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Grünwald-Leitnikov fractional derivative for a product of two functions

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Abstract

In this paper, we introduce a procedure to approximate the Grünwald-Leitnikov fractional derivative of order $\alpha \in (0, 1)$ for a product of two functions. This procedure uses the Laplace transform for a product of functions.

1 Introduction

In fractional calculus, one can take the derivative a non-integer number of times. An introduction for fractional derivatives was given in [4]. Historically, the concept of the fractional calculus is more than 300 years old. Oliver Heaviside introduced the practical use of fractional differential operators in electrical transmission back in 1890. Recently, scientists and engineers have come to realize the potential fractional calculus presents in applied science. Research has been done successfully on applications using models and tools from fractional calculus , see [7], [8], [9], and [10]. The Grünwald-Letnikov derivative was introduced by Anton Karl Grünwald in 1867 and by Aleksey Vasilievich Letnikov in 1868. The Grünwald-Letnikov derivative [10] is defined as:

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Definition 1.1. *For $f : \boldsymbol{R} \to \boldsymbol{R}$, the Grünwald-Letnikov fractional derivative of order α of the function f is given by the equation:*

$$(D^{\alpha} f)(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\infty} \frac{(-\alpha)_j}{j!} f(t - jh).$$

The above derivative is also called Forward Grünwald-Letnikov fractional derivative. More details were given by Ortigeira et al. in [11]. The Grünwald-Letnikov derivative was one of the proposed derivatives to answer Leibniz's question related to the definition of the fractional derivative; many investigations were done related to the effective use of this derivative.

Suppose that f is a real-valued function and s is a complex variable. The Laplace transform of $f(t)$ is defined as

$$\mathcal{L}_s\left(f(t)\right) = (\mathcal{L}f)(s) = F(s) = \int_0^{\infty} e^{-st} f(t)dt. \tag{1.1}$$

We note that the Laplace transform of the product of two functions is not the product of their Laplace transforms. However, the Laplace transform of a convolution of two functions equals the product of their Laplace transforms.

1.1 The Mittag-Leffler functions

Another important tool to deal with fractional calculus is the Mittag-Leffler function $E_{a,b}(t)$ and it is defined as

$$E_{a,b}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(an + b)}. \tag{1.2}$$

Proposition 1.2. *We have*

1. $E_{0,1}(t) = \frac{1}{1-t}, |t| < 1,$
2. $E_{1,1}(t) = e^t,$
3. $E_{2,1}(-t^2) = \cos(t)$ *and* $E_{2,1}(t^2) = \cosh(t),$
4. $E_{2,2}(-t^2) = \frac{\sin(t)}{t}$ *and* $E_{2,2}(t^2) = \frac{\sinh(t)}{t}.$

Definition 1.3. *Define $\psi_{a,b}(t)$ to be the antiderivative of $E_{a,b}(t)$. In this case,*

$$\psi_{a,b}(t) = \sum_{n=0}^{\infty} \frac{t^{n+1}}{(n+1)\Gamma(an+b)} = \sum_{n=1}^{\infty} \frac{t^n}{n\Gamma(an+b-a)}.$$

For example,

$$\psi_{1,1}(t) = \sum_{n=1}^{\infty} \frac{t^n}{n\Gamma(n)} = \sum_{n=1}^{\infty} \frac{t^n}{\Gamma(n+1)}$$
$$= \sum_{n=1}^{\infty} \frac{t^n}{n!} = e^t - 1,$$

which is an antiderivative for $E_{1,1}(t)$.

For further discussion on the Mittag-Leffler function properties, see [9].

2 Preliminaries

In this section, we will give the fractional derivative of order α for products of two functions. Also, using these results, Lemma 2.1,Theorem 2.2 and differential transform allows us to solve fractional differential equations.

Lemma 2.1. *[10] For $\alpha \in (0,1)$, the Laplace transform of Grünwald-Letnikov derivative of $(D^\alpha f)(t)$ is $s^\alpha F(s)$.*

In [3], an explicit formula for the Laplace transform of a product of two functions is given as follows:

Theorem 2.2. *Assume that $\mathcal{L}(f(t)) = F(s)$ and $\mathcal{L}(g(t)) = G(s)$. If $\int_0^\infty \int_0^\infty e^{-(s+\xi)t} g(\xi) f(t) dt d\xi$ converges absolutely for $s > b$, then $\mathcal{L}_s(f(t)G(t))$ is given as*

$$\mathcal{L}_s\Big(f(t)G(t)\Big) = \mathcal{L}_s\Big(f(t)\mathcal{L}(g(t))\Big) = \int_0^\infty g(\xi)F(\xi+s)d\xi = \int_s^\infty g(\xi-s)F(\xi)d\xi \text{ for } s > b. \tag{2.3}$$

This theorem can be stated as: if $\int_0^\infty \int_0^\infty e^{-(s+\xi)t}(\mathcal{L}^{-1}h)(\xi) f(t) dt d\xi$ converges absolutely for $s > b$, then

$$\mathcal{L}_s\Big(f(t)h(t)\Big) = \int_0^\infty (\mathcal{L}^{-1}h)(\xi)(\mathcal{L}f)(\xi+s)d\xi = \int_s^\infty (\mathcal{L}^{-1}h)(\xi-s)(\mathcal{L}f)(\xi)d\xi \text{ for } s > b. \tag{2.4}$$

Example 2.3. *Using the fact that* $\mathcal{L}\left(\cos(at)\right) = \frac{s}{s^2+a^2}$, *we get*

$$\begin{aligned}\mathcal{L}_s\left(\frac{tf(t)}{t^2+a^2}\right) &= \mathcal{L}\left(f(t)\mathcal{L}(\cos(at))\right)\\ &= \int_s^{\infty} \cos(\xi - s)F(\xi)d\xi\\ &= \cos(s)\int_s^{\infty}\cos(\xi)F(\xi)d\xi - \sin(s)\int_s^{\infty}\sin(\xi)F(\xi)d\xi.\end{aligned}$$

3 Main result

Using Theorem 2.2 and Lemma 2.1, we have the following result:

Theorem 3.1. *The Grünwald-Letnikov derivative of order* $\alpha \in (0,1)$ *of* fg *is given by:*

$$(D^{\alpha}fg)(t) = \mathcal{L}^{-1}\{s^{\alpha}\int_0^{\infty}(\mathcal{L}^{-1}g)(\xi)(\mathcal{L}f)(\xi+s)d\xi\} = \mathcal{L}^{-1}\{s^{\alpha}\int_s^{\infty}(\mathcal{L}^{-1}g)(\xi-s)(\mathcal{L}f)(\xi)d\xi\}.$$

Proof. According to Lemma 2.1, the Laplace transform of $(D^{\alpha}fg)(t)$ is $s^{\alpha}\mathcal{L}\Big(f(t)g(t)\Big)$. Therefore,

$$(D^{\alpha}fg)(t) = \mathcal{L}^{-1}\{s^{\alpha}\mathcal{L}\Big(f(t)g(t)\big)\}. \tag{3.5}$$

Now, using Theorem 2.2, we get:

$$\mathcal{L}_s\Big(f(t)g(t)\Big) = \int_0^{\infty}(\mathcal{L}^{-1}g)(\xi)(\mathcal{L}f)(\xi+s)d\xi = \int_s^{\infty}(\mathcal{L}^{-1}g)(\xi-s)(\mathcal{L}f)(\xi)d\xi. \tag{3.6}$$

Combining (3.5) and (3.6), the result follows. □

As applications we have the following examples:

Example 3.2. *For* $\alpha \in (0,1)$, *we prove that*

$$D^{\alpha}\Big(te^{-t}\Big) = t^{1-\alpha}\Big(E_{1,1-\alpha}(t) + \alpha E_{1,2-\alpha}(t)\Big).$$

Now,

$$\begin{aligned}D^{\alpha}\Big(te^{-t}\Big) &= \mathcal{L}^{-1}\Big(s^{\alpha}\int_s^{\infty}u^{-2}\delta(u-s-1)du\Big)\\ &= \mathcal{L}^{-1}\Big(s^{\alpha}(1+s)^{-2}\Big).\end{aligned}$$

Now, for $|s| > 1$,

$$\begin{aligned}\frac{1}{s+1} &= \frac{1}{s}\Big(\frac{1}{\frac{1}{s}+1}\Big)\\ &= \sum_{n=0}^{\infty}(-1)^n s^{-n-1}.\end{aligned}$$

Differentiating both sides gives:

$$\begin{aligned}\frac{1}{(s+1)^2} &= \frac{1}{s}\Big(\frac{1}{\frac{1}{s}+1}\Big)\\ &= \sum_{n=0}^{\infty}(-1)^n (n+1) s^{-n-2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{s^\alpha}{(s+1)^2} &= \sum_{n=0}^{\infty}(-1)^n \frac{n+1}{s^{n+2-\alpha}}\\ &= \sum_{n=0}^{\infty}(-1)^n \frac{(n+1)\Gamma(n+2-\alpha)}{\Gamma(n+2-\alpha)s^{n+2-\alpha}}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{L}^{-1}\Big(\frac{s^\alpha}{(s+1)^2}\Big) &= \sum_{n=0}^{\infty}(-1)^n \frac{(n+1)t^{n+1-\alpha}}{\Gamma(n+2-\alpha)}\\ &= \sum_{n=0}^{\infty}(-1)^n \frac{(n+1)t^{n+1-\alpha}}{(n+1-\alpha)\Gamma(n+1-\alpha)}\\ &= \sum_{n=0}^{\infty}(-1)^n \frac{(n+1-\alpha+\alpha)t^{n+1-\alpha}}{(n+1-\alpha)\Gamma(n+1-\alpha)}\\ &= \sum_{n=0}^{\infty}(-1)^n \frac{t^{n+1-\alpha}}{\Gamma(n+1-\alpha)} + \alpha\sum_{n=0}^{\infty}(-1)^n \frac{t^{n+1-\alpha}}{(n+1-\alpha)\Gamma(n+1-\alpha)}\\ &= t^{1-\alpha}\sum_{n=0}^{\infty}\frac{(-t)^n}{\Gamma(n+1-\alpha)} + \alpha t^{1-\alpha}\sum_{n=0}^{\infty}\frac{(-t)^n}{\Gamma(n+2-\alpha)}\\ &= t^{1-\alpha}\Big(E_{1,1-\alpha}(t) + \alpha E_{1,2-\alpha}(t)\Big).\end{aligned}$$

Therefore, $D^\alpha\Big(te^{-t}\Big) = t^{1-\alpha}\Big(E_{1,1-\alpha}(t) + \alpha E_{1,2-\alpha}(t)\Big)$.

Interestingly, $\lim_{\alpha\longrightarrow 1} D^\alpha\Big(te^{-t}\Big) = -te^{-t} + e^{-t} = D\Big(te^{-t}\Big)$.

Example 3.3. *For $\alpha \in (0,1)$, we prove that*

$$D^{\alpha}\Big(\frac{e^{-t}+t-1}{t^2}\Big) = t^{-\alpha-2}\Big(\psi_{1,-\alpha}(-t)+\frac{t}{\Gamma(-\alpha)}\Big) - t^{-\alpha-1}\psi_{1,1-\alpha}(-t).$$

Now,

$$\begin{aligned} D^{\alpha}\Big(\frac{e^{-t}+t-1}{t^2}\Big) &= \mathcal{L}^{-1}\Big(s^{\alpha}\int_s^{\infty}(u-s)(-\frac{1}{u}+\frac{1}{u^2}+\frac{1}{1+u})du\Big) \\ &= \mathcal{L}^{-1}\Big(s^{\alpha}\Big((s+1)\ln(1+\frac{1}{s})-1\Big)\Big). \end{aligned}$$

Since

$$\ln(1+x) = -\sum_{k=1}^{\infty}\frac{(-1)^k x^k}{k} \text{ for } |x|<1,$$

we have

$$\begin{aligned} (s+1)\ln(1+\frac{1}{s})-1 &= -(s+1)\sum_{k=1}^{\infty}\frac{(-1)^k}{ks^k}-1 \\ &= -\sum_{k=2}^{\infty}\frac{(-1)^k}{ks^{k-1}} - \sum_{k=1}^{\infty}\frac{(-1)^k}{ks^k}. \end{aligned}$$

Hence,

$$s^{\alpha}\Big((s+1)\ln(1+\frac{1}{s})-1\Big) = -\sum_{k=2}^{\infty}\frac{(-1)^k}{ks^{k-1-\alpha}} - \sum_{k=1}^{\infty}\frac{(-1)^k}{ks^{k-\alpha}}.$$

$$\begin{aligned} D^{\alpha}\Big(\frac{e^{-t}+t-1}{t^2}\Big) &= \mathcal{L}^{-1}\Big(-\sum_{k=2}^{\infty}\frac{(-1)^k}{ks^{k-1-\alpha}} - \sum_{k=1}^{\infty}\frac{(-1)^k}{ks^{k-\alpha}}\Big) \\ &= \sum_{k=2}^{\infty}\frac{(-1)^k t^{k-\alpha-2}}{k\Gamma(k-1-\alpha)} - \sum_{k=1}^{\infty}\frac{(-1)^k t^{k-\alpha-1}}{k\Gamma(k-\alpha)} \\ &= t^{-\alpha-2}\sum_{k=2}^{\infty}\frac{(-t)^k}{k\Gamma(k-1-\alpha)} - t^{-\alpha-1}\sum_{k=1}^{\infty}\frac{(-t)^k}{k\Gamma(k-\alpha)}. \end{aligned}$$

Using Definition 1.3, we get

$$D^{\alpha}\Big(\frac{e^{-t}+t-1}{t^2}\Big) = t^{-\alpha-2}\Big(\psi_{1,-\alpha}(-t)+\frac{t}{\Gamma(-\alpha)}\Big) - t^{-\alpha-1}\psi_{1,1-\alpha}(-t).$$

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