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Positive Weighted Composition Operators on the Fock space

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Abstract

In this paper, we obtain a characterization for the bounded positive weighted composition operators and their spectrum on the Fock space of $\mathbb{C}^n$.

1 Introduction

The Fock space $\mathcal{F}^2$ also known as the Segal-Bargmann space consists of holomorphic functions on $\mathbb{C}^n$ that are square-integrable with respect to the Gaussian measure $\mu(z)\,dz = (2\pi)^{-n}e^{-|z|^2/2}\,dz$, where $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$. The inner product on $\mathcal{F}^2$ is given by

$$\langle f, g\rangle = \int_{\mathbb{C}^n} f(z)\overline{g(z)}(2\pi)^{-n}e^{-|z|^2/2}dz.$$

Then set $\{\frac{z^n}{\sqrt{2^n n!}}\}_{n=0}^{\infty}$ is an orthonormal basis for $\mathcal{F}^2$ and the reproducing function is

$$K_w(z) = e^{\frac{\langle z,w\rangle}{2}} \qquad \text{for all } w, z \in \mathbb{C}^n$$

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where $\langle z, w\rangle$ denotes the inner product for $w, z \in \mathbb{C}^n$. Let k_w be the normalization of K_w. Then

$$k_w(z) = e^{\frac{\langle z,w\rangle}{2} - \frac{|w|^2}{4}}.$$

For each fixed $z \in \mathbb{C}^n$ and $f \in \mathcal{F}^2$,

$$f(z) = \int_{\mathbb{C}^n} K_w(z) f(w)(2\pi)^{-n} e^{-|w|^2/2} dw.$$

For a given holomorphic function $\varphi\colon \mathbb{C}^n \to \mathbb{C}^n$, the composition operator C_φ on the Fock space is defined by $C_\varphi(f) = fo\varphi$. In 2003 Carswell et at [2] first studied composition operators on the class of Fock space. They provided a complete description of the boundedness and compactness together with a formular for the norm of C_φ. It has been extensively studied on the Hardy Bargmann and Block spaces on the unit disk of the complex plane.

In 2007 Ueki [8] defined a weighted composition operator on the Fock space by $C_{\psi,\varphi}(f) = \psi \cdot (fo\varphi)$ where ψ is a holomorphic function on $\mathbb{C}^n$. The domain of $C_{\psi,\varphi}$ consists of all $f \in \mathcal{F}^2$ for which $C_{\psi,\varphi}(f)$ also belongs to $\mathcal{F}^2$. When the weighted function ψ is identically one the operator $C_{\psi,\varphi}$ reduced to the composition opertor C_φ. In recent years, the study of composition operator and weighted composition operator on the Fock space has attracted a lot of attention [2]-[11]. Zhao [9] -[11] and Le [6]-[7] describe some properties of isometric weighted composition operator on the Fock space and characterized a class of a self adjoint, unitary and normal weighted composition operator on the Fock space.

In this paper we characterize the bounded positive weighted composition operators and their spectrum on the Fock space of $\mathbb{C}^n$.

2 Main Results

In 2003, Carswell et al. studied composition operators on the Segal-Bargmann space. They found the important theorem.

Theorem 2.1. *Suppose $\varphi\colon \mathbb{C}^n \to \mathbb{C}^n$ is a holomorphic mapping.*

(i) C_φ is bounded on the Segal-Bargmann space if and only if $\varphi(z) = Az+b$ where A is an $n \times n$ matrix and $b \in \mathbb{C}^n$. Furthermore, $\|A\| \le 1$, and $\langle Ap, b\rangle = 0$ if $|Ap| = |p|$ for some $p \in \mathbb{C}^n$.

(ii) C_φ is compact on the Segal-Bargmann space if and only if $\varphi(z) = Az+b$ where $\|A\| < 1$.

In this paper we characterizes a class of positive weighted composition operators on the Segal-Bargmann space. For any $p \in \mathbb{C}^n$, denote $\varphi_p(z) = z-p$ and $U_p = C_{k_p,\varphi_p}$. The following results are well know.

Lemma 2.2. *U_p is a unitary operator on $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$, with $U_p^{-1} = U_{-p}$ and $U_{-p}C_{\psi,\varphi}U_p = C_{\Psi_p,\Phi_p}$ where $\Phi_p(z) = (\varphi_p o\varphi o\varphi_{-p})(z)$ and $\Psi_p(z) = (k_{-p}(\psi o\varphi_{-p}))\dot{c}(k_p o\varphi o\varphi_{-p}))(z)$.*

Lemma 2.3. $C^*_{\psi,\varphi}K_w = \overline{\psi(w)}K_{\varphi(w)}$.

If $\varphi(z) = Az + B$ and $\psi(z) = k_c(z)$, then $U_{-c}C_{\psi,\varphi} = C_{\varphi o\varphi_{-c}}$ with $C_{\varphi o\varphi_{-c}}(z) = Az + Ac + b$. Then from Theorem 2.1 we have that $C_{\varphi o\varphi_{-c}}$ is bounded if and only if

$$\|A\| \leq 1 \quad \text{and } \langle Az, b + Ac\rangle = 0$$

and $C_{\psi,\varphi}$ is compact if and only if $\|A\| < 1$.

Theorem 2.4. *Let φ be a holomorphic function on $\mathbb{C}^n$. If C_φ is bounded on the Segal-Bargmann space with $\varphi(z) = Az + b$, then the spectrum of C_φ is*

$$\sigma(C_\varphi) = \overline{\{\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\ldots\lambda_n^{\alpha_n} \colon (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n\}}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A.

In this section we investigate bounded positive weighted composition operators on Fock space. Recall that an operator T on a Hilbert space H is positive if and only if $\langle T(f), f\rangle > 0$. for all $f \in H$. The operator is self adjoint if and only if $T = T^*$. Then all spectra of self adjoint operator are real and the eigenspace associated with distinct eigenvalues are orthogonal. The spectral radius of self adjoint operator T is equal to $\|T\|$. The main results of this paper read as follows.

Theorem 2.5. *Let φ be a holomorphic function on $\mathbb{C}^n$. Assume that C_φ is bounded on the Segal-Bargmann space. Then C_φ is positive if and only if $\varphi(z) = Az$ with A is a self adjoint $n \times n$ matrix.*

Proof. First, since C_φ is bounded, $\varphi(z) = Az + B$ where A is an$n \times n$ matrix with $\|A\| \leq 1$ and B is an $n \times 1$ vector such that $\langle Az, B\rangle = 0$. whenever $|Az| = |z|$. Assume C_φ is positive. Then

$$1 = \|C_\varphi 1\|^2 = \|C^*_\varphi 1\|^2 = \|C^*_\varphi K_0\|^2 = e^{|\varphi(0)|^2/2},$$

which implies that $\varphi(0) = 0$. Then we have $B = 0$ and $\varphi(z) = Az$. Since C_φ is positive definite, C_φ is self adjoint. By [5] shows that

$$C_\varphi = C_\varphi^* = C_\lambda, \qquad \text{where } \lambda(z) = A^* z.$$

Therefore, we have $A = A^*$. The other direction assume $\varphi(z) = Az$ with A is a self adjoint. Then for any $z \in \mathbb{C}^n$

$$\begin{aligned}
\langle C_\varphi K_z, K_z \rangle &= \int_{\mathbb{C}^n} K_z(Aw)\overline{K_z(w)}d\mu(w) \\
&= \int_{\mathbb{C}^n} e^{\langle Aw,z\rangle/2}\overline{e^{\langle w,z\rangle/2}}d\mu(w) \\
&= \int_{\mathbb{C}^n} e^{\langle w,A^*z\rangle/2}\overline{e^{\langle w,z\rangle/2}}d\mu(w) \\
&= \int_{\mathbb{C}^n} K_A^* z(w)\overline{K_z(w)}d\mu(w) \\
&= \langle K_{A^*z}, K_z\rangle \\
&= K_{A^*z}(z) \\
&= e^{\langle Az,z\rangle/2}.
\end{aligned}$$

Since A is self adjoint, for any z, $< Az, z > \in \mathbb{R}$. Therefore $\langle C_\varphi K_z, K_z\rangle > 0$ for all $z \in \mathbb{C}^n$ which implies C_φ is positive. □

Theorem 2.6. *Let $\varphi(z) = Az+b$ and $\psi(z) = k_c(z)$ with A an $n\times n$ complex matrix and $b, c \in \mathbb{C}^n$. Then $C_{\psi,\varphi}$ is a bounded positive operator on the Segal-Bargmann space if and only if A is self adjoint with $\|A\| \le 1$ and $b = c$.*

Proof. Obviously, $C_{\psi,\varphi}$ is a bounded positive operator on the Segal-Bargmann space if and only if $C_{\psi,\varphi}$ is a bounded and for $w \in \mathbb{C}^n$,

$$\langle C_{\psi,\varphi}(K_w), K_w\rangle = \langle C^*_{\psi,\varphi}(K_w), K_w\rangle > 0.$$

We have that $C_{\psi,\varphi}$ is bounded on $\mathcal{F}^2$ if and only if $\|A\| \le 1$,

$$\langle Az, b + Ac\rangle = 0$$

wherever $|Az| = |z|$ for $z \in \mathbb{C}^n$. Then by the directly computation we have $(C_{\psi,\varphi}K_w)(z) = \exp\left(\frac{\langle b,w\rangle}{2} - \frac{|c|^2}{4}\right)K_{A^*w+c}(z)$, and

$$\begin{aligned}
\langle C_{\psi,\varphi}(K_w), K_w\rangle &= \langle \exp\left(\frac{\langle b,w\rangle}{2} - \frac{|c|^2}{4}\right)K_{A^*w+c}, K_w\rangle \\
&= \exp\left(\frac{\langle b,w\rangle}{2} - \frac{|c|^2}{4}\right)K_{A^*w+c}(w) \\
&= \exp\left(\frac{\langle b,w\rangle}{2} - \frac{|c|^2}{4} + \frac{\langle w, A^*w+c\rangle}{2}\right), \qquad \text{and} \qquad (2.1)
\end{aligned}$$

$$\begin{aligned}\langle C^*_{\psi,\varphi}(K_w), K_w\rangle &= \langle \overline{\psi(w)} K_{\varphi(w)}, K_w\rangle \\ &= \overline{\psi(w)} K_{\varphi(w)}(w) \\ &= exp\Big(\frac{\langle c, w\rangle}{2} - \frac{|c|^2}{4} + \frac{\langle w, Aw+b\rangle}{2}\Big). \end{aligned} \tag{2.2}$$

Thus $\langle C_{\psi,\varphi}(K_w), K_w\rangle > 0$ and $\langle C_{\psi,\varphi}(K_w), K_w\rangle = \langle C^*_{\psi,\varphi}(K_w), K_w\rangle$ if and only if $\frac{\langle b,w\rangle}{2} - \frac{|c|^2}{4} + \frac{\langle w,A^*w+c\rangle}{2} \in \mathbb{R}$ and $\frac{\langle b,w\rangle}{2} - \frac{|c|^2}{4} + \frac{\langle w,A^*w+c\rangle}{2} = \frac{\langle c,w\rangle}{2} - \frac{|c|^2}{4} + \frac{\langle w,Aw+b\rangle}{2}$. Then $\frac{\langle b,w\rangle}{2} - \frac{|c|^2}{4} + \frac{\langle w,A^*w+c\rangle}{2} = \frac{\langle c,w\rangle}{2} - \frac{|c|^2}{4} + \frac{\langle w,Aw+b\rangle}{2}$ if and only if

$$\begin{aligned}\langle w, (A^* - A)w\rangle &= \langle w, b\rangle - \langle b, w\rangle + \langle c, w\rangle - \langle w, c\rangle \\ &= 2re(\langle w, b\rangle) + 2re(\langle c, w\rangle).\end{aligned}$$

But $\langle w, (A^* - A)w\rangle = \langle w, A^*w\rangle - \langle w, Aw\rangle \in i\mathbb{R}$ which implies that $A = A^*$. Then $\frac{\langle b,w\rangle}{2} - \frac{|c|^2}{4} + \frac{\langle w,A^*w+c\rangle}{2} \in \mathbb{R}$ if and only if $\frac{\langle b,w\rangle}{2} + \frac{\langle w,+c\rangle}{2} \in \mathbb{R}$. Letting $w = e_j$ and $w = ie_j$ in the above formula where $\{e_1, e_2, \dots, e_n\}$ is the standard basis for $\mathbb{C}^n$, we have $im(b_j) = im(c_j)$ and $re(b_j) = re(c_j)$ for all $j = 1, \dots n$ so $b = c$. □

Theorem 2.7. *Let $C_{\psi,\varphi}$ is a bounded positive operator on the Segal-Bargmann space with*

$$\phi(z) = Az + b \qquad and\ \varphi(z) = k_c(z).$$

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of A.

(1) If $\|A\| = 1$, then

$$\sigma(C_{\psi,\varphi}) = \beta\overline{\{\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\dots\lambda_n^{\alpha_n} : (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n\}}$$

where $\beta = \exp\Big(\frac{1\langle (P+I-A)^{-1}b, (I+A-2P)b\rangle}{4}\Big)$.

(2) If $\|A\| < 1$, then

$$\sigma(C_{\psi,\varphi}) = \gamma\overline{\{\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\dots\lambda_n^{\alpha_n} : (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n\}}$$

where $\gamma = \exp\Big(\frac{\langle (I-A)^{-1}b, b\rangle}{2} - \frac{|b|^2}{4}\Big)$.

Proof. From theorem2.6 we have that, $A = A^*$, $\|A\| \le 1$ and $b = c$.
(1) Assume that $\|A\| = 1$. Then $\|A\| = \max\{|\lambda_1|, |\lambda_2|, \dots |\lambda_n|\}$. There is

an eigenvalue λ_i of A with $\lambda_i = 1$ or $\lambda_i = -1$. Without loss of generality, suppose that $\lambda_i = 1$. Then there is $p \neq 0$ such that $Ap = p$. Thus

$$\begin{aligned} 0 = \langle Ap, Ac + b\rangle &= \langle p, Ab + b\rangle \\ &= \langle p, Ab\rangle + \langle p, b\rangle \\ &= \langle A^* p, b\rangle + \langle p, b\rangle \\ &= \langle p, b\rangle + \langle p, b\rangle \end{aligned}$$

implies that $\langle p, b\rangle = 0$. If $b \in \ker(I - A)$, then $b = 0$ with implies that $C_{\psi,\varphi}{=}C_A$. Assume that $b \notin \ker(I - A)$. Then $b \in \ker(I - A)^{\perp}$. Let $\{x_1, x_2, \dots, x_k\}$ be an orthonormal basis for $\ker(I - A)^{\perp}$ with for each x_j an eigenvalue corresponding to $\lambda_j \neq 1$. There are unique scalars $\beta_1, \beta_2, \dots, \beta_k$ such that $b = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k$. Then

$$Ab = \beta_1 \lambda_1 x_1 + \beta_2 \lambda_2 x_2 + \cdots + \beta_k \lambda_k x_k.$$

For any $i = 1, 2, \dots, k$, define $q_i = \frac{\beta_i x_i}{1-\lambda_i}$ and $q = q_1 + q_2 + \cdots + q_k$. Then we have $(I - A)q = b$ which implies that $\Phi_q(z) = Az$ and $\Psi_q(z) = \exp\left(\frac{\langle q,c\rangle}{2} - \frac{|b|^2}{4}\right)$. A direct computation show that

$$\frac{\langle q, c\rangle}{2} - \frac{|b|^2}{4} = \frac{1}{4}\langle (P + I - A)^{-1} b, (I + A - 2P)b\rangle$$

where P is the orthonormal projection on the eigenvector subspace of A corresponding to the eigenvalue $\lambda = 1$. Thus we have that

$$\sigma(C_{\psi,\varphi}) = \exp\left(\frac{1}{4}\langle (P{+}I{-}A)^{-1} b, (I{+}A{-}2P)b\rangle\right)\overline{\{\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\dots\lambda_n^{\alpha_n} : (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n\}}.$$

(2.) Suppose that $\|A\| < 1$. Then $1 \notin \sigma(A)$, so $(I - A)$ is invertible. Denote $(I - A)^{-1} b = p$, so $\varphi(p) = p$. Therefore we have $\Phi_p(z) = Az$ and

$$\Psi_p(z) = \exp\left(\frac{\langle p, b\rangle}{2} - \frac{|b|^2}{4}\right) = \exp\left(\frac{\langle (I - A)^{-1} b, b\rangle}{2} - \frac{|b|^2}{4}\right).$$

Thus

$$U_{-p} C_{\psi,\varphi} U_p = \exp\left(\frac{\langle (I - A)^{-1} b, b\rangle}{2} - \frac{|b|^2}{4}\right) C_A.$$

Since U_p is unitary and $U_p^{-1} = U_{-p}$,

$$\sigma(C_{\psi,\varphi}) = \exp\left(\frac{\langle (I - A)^{-1} b, b\rangle}{2} - \frac{|b|^2}{4}\right)\overline{\{\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\dots\lambda_n^{\alpha_n} : (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n\}}.$$

□

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