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On Cayley Graphs of Rees Matrix Semigroups Relative to Green's Equivalence $\mathcal{L}$-Class

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Abstract

In 1878, Cayley introduced graphs of groups. Later, graphs of semigroups were introduced as generalization of Cayley graphs of groups. In 1940, Rees introduced matrix semigroups. In this paper, we describe some properties of Cayley graphs of Rees matrix semigroups. We see that, for a group G and for any arbitrary non-empty finite sets I and Λ, the Cayley graph of the Rees matrix semigroup $S = M(G; I, \Lambda; P)$ relative to any Green's equivalence $\mathcal{L}$-class of S has a Hamiltonian decomposition consisting of $|\Lambda|$ components.Moreover, when $|G| \times |I|$ is odd, we show that the decomposition is Eulerian.

1 Introduction

Let G be a group, I and Λ be two arbitrary non empty sets, and P be $\Lambda \times I$ matrix with entries $p_{\lambda i}$ from $G^0 = G \cup \{0\}$. Define $S = M(G; I, \Lambda; P)$, the set of triples with composition

$$(g_1, i_1, \lambda_1)(g_2, i_2, \lambda_2) = \begin{cases} (g_1 p_{\lambda_1 i_2} g_2, i_1, \lambda_2) & , \text{ if } p_{\lambda_1 i_2} \neq 0 \\ 0 & , \text{ if } p_{\lambda_1 i_2} = 0 \end{cases}$$

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The resulting semigroup is a Rees matrix semigroup and is regular. In 1964, Bosák [1] studied certain graphs over semigroups. Lawson [5] provided a new, abstract characterization of Rees matrix semigroups over monoids with a zero. The principal ideal graph of a Rees matrix semigroup was introduced by Indu and John [4]. Here we focus on some properties of Cayley graphs of Rees matrix semigroup. Throughout this paper, I and Λ are considered as finite sets.

2 Preliminaries

In this section, we describe some basic definitions and results which are needed in the sequel. These appear in [3, 6] in connection with semigroup theory and in [2] in connection with Graph Theory.

Definition 2.1. *Let S be a semigroup. We define $a\mathcal{L}b$ $(a, b \in S)$ if and only if a and b generate the same principal left ideal; that is, if and only if $S^1a = S^1b$.*

Lemma 2.2. *Let a and b be elements of a semigroup S. Then $a\mathcal{L}b$ if and only if there exist $x, y \in S^1$ such that $xa = b$, $yb = a$.*

Definition 2.3. *If for a pair of vertices u, v of a digraph D both (u, v) and (v, u) are arcs of D, then (u, v) and (v, u) are symmetric pair of arcs and denoted it by (uv). A symmetric digraph is one in which every arc occurs as a symmetric pair. A complete symmetric digraph is the digraph with vertex set V and every symmetric pair (u, v) for $u, v \in V$. This will be denoted by K_n^*, where $n = |V|$.*

Definition 2.4. *For any $v \in V$, the number of arcs adjacent to v is the invalence of v and the number of arcs adjacent from v is the outvalence of v. We denote these by $d^-(v)$ and $d^+(v)$, respectively. The total valence or, simply, the valence of v is $d(v) = d^-(v) + d^+(v)$. If $d(v) = k$ for every $v \in V$, then D is said to be a k-regular digraph. If for every $v \in V$, $d^-(v) = d^+(v)$, the digraph is said to be an isograph.*

Definition 2.5. *Two vertices v and u of D are strongly connected if they are mutually reachable: there is a path from v to u and a path from u to v.*

Definition 2.6. *A spanning path (cycle) of a digraph is a path (cycle) that visits each vertex exactly once. A spanning path (cycle) of a digraph is called a Hamiltonian path (cycle). A digraph with a Hamiltonian cycle is called*

Hamiltonian. A graph is Hamiltonian connected if, for every pair of vertices, there is a Hamiltonian path between them.

Definition 2.7. *A tour traversing each arc of a digraph exactly once is called an Euler tour and a digraph with Euler tour is called Eulerian.*

Theorem 2.8. *A directed graph G has an Eulerian circuit if and only if it is connected and its vertices all have even invalence.*

Definition 2.9. *A sub digraph $D' = (U, B)$ of a digraph $D = (V, A)$ is said to be vertex induced subgraph or induced subgraph if B consists of all the arcs of D joining pairs of vertices of U. A decomposition of a graph $D = (V, A)$ is a set of subgraphs $H_1, H_2, \ldots, H_K$ that partition the arcs of D.*

Definition 2.10. *Let S be a finite semigroup and let H be a non-empty subset of S. The Cayley graph $Cay(S, H)$ of S relative to H is defined as the graph with vertex set S and arc set $\{(x, y) : hx = y \text{ for some } h \in H, x \neq y\}$.*

3 Results

Proposition 3.1. *Let $S = M(G; I, \Lambda; P)$ be a Rees matrix semigroup and $s_1 = (g_1, i_1, \lambda_1), s_2 = (g_2, i_2, \lambda_2) \in S$. Then*
(i) $s_1 \mathcal{L} s_2$ if and only if $\lambda_1 = \lambda_2$ (ii) $s_1 \mathcal{R} s_2$ if and only if $i_1 = i_2$

Remark 3.2. *The $\mathcal{L}$-class of a Rees matrix semigroup $S = M(G; I, \Lambda; P)$ are of the form $L_{\lambda_n} = \{(g, i_m, \lambda_n) \in S : i_m \in I\}$, for some $\lambda_n \in \Lambda$.*

Proposition 3.3. *Let $S = M(G; I, \Lambda; P)$ be a Rees matrix semigroup and L_{λ_n} be any $\mathcal{L}$-class of S. Then for any subset S' of S containing L_{λ_n}, the Cayley graph $Cay(S, S')$ is symmetric.*

Proof. Let $x \in S$ and $y \in S'$. Then, by the group operations on S, $y' L_{\lambda_n} x$, where $y' = yx$. Since $y' L_{\lambda_n} x$, there exist $u, v \in L_{\lambda_k}, 1 \leq k \leq n$ such that $uy' = x$ and $vx = y'$. Then, by Definition 2.10, there exists a bidirected arc between x and y' in $Cay(S, S')$. Since y is arbitrary, there is a bidirectd arc between x, to every $y' \in S$ and so for any arc in $Cay(S, S')$ occurs as a symmetric pair. Consequently, the graph is symmetric. □

Proposition 3.4. *Let $S = M(G; I, \Lambda; P)$ be a Rees matrix semigroup and L_{λ_n} be any $\mathcal{L}$-class of S. Then for any subset S' of S containing L_{λ_n}, the Cayley graph $Cay(S, S')$ is i) an isograph ii)2(p-1)regular, $p = |G| \times |I|$.*

Proof. Since every symmetric graph is an isograph, the proof of (i) is obvious. In proof of Proposition 3.3, we see that for any $x \in S$ in $Cay(S, S')$ is connected to every $y' \in S$, where $xL_{\lambda_n}y'$. Also, by Remark 3.2, we get $|L_{\lambda_n}| = |G| \times |I| = p$. The proof of (ii) is complete. □

Corollary 3.5. *Let $S = M(G; I, \Lambda; P)$ be a Rees matrix semigroup and L_{λ_n} be any $\mathcal{L}$-class of S. Then the graph induced by the vertex set $L_{\lambda_n'}$ of $Cay(S, L_{\lambda_n})$ is i) symmetric ii) an isograph iii) 2(p-1)regular, $p = |G| \times |I|$.*

Proposition 3.6. *Let $S = M(G; I, \Lambda; P)$ be a Rees matrix semigroup and L_{λ_n} be any $\mathcal{L}$-class of S. Then the graph induced by the vertex set $L_{\lambda_n'}$ of $Cay(S, L_{\lambda_n})$ is i) strongly connected ii) Hamiltonian connected iii) Eulerian, when $p = |G| \times |I|$ is odd.*

Proof. By Corollary 3.5, the graph induced by the vertex set $L_{\lambda_n'}$ of $Cay(S, L_{\lambda_n})$ is $2(p-1)$ regular. Since $|L_{\lambda_n}| = p$, it is K_p^*. Hence (i) and (ii) are trivial. When p is odd every vertex in this graph have even in valence. By Theorem 2.8, the proof of (iii) follows. □

Proposition 3.7. *Let $S = M(G; I, \Lambda; P)$ be a Rees matrix semigroup and L_{λ_n} be any $\mathcal{L}$-class of S. Then $Cay(S, L_{\lambda_n})$ has a Hamiltonian decomposition consisting of q subgraphs of p vertices where $p = |G| \times |I|$ and $q = |G|$.*

Proof. Let $x, y \in S$ with $x = (g_1, i_1, \lambda_1)$ and $y = (g_2, i_2, \lambda_2)$. Suppose there is an arc from x to y in $Cay(S, L_{\lambda_n})$. Then, by Definition 2.10, there exist an $(g_3, i_j, \lambda_n) \in L_{\lambda_n}$ such that $(g_3, i_j, \lambda_n)x = y$, which implies $\lambda_1 = \lambda_2$ and so $xL_{\lambda_n}y$. Thus $x, y \in L_{\lambda_n'}$, for some fixed $\lambda_n' \in \Lambda$. Since the graph induced by the vertex set $L_{\lambda_n'}$ of $Cay(S, L_{\lambda_n})$ is K_p^*, the occurrence of arc from x to y in the graph induced by the vertex set $L_{\lambda_n'}$ of $Cay(S, L_{\lambda_n})$ ensures the occurrence of the arc from x to y in the union of such induced subgraphs with vertex set $L_{\lambda_n} = \bigcup_{\lambda_n' \in \Lambda} L_{\lambda_n'}$. Since $L_{\lambda_n'} \cap L_{\lambda_n''} = \phi$, each of such induced subgraphs are disjoint.

On the other hand, there is an arc from x to y in the disjoint union of all such induced subgraphs. Hence $x, y \in L_{\lambda_n'}$, for some $\lambda_n' \in \Lambda$ and so $xL_{\lambda_n}y$. Since $xL_{\lambda_n}y$, by Proposition 3.1, we have $\lambda_1 = \lambda_2 = \lambda$. Thus $x = (g_1, i_1, \lambda)$ and $y = (g_2, i_2, \lambda)$. Also, there exist $l_1 = (h_1, i_1, \lambda_n), l_2 = (h_2, i_2, \lambda_n) \in L_{\lambda_n}$ which implies $x = l_1 y$ and $y = l_2 x$. Then, by Definition 2.10, there is a bidirected arc between x and y in $Cay(S, L_{\lambda_n})$. Hence $Cay(S, L_{\lambda_n})$ is the disjoint union of q subgraphs of p vertices, each of which is hamiltonian.

□

Corollary 3.8. *Let $S = M(G; I, \Lambda; P)$ be a Rees matrix semigroup and L_{λ_n} be any $\mathcal{L}$-class of S. Then $Cay(S, L_{\lambda_n})$ has an Eulerian decomposition when p is odd.*

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