b_2, respectively, $f_1(x) = 2x^2 - x$ for $0 \leq x \leq 6$, and $f_2(x) = 3x^3 - 2x$ for $0 \leq x \leq 4$. Let $F = f_1 \circ f_2$. Then, for each $x \in \mathbb{N}$, the sequence $(F^{(n)}(x))_{n\geq 0}$ contains 1, 6, 43, 56, or 61. Moreover, 1, 6, and 43 are the only fixed points of F and if the sequence $(F^{(n)}(x))_{n\geq 0}$ does not contain 1, 6, or 43, then it eventually enters into the cycles $(56, 16, 82, 112)$ or $(61, 111, 35, 15)$.*

Proof. We first show that $F(x) < x$ for all $x \geq 1030$. Let $x \geq 1030$. Since $x > 5^4$, we write $x = (a_k a_{k-1} \cdots a_0)_5$, where $k \geq 4$, $0 \leq a_i \leq 4$ for every i, and $a_k \neq 0$. Then $f_2(x) \leq (k+1)f_2(4) = 184(k+1)$ and it is easy to prove by induction on k that $184(k+1) < 7^k$ for all $k \geq 4$. Then

$$F(x) \leq \max\{f_1(x) \mid 0 \leq x < 7^k\} = f_1((\underbrace{66\cdots6}_{k \text{ digits}})_7) = 66k.$$

Since $5^k \leq a_k 5^k \leq x$, it follows that $k \leq \frac{\log x}{\log 5}$. Since the function $y \to \frac{\log y}{y}$ is decreasing on $[3, \infty)$ and $x \geq 1030$, we obtain

$$F(x) \leq 66k \leq 66\left(\frac{\log x}{\log 5}\right) \leq \frac{66}{\log 5}\left(\frac{\log x}{x}\right)x \leq \frac{66}{\log 5}\left(\frac{\log 1030}{1030}\right)x < x.$$

Similar to Example 2.9, the rest can be verified using a computer. □

Other examples of compositions of digit maps and their fixed points and cycles are shown in Table 1. For instance, Line 5 of Table 1 means that if f_1, f_2, f_3 are digit maps such that $f_1(x) = 2x^2 - x$ for $0 \leq x \leq 3$, $f_2(x) = 3x^3 - x^2 - x$ for $0 \leq x \leq 4$, and $f_3(x) = 3x^4 + 2x^2 - 4x$ for $0 \leq x \leq 6$, then the fixed points of $F = f_1 \circ f_2 \circ f_3$ are 1, 7, and 53, and for any $x \in \mathbb{N}$, $(F^{(n)}(x))_{n \geq 0}$ converges to 1, 7, or 53. Note that zero is also a fixed point of F but we are not interested in this fixed point since in our example $(F^{(n)}(x))_{n \geq 0}$ does not converge to zero for any $x \in \mathbb{N}$.

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