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On p series

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Abstract

We present an algorithm for generating a series of the form $y = \sum_{h=1}^{\infty} c_h p^h$ with coefficients either 0 or 1 and prove that it always converges to y when $\frac{1}{2} \leq p < 1$. We also demonstrate that there are infinitely many intervals for which the series fails to converge when $0 < p < \frac{1}{2}$.

1 Introduction

Definition 1.1. *We call a series of the form*

$$y = \sum_{h=1}^{\infty} c_h p^h$$

a p series provided each coefficient c_h is either 0 or 1.

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Remark 1.2. *When only a finite number of the c_h are 1 the series reduces to a polynomial and so there is, of course, no question regarding convergence of the series. Here though, we wish to also consider cases for which infinitely many c_h are 1 and the number p is real. In these cases, such a non-trivial series (not all c_h are 0) converges provided -1 <p <1.*

In this article we will consider only cases such that

$$0 < p < 1$$

and we address the question "What real numbers y are represented by such a p series?"

In this paper we present an algorithm for generating p series coefficients. We prove that when $p \geq \frac{1}{2}$ the algorithm always produces a series converging to y (for $p = \frac{1}{2}$, any $y \in [0, 1]$ can be so expressed some with two representations-this is just the base 2 representation). We also show that there are infinitely many intervals for which the algorithm fails to produce a series converging to y when $p < \frac{1}{2}$.

2 Results

Lemma 2.1. *If $0 < p < 1$, then the partial sums*

$$L\,(k) = \sum_{h=1}^{k} c_h p^h$$

for a p series are a non-decreasing sequence which converges to a number $y \leq \frac{p}{q}$, where $q = 1 - p$.

Proof. Since the terms in the partial sums are non-negative, the non-decreasing character of the sequence is clear. Clearly, $c_h p^h \leq p^h$. So, for all k, we have

$$L(k) \leq S(k) = \sum_{n=1}^{k} p^k = \frac{p - p^{k+1}}{1 - p} < \frac{p}{q}.$$

Clearly then the sequence is convergent to some number y^* which satisfies

$$L(k) \leq y^* \leq \frac{p}{q},$$

for all positive integers k.

ALGORITHM. Given a number y such that
$0 < y < \frac{p}{q}$,
we define the p series by initiating
$L(0) = 0$,
and for positive integers h we use the recursive rule
$c_h \quad = 1$ if $L(h\text{-}1){+}p^h \leqslant y$
$c_h \quad = 0$ if $L(h\text{-}1){+}p^h > y$
$L(h) \quad = L(h\text{-}1){+}c_h p^h$.

Theorem 2.2. *If* $0 < y < \frac{p}{q}$ *and* $p \geq \frac{1}{2}$, *then the sequence* $L(k)$ *defined by the algorithm converges to* y *as* $k \to \infty$.

Proof. We first establish, by induction, that $L(k) \leqslant y$. When $k = 0$, this is true by the initial rule of the algorithm. Suppose that it is true for some integer k. Then by the recursive rule for the algorithm it must be true for $k + 1$.

Clearly, the $L(k)$ tends to some limit y^* as $k \to \infty$ and we must have $0 < y^* \leq y$. To prove that $y^* = y$, we introduce a related sequence $R(k)$ by the rules

$$R(0) = \frac{p}{q},$$

$$R(h) = R(h-1) - (1 - c_h)p^h$$

for $h > 0$.

We first show

$$y \leq R(k)$$

for all integers $k \geq 0$.

Again, this is established by induction. The result is clearly true for $k = 0$. Suppose that the result holds for some non-negative integer k. Add the differences $R(h) - R(h-1)$ from $h = 1$ to $h = k + 1$ to get

$$R(k+1) - R(0) = \sum_{h=1}^{k+1}[R(h) - R(h-1)] = \sum_{h=1}^{k+1}(c_h p^h - p^k) = L(k+1) - \sum_{h=1}^{k+1} p^h$$

The expression $\sum_{h=1}^{k+1} p^h$ is a geometric progression with first term p, last term p^{k+1} and ratio p so its sum is $\frac{p-p^{k+2}}{q}$. Thus

$$R(k{+}1) = R(0){+}L(k{+}1){-}\frac{p-p^{k+2}}{q} = \frac{p}{q}{+}L(k{+}1){-}\frac{p-p^{k+2}}{q} = L(k{+}1){+}\frac{p^{k+2}}{q}.$$

With $h = k + 1$, we get

$$R(k+1) = R(k) - (1 - c_{k+1})p^{k+1}.$$

Now if we suppose that $c_{k+1} = 1$, then $R(k+1) = R(k)$. So clearly the result holds for $k + 1$. On the other hand, if $c_{k+1} = 0$ and we assume that $y > R(k+1)$ we conclude that

$$y > L(k+1) + \frac{p^{k+2}}{q}$$

or, since $L(k+1) = L(k)$,

$y > L(k) + \frac{p^{k+2}}{q}$.

However, we are still in the case where $c_{k+1} = 0$. So

$y < L(k) + p^{k+1}$

with $h = k + 1$. Putting these last two inequalities together, we have

$$L(k) + p^{k+1} > L(k) + \frac{p^{k+2}}{q}$$

or

$$p^{k+1} > \frac{p^{k+2}}{q}$$

or

$q > p$ or $1-p > p$ or $p < \frac{1}{2}$, contradicting our hypothesis $p \geq \frac{1}{2}$. It follows that our assumption $y > R(k{+}1)$ cannot be true and the result holds for $k{+}1$.

We now know $L(k) \leq y \leq R(k)$ for all k, and we proceed to show that the sequence *R(k) - L(k)* converges to 0 as $k \to \infty$. We have $R(k) - L(k) = \frac{p^{k+1}}{q}$ and since p<1, this clearly converges to 0. This concludes the proof that $L(k)$ converges to y.

We now turn to the case $p < \frac{1}{2}$. For this case we present a result that establishes that the algorithm will not always generate a p series that converges to y.

We adopt the following notation

$$S(0) = 0$$

and

$$S(k) = \sum_{n=1}^{k} p^h$$

for any positive integer k.

Obviously, $S(k) = L(k)$ when all coefficients of $L(k)$ are 1. There are values of p and k for which the open interval $(S(k-1) + \frac{p^{k+1}}{q},\ S(k))$ is non-empty. For example, consider $p = \frac{1}{3}$ and k =1, for which the interval is $(\frac{1}{6}, \frac{1}{3})$. In fact, if $p < \frac{1}{2}$, then the open interval $(S(k-1) + \frac{p^{k+1}}{q},\ S(k))$ is always non-empty, as is clear from the following, even stronger, lemma.

Lemma 2.3. *For every positive integer* k*, the open interval* $(S(k-1) + \frac{p^{k+1}}{q}, S(k))$ *is non-empty if and only if* $p < \frac{1}{2}$.

Proof. The non-empty interval means $S(k-1) + \frac{p^{k+1}}{q} < S(k)$, which is equivalent to each of the following:

$$S(k-1) + \frac{p^{k+1}}{q} < S(k-1) + p^k,$$

or

$$\frac{p^{k+1}}{q} < p^k$$

or $\frac{p}{q} < 1$ or $p < 1 - p$ or $p < \frac{1}{2}$.

Theorem 2.4. *If there is a number* y *such that*

$$S(k-1) + \frac{p^{k+1}}{q} < y < S(k),$$

for a positive integer k*, then the algorithm will not generate a* p*-series for* y.

Proof. We first establish, by induction, that $L(k-1) = S(k-1)$. Clearly this is true for k =1 since $L(0) = S(0) = 0$. Suppose that $L(m-1) = S(m-1)$ for $m<k$. Then

$$L(m-1) + p^m = S(m-1) + p^m = S(m).$$

Since $m \le k-1, S(m) \le S(k-1)$, we now have

$$L(m-1) + p^m \le S(k-1) < S(k-1) + \frac{p^{k+1}}{q} < y.$$

So $c_m = 1$ and $L(m) = S(m)$.

Next, we establish that $c_k = 0$. This is true since if $c_k = 1$, then $L(k-1) + c_k p^k < y$ is equivalent to $S(k-1) + p^k < y$ or $S(k) < y$ which contradicts the hypothesis that $y < S(k)$.

Finally, if $m > k$,

$$\begin{aligned} L(m) &= L(k-1) + 0p^k + \sum_{h=k+1}^{m} c_h p^h \\ &\le S(k-1) + \sum_{h=k+1}^{m} c_h p^h \\ &\le S(k-1) + \sum_{h=k+1}^{\infty} c_h p^h \\ &\le S(k-1) + \frac{p^{k+1}}{q}. \end{aligned}$$

Since this is strictly less than y by hypothesis, the algorithm will not generate a p series for y. This concludes the proof of the theorem.

Remark 2.5. *This theorem establishes that there are many intervals for which the algorithm fails to converge, for example, the interval* $(\frac{1}{6}, \frac{1}{3})$ *when* k *=1 and* $p = \frac{1}{3}$. *Note that it is implicit in the proof that the algorithm does converge in this example for* $y = \frac{1}{6}$, *and more generally to the value* $S(k-1) + \frac{p^{k+1}}{q}$.