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On Different Types of Monotonically $\mu_\omega-$ Spaces in Generalized Topological Spaces

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Abstract

In this paper, we introduce monotonically $\omega - T_2-$space , monotonically $\omega-$normal space in generalized topological spaces. Moreover, we define $\omega-$stratifiable and $\omega-$ semistratifiable in generalized topological spaces. Furthermore, we give some characterizations of these notions and related results.

1 Introduction

Buck [13] defined monotonically T_2-space and gave relations between monotonically T_2-space and m_k-spaces for $k = 1, 2, 3$. Later, Al-Bsoul [2] characterized this definition and Abushaheen [1] investigated monotonically T_2-space in bi-topological spaces.

Heath [24] introduced a monotonically normal space as a generalization of stratifiable space that defined Borges [5]. Afterwards, various papers were written on monotonically normal spaces [19, 20, 21, 25, 26, 31, 32].

Császár [14] defined a generalized topological space (X, μ) as a collection of nonempty subsets of X with $\phi \in \mu$ and $\bigcup_{U\in\mu} U \in \mu$. Elements of μ are

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called $\mu-$ open. A set $A \subseteq X$ is called $\mu-$ closed if $X - A$ is $\mu-$open. A space (X, μ) is called a strong generalized topological space if $X \in \mu$. Further studies were done on generalized topological space (for example see [11,15,16,17,27,29,30]).

For a space (X, τ), a point $x \in X$ is called a condensation point of a set $A \subseteq X(Cond(A))$ if there exists an open set U_x containing x such that $U_x \cap A$ is uncountable set.

Hdeib [22] gave a weak form of a closed set called $\omega-$ closed subset: a subset of X is called $\omega-$ closed if it contains all its condensation points. The complement of $\omega-$ closed is called $\omega-$ open. Note that a set A is an $\omega-$ open set if and only if for every $x \in A$ there exists an open set U_x containing x with $U_x - A$ being a countable set. Clearly, the family of $\omega-$ open sets forms a topology, denoted by (X, τ_ω), which is finer than (X, τ); i.e., every open set is $\omega-$open set. Many articles have been published on (X, τ_ω) (for example, see [3,7,8,9,10,23,28]).

In 2016, Al Ghour [4] extended $\omega-$open into a generalized topological space and gave many generalizations of the known topological spaces (for example. Lindelöf, compact, countably compact spaces and continuous functions).

Definition 1.1. [4] *Let (X, μ) be a generalized topological space and let B be a subset of X.*

(a) A point $x \in X$ is a condensation point of B if for all $A \in \mu$ such that $x \in A$, $A \cap B$ is uncountable. (The set of all condensation points of B is denoted by $Cond(B)$).

(b) B is $\mu_\omega-$closed if $Cond(B) \subseteq B$.

(c) B is $\mu_\omega-$open if $X - B$ is $\mu_\omega-$closed.

(d) The family of all $u_\omega-$open sets of (X, μ) will be denoted by μ_ω.

In [5,6], the authors studied this notion further. We will use the following definitions and Theorem in our paper.

Definition 1.2. [5] *A generalized topological space (X, μ) is called $\mu_\omega T_1-$space if, for all $x \neq y \in X$, there exists $U, V \in \mu_\omega$ such that $x \in U - V$ and $y \in V - U$.*

Definition 1.3. [5] *A generalized topological space (X,μ) is called $\mu_\omega T_2-$ space if, for all $x \neq y \in X$, there exists $U, V \in \mu_\omega$ such that $x \in U$ and $y \in V$ and $U \cap V = \phi$.*

Definition 1.4. [5] *Let (X,μ) be a generalized topological space and let A be a subset of X. Then we define $\mu_\omega-$closure of A ; denoted by $\overline{A}^{\mu_\omega}$ as :*

$$\overline{A}^{\mu_\omega} = \bigcap \{B : B \quad is \quad \mu_\omega - closed\ in \quad X \quad and \quad A \subseteq B\}.$$

Definition 1.5. [5] *Let (X,μ) a generalized topological space and let A be a subset of X. Define $\mu_\omega-$ interior of A, denoted by $Int_{\mu_\omega}(A)$, as*

$$Int_{\mu_\omega}(A) = \bigcup \{B : B \quad is \quad \mu_\omega - open\ in \quad X \quad and \quad B \subseteq A\}.$$

Theorem 1.6. [5] *Let $f : (X,\mu_1) \to (Y,\mu_2)$ be a function. Then the following are equivalent*

(a) f is $\omega - (\mu_1,\mu_2)-$irresolute;

(b) For each $\mu_{2\omega}-$ closed subset C of Y, $f^{-1}(C)$ is a $\mu_{1\omega}-$ closed subset of X;

(c) For each subset A of X, $f\overline{(A)}^{(\mu_{1\omega})} \subseteq \overline{f(A)}^{(\mu_{2\omega})}$;

(d) For each subset B of Y, $\overline{f^{-1}(B)}^{(\mu_{2\omega})} \subseteq f^{-1}\overline{(B)}^{(\mu_{2\omega})}$.

In section 2, we will introduce monotonically $\omega - T_2-$space in a generalized topological space and give some related results. In section 3, we define monotonically $\omega-$normal space in a generalized topological space and investigate some equivalent statements. In section 4, we introduce $\omega-$stratifiable and $\omega-$semistratifiable in generalized topological spaces and study the relations between these definitions and monotonically $\omega-$normal space in a generalized topological space. Finally, a product theorem is given. We will adopt the terms and notations in [18].

2 Monotonically $\omega - T_2-$space in a Generalized Topological Spaces

Definition 2.1. *A generalized topological space (X,μ) is called monotonically $\omega - T_2-$ space ($\mu_\omega MT_2-$ space) if there is a function $U : X \times X \to \mu_\omega$ assigning to an ordered pair (x,y) of distinct points in X an $\mu_\omega-$ open $U(x,y) \subset X$ of x such that* *(a)* $U(x,y) \cap U(y,x) = \phi$;

(b) *For each* $M \subseteq X$, *if* $x \in \overline{\bigcup \{U(y,x) | y \in M\}}^{\mu_\omega}$, *then* $x \in \overline{M}^{\mu_\omega}$. *In addition, if*

(c) *For* $z \in U(x,y)$, *then* $U(z,y) \subseteq U(x,y)$,

then (X,μ) *is strongly monotonically* $\omega - T_2-$ *space (strongly-*$\mu_\omega MT_2-$ *space).*

Clearly, strongly$-\mu_\omega M - T_2-$ space implies $\mu_\omega MT_2-$ space and hence $\mu_\omega T_2-$ space.

The following two theorems are about subspaces of $\mu_\omega MT_2-$space. The proof of the first one is obvious. The proof of the second appears as Theorem 2.13 in [1].

Theorem 2.2. *Let* (X,μ) *be* $\mu_\omega MT_2-$ *space and let* $A \subseteq X$. *Then* A *is* $\mu_\omega MT_2-$ *space.*

Theorem 2.3. *If every proper subspace of* (X,μ) *is a* $\mu_\omega MT_2-$ *space, then* (X,μ) *is* $\mu_\omega MT_2-$ *space.*

Definition 2.4. *A generalized topological space* (X,μ) *is called* $\mu_\omega T_3-$ *space if, for each point* $x \in X$ *and each* $\mu_\omega-$*closed set* A *such that* $x \notin A$, *there are disjoint* $\mu_\omega-$*open sets* U, V *with* $x \in U$ *and* $A \subset V$.

Theorem 2.5. *Let* (X,μ) *be a generalized topological space. If* (X,μ) *is a* $\mu_\omega MT_2-$ *space, then* (X,μ) *is* $\mu_\omega T_3-$ *space.*

Proof. Let A be $\mu_\omega-$closed and $x \notin A = \overline{A}^{\mu_\omega}$ and let $y \in A$. Since (X,μ) is $\mu_\omega MT_2-$ space, there is a function $U : X \times X \to \mu_\omega$ such that:

$$x \notin \overline{\bigcup \{U(y,x) | y \in A\}}^{\mu_\omega},$$

then

$$A \subseteq \bigcup \{U(y,x) | y \in A\},$$

and

$$x \in U(x,y) \quad \text{with} \quad U(x,y) \cap \bigcup \{U(y,x) | y \in A\} = \phi.$$

□

In general, the converse is not true. However, by adding the following condition, we get an equivalent statement.

Definition 2.6. *A generalized topological space (X, μ) is said to have C^* property if, for all $x \in X$, there exists a countable nested $\mu_\omega-$open local base.*

Theorem 2.7. *For a generalized topological space (X, μ) with the C^* property, (X, μ) is $\mu_\omega T_3-$ space if and only if (X, μ) is $\mu_\omega MT_2-$ space.*

Proof. $(\Leftarrow)$ From Theorem 2.5.

$(\Rightarrow)$ Let $x \neq y \in X$. Since X has the C^* property, there exists a countable nested $\mu_\omega-$open local base for each x and y, say $\{V_n(x)\}_{n=1}^{\infty}$ of x and $\{V_n(y)\}_{n=1}^{\infty}$ of y. Let $j(x)$ be the minimum index such that $x \notin \overline{V_{j(x)}(y)}^{\mu_\omega}$ and let $i(x, y)$ be the smallest number such that $V_{i(x,y)}(x) \cap V_{j(x)}(y) = \phi$. Define

$$U(x, y) = V_{i(x,y)}(x).$$

We have

$$U(x, y) \cap U(y, x) = \phi.$$

Now, let $M \subset \overline{M}^{\mu_\omega} \subset X - \{x\}$ and let i be the minimum element such that $V_i(x) \cap M = \phi$ for $y \in M$. Then

$$V_i(x) \subset V_{j(y)}(x) \subset X - V_{i(y,x)}(y) = X - U(y, x).$$

Consequently,

$$x \in V_i(x) \subset X - \cup\{U(y, x) : y \in M\}.$$

□

3 Monotonically $\omega-$normal in a Generalized Topological Spaces

Definition 3.1. *A $\mu_\omega T_1-$ space is called monotonically $\omega-$ normal in a generalized topological space (X, μ) ($\mu_\omega M-$ normal) if for each disjoint $\mu_\omega-$closed sets E, F, there exists a $\mu_\omega-$ open set $U(E, F)$ such that*

(a) $E \subseteq U(E, F) \subseteq \overline{U(E, F)}^{\mu_\omega} \subseteq X - F$,

(b) If E', F' are disjoint $\mu_\omega-$closed subsets of X with $E \subseteq E'$ and $F' \subseteq F$, then $U(E, F) \subseteq U(E', F')$.

The following Lemma is important for the rest of this paper.

Lemma 3.2. *Let (X,μ) be a generalized topological space. If (X,μ) is a $\mu_\omega M-$ normal space, then there exists $\mu_\omega-$open sets $U'(E,F)$ and $U'(F,E)$ such that $U'(E,F)\cap U'(F,E)=\phi$ for each pair of disjoint $\mu_\omega-$ closed sets E,F.*

Proof. Let E,F be disjoint $\mu_\omega-$ closed sets. Let

$$U'(E,F)=U(E,F)-\overline{U(F,E)}^{\mu_\omega},$$

and

$$U'(F,E)=U(F,E)-\overline{U(E,F)}^{\mu_\omega},$$

where $U(E,F)$ and $U(F,E)$ are $\mu_\omega-$open sets with

$$E\subseteq U(E,F)\subseteq\overline{U(E,F)}^{\mu_\omega}\subseteq X-F,$$

and

$$F\subseteq U(F,E)\subseteq\overline{U(F,E)}^{\mu_\omega}\subseteq X-E.$$

Clearly, $U'(E,F)\cap U'(F,E)=\phi$. □

Definition 3.3. *The ordered pair (S,T) of subsets of (X,μ) is called $\mu_\omega-$ separated if $\overline{S}^{\mu_\omega}\cap T=S\cap\overline{T}^{\mu_\omega}=\phi$.*

Lemma 3.4. *Let (X,μ) be a generalized topological space. Then (X,μ) is $\mu_\omega T_1-$ space if and only if, for each $x\in X$, $\{x\}$ is $\mu_\omega-$ closed.*

Theorem 3.5. *Let (X,μ) be a generalized topological $\mu_\omega T_1-$ space. For a $\mu_\omega M-$ normal space (X,μ), the following are equivalent:*

(i) For each ordered pair (S,T) of $\mu_\omega-$ separated sets, there exists a $\mu_\omega-$open set $U(S,T)$ such that

(1) $S\subseteq U(S,T)\subseteq\overline{U(S,T)}^{\mu_\omega}\subseteq X-T$,

(2) If (S',T') is a pair of $\mu_\omega-$ separated subsets of X with $S\subseteq S'$ and $T'\subseteq T$, then $U(S,T)\subseteq U(S',T')$ where $U(S',T')$ is $\mu_\omega-$open set with $S'\subseteq U(S',T')\subseteq\overline{U(S',T')}^{\mu_\omega}\subseteq X-T'$.

(ii) For each $\mu-$ closed set C and $p\in X-C$, there exists a $\mu_\omega-$ open set $U'(p,C)$ such that

(a) $p\in U'(p,C)\subseteq X-C$,

(b) *If D is a $\mu-$ closed subset with $D \subseteq C$ and $p \notin C$, then $U'(p,C) \subseteq U'(p,D)$, where $U'(p,D)$ is a $\mu_\omega-$open subset of X with $p \in U'(p,D) \subseteq X - D$,*

(c) *If $p \neq q \in X$, then $U(p,q) \cap U(q,p) = \phi$, where $U(p,q)$ and $U(q,p)$ are $\mu_\omega-$open sets with $p \in U(p,q) \subseteq X - \{q\}$ and $q \in U(q,p) \subseteq X - \{p\}$.*

Proof. $(i) \Rightarrow (ii)$ Let C be a $\mu-$ closed set and let $p \in X - C$. Then $\{p\}$ is a $\mu_\omega-$ closed set. Hence the result follows from Lemma 3.2.

$(ii) \Rightarrow (i)$ Let (S,T) be a pair of $\mu_\omega-$ separated subsets. Let

$$U(S,T) = \bigcup\{U'(p, \overline{T}^{\mu_\omega}) : p \in S\}.$$

Now, for each $q \in T$, the set $U'(q, \overline{S}^{\mu_\omega})$ is a $\mu_\omega-$ open set with

$$U'(q, \overline{S}^{\mu_\omega}) \bigcap (\bigcup U'(p, \overline{T}^{\mu_\omega})) = \phi,$$

then $\overline{U(S,T)}^{\mu_\omega} \subseteq X - T$ and $S \subseteq U(S,T)$.
For (2), let (S',T') be a pair of $\mu_\omega-$ separated with $S \subseteq S'$ and $T' \subseteq T$. Then

$$U(S,T) = \bigcup\left\{U'(p, \overline{T}^{\mu_\omega}) : p \in S\right\} \subseteq \bigcup\left\{U'(p, \overline{T}^{\mu_\omega}) : p \in S'\right\} = U(S',T').$$

□

Definition 3.6. *For a generalized topological spaces (X,μ_1) and (Y,μ_2), a function $f : (X,\mu_1) \to (Y,\mu_2)$ is called $\mu_\omega-$closed function if the image of $\mu_\omega-$closed subset of X is $\mu_\omega-$closed subset of Y.*

Theorem 3.7. *Let $f : (X,\mu_1) \to (Y,\mu_2)$ be $\omega - (\mu_1,\mu_2)-$irresolute and $\mu_\omega-$closed onto function. If X is a $\mu_\omega M-$ normal space, then so is Y.*

Proof. Let E, F be two disjoint $\mu_\omega-$closed subsets. Then there exists a $\mu_\omega-$open set $U'(f^{-1}(E), f^{-1}(F))$ such that

$$f^{-1}(E) \subseteq U'(f^{-1}(E), f^{-1}(F)) \subseteq \overline{U'(f^{-1}(E), f^{-1}(F))}^{\mu_\omega} \subseteq X - f^{-1}(F).$$

Consider the set

$$U(E,F) = Y - f(X - U'(f^{-1}(E), f^{-1}(F)).$$

Then

$$E \subseteq U(E,F) \subseteq \overline{U(E,F)}^{\mu_\omega} \subseteq X - F.$$

Consequently, Y is a $\mu_\omega M-$ normal space.

□

Theorem 3.8. *For a generalized topological $\mu_\omega T_1-$space (X, μ), the following statements are equivalent*

(i) (X, μ) *is a* $\mu_\omega M-$ *normal space,*

(ii) *For each pair* (A, U) *of* X *, where* A *is* $\mu_\omega-$*closed set and* U *is* $\mu_\omega-$*open set with* $A \subseteq U$*, there exists a* $\mu_\omega-$ *open set* $U(A)$ *with* $A \subseteq U_\omega(A)$ *such that*

(1) $U \subseteq V(B)$ *whenever* $A \subseteq B$ *and* $U \subseteq V$*,*

(2) $U(A) \cap U(X - U) = \phi$*,*

(iii) *For each* $\mu_\omega-$*open set* $U \subseteq X$ *and* $x \in X$*, there exists a* $\mu_\omega-$ *open set* $U(x)$ *with* $U(x) \cap V(y) \neq \phi$*, then* $x \in V(y)$ *or* $y \in U(x)$ *where* $V(y)$ *is a* $\mu_\omega-$*open set with* $y \in V(y)$*,*

(iv) *For each pair* (A, U) *of* X *where* A *is* $\mu_\omega-$*closed set and* U *is* $\mu_\omega-$*open set, then there exists a* $\mu_\omega-$ *open* $U(A) \subseteq U$ *such that*

(1) $U(A) \subseteq V(B)$ *whenever* $A \subseteq B$ *and* $U \subseteq V$*,*

(2) $A \subseteq U(A) \subseteq \overline{U(A)}^{\mu_\omega} \subseteq U$ *for* $A \subseteq U$*,*

(v) *For each pair* (A, U) *of* X *where* A *is* $\mu_\omega-$*closed set and* U *is* $\mu_\omega-$*open set, there exists* $\omega-(\mu_1, \mu_2)-$*irresolute function* $f_{U(A)} : (X, \mu_1) \to \mathbb{R}$ *such that* $f_{U(A)}(A) = 0$ *and* $f_{U_{(A)}}(X - U) = 1$ *and* $f_{U(A)} > f_{V(B)}$ *for* $A \subseteq B$ *and* $U \subseteq V$*.*

Proof. *(i)* $\Rightarrow$ *(ii)* Let (A, U) be a pair of $\mu_\omega-$closed set A and $\mu_\omega-$open set U with $A \subseteq U \subseteq X$. By Lemma 3.2, there exists a $\mu_\omega-$ open set $U(A) = U(A, X - U)$. Clearly, $U(A)$ satisfies (1) and (2) in *(ii)*.

(ii) $\Rightarrow$ *(iii)* Let $U \subseteq X$ and let $x \in U$. Assume that $U(x) \cap V(y) \neq \phi$ with $x \notin V(y)$ and $y \notin U(x)$. Then $U(x) \cap U(X - x) = \phi$. However, in this case, $U(x) \cap V(y) = \phi$. This is a contradiction. As a result, $x \in V(y)$ or $y \in U(x)$. *(iii)* $\Rightarrow$ *(iv)* Let (A, U) be a pair of a $\mu_\omega-$closed subset A and a $\mu_\omega-$open set U of X . Define

$$U(A) = \bigcup \{V(x) : x \in A, V \subseteq U\},$$

then $U(A)$ is a $\mu_\omega-$ open subset. For (2) , enough to show $\overline{U(A)}^{\mu_\omega} \subseteq U$. let $x \notin A$, therefore $x \notin U(A)$ and $U(A) \cap U(x) = \phi$, hence $x \notin \overline{U(A)}^{\mu_\omega}$. For $A \subseteq X$, let

$$(X-A)_{(X-U)} = \bigcup \{W_y : y \in X-U, W \subseteq X-A\},$$

assume $U(A) \cap (X-A)_{(X-U)} \neq \phi$, then $V_x \cap W_y \neq \phi$, so $x \in W$ or $y \in V$, which is a contradiction.

$(iv) \Rightarrow (v)$ Let $A \subseteq U$. Define

$$U_0 = A \quad \text{and} \quad U_1 = U,$$

and for $r_i \in (0,1) \cap \mathbb{Q}$, $\quad i = 2,3,4,\cdots,(r_0 = 0, r_1 = 1)$. We need to construct U_{r_i} such that $(*)$

$$\overline{U_{r_i}}^{\mu_\omega} \subseteq U_{r_j} \quad \text{whenever} \quad r_i < r_j \in (0,1) \cap \mathbb{Q} \qquad (*).$$

For U_{r_0} and U_{r_1}, $\overline{U_{r_0}}^{\mu_\omega} \subseteq U_{r_1}$. We use induction on i. Assume the result is true for $i = 1,2,...,n-1$. Let α, β denote the nearest number to r_n from left and right taken from $r_0, r_1, r_2, \cdots, r_n$. Clearly, $\overline{U_\alpha}^{\mu_\omega} \subseteq U_\beta$. Now let

$$U_{r_n} = U(A) = U(A, X - U_{r_n}).$$

Then

$$\overline{U_\alpha}^{\mu_\omega} \subseteq U_{r_n} \subseteq \overline{U_{r_n}}^{\mu_\omega} \subseteq U_\beta.$$

So $(*)$ is satisfied for $r_0, r_1, \cdots, r_n$. For $r > 1$, let $U_r = X$ and for $r < 0, U_r = \phi$. That means U_r is defined for all $r \in \mathbb{Q}$.

Now, consider the function $f : (X,\mu) \to \mathbb{R}$ as:

$$f_{(A,U)}(x) = inf\{r \in \mathbb{Q} : x \in U_r(A)\},$$

by the values of r. We have $0 \leq f_{(A,U)}(x) \leq 1$, $f_{(A,U)}(x) = 0$ if $x \in A$ and $f_{(A,U)}(x) = 1$ if $x \in X - U$. To show f is an $\omega-(\mu_1,\mu_2)-$irresolute function, we need to prove that

(a) For $f_{(A,U)}(x) > r$, we have $x \notin \overline{U_r}^{\mu_\omega}$,

(b) For $f_{(A,U)}(x) < r$, we have $x \in U_r$.

To prove (a), let $x \in \overline{U_r}^{\mu_\omega}$. Then $x \in U_{r_1}$ for all $r_1 > r$. So $f_{(A,U)}(x) \leq r_1$ for all $r_1 > r$. Hence, $f_{(A,U)}(x) \leq r$. For (b), since $f_{(A,U)}(x) < r$, there exists r_1 with $r_1 < r$ such that $x \in U_{r_1}$ but $r_1 < r$. So $x \in U_r$. Now, let $x \in f^{-1}_{(A,U)}(-\infty, r_1)$. Then $f_{(A,U)}(x) < r_1$. So there exists a rational number r such that $f_{(A,U)}(x) \leq r < r_1$, so $x \in U_r$, then $x \in U_r \subseteq f^{-1}_{(A,U)}(-\infty, r_1)$. Hence, $f^{-1}_{(A,U)}(-\infty, r_1)$ is $\mu_\omega-$open. For $x \in f^{-1}_{(A,U)}(r, \infty)$. by our construction of f, one can find a rational number t such that $f_{(A,U)}(x) > t > r$. Then $x \notin \overline{U_t}^{\mu_\omega}$. Now, let $z \in X - \overline{U_t}^{\mu_\omega}$. Then $z \notin U_t$ with $f_{(A,U)}(z) \geq t > r$. Therefore,

$$x \in X - \overline{U_t}^{\mu_\omega} \subseteq f^{-1}_{(A,U)}(r, \infty).$$

Consequently, $f^{-1}_{(A,U)}(r, \infty)$ is a $\mu_\omega-$open subset of X.

In addition, as in [31], $f_{(A,U)} \geq f_{(B,V)}$ for $A \subseteq B$ and $U \subseteq V$.

$(v) \Rightarrow (i)$ Let E, F be two disjoint $\mu_\omega-$ closed subsets of X. By (v), there exists an $\omega - (\mu_1, \mu_2)-$irresolute function $f_{(E,X-F)} : X \to \mathbb{R}$ such that $f_{(E,X-F)}(E) = 0$ and $f_{(E,X-F)}(F) = 1$. Now, let $U(E, F) = f^{-1}_{(E,X-F)}(-\infty, r)$ for fixed $0 < r < 1$. Then clearly $U(E, F)$ is $\mu_\omega-$open and $E \subseteq U(E, F) \subseteq \overline{U(E, F)}^{\mu_\omega}$ and

$$\overline{U(E, F)}^{\mu_\omega} = \overline{f^{-1}(-\infty, r)}^{\mu_\omega} \subseteq f^{-1}\overline{((-\infty, r))}^{\mu_\omega} = f^{-1}(-\infty, r] \subseteq X - F.$$

The second condition follows directly from the fact that $f_{U(A)} > f_{V(B)}$. □

Corollary 3.9. *For a generalized topological $\mu_\omega T_1-$space (X, μ), the following are equivalent*

(i) (X, μ) is a $\mu_\omega M-$ normal space,

(ii) for each $x \in X$ and for $\mu_\omega-$ open set U containing $\overline{\{x\}}^{\mu_\omega}$, there exists a $\mu_\omega-$ open set $U(x, U)$ such that:

(1) $\overline{\{x\}}^{\mu_\omega} \subseteq U(x, U) \subseteq U$,

(2) (a) if V is $\mu_\omega-$open and $\overline{\{x\}}^{\mu_\omega} \subseteq U \subseteq V$, then $U(x, U) \subseteq U(x, V)$,

(b) if $x \in \overline{\{y\}}^{\mu_\omega} \subseteq U$, then $U(x, U) \subseteq U(y, U)$,

(3) if $\overline{\{x\}}^{\mu_\omega} \cap \overline{\{y\}}^{\mu_\omega} = \phi$, *then* $U(x, X - \overline{\{y\}}^{\mu_\omega}) \cap U(y, X - \overline{\{x\}}^{\mu_\omega}) = \phi$.

Definition 3.10. *A generalized topological space* (X, μ) *is called collectionwise* $\mu_\omega-$*normal if, for each discrete collection* $\mathcal{H}$ *of* $\mu_\omega-$*closed subsets of* X, *there exists a disjoint collection* $\mathcal{U}'_\omega = \{U_\omega(H) : H \in \mathcal{H}\}$ *of* $\mu_\omega-$*open subsets of* X *with* $H \subseteq U_\omega(H)$ *for each* $H \in \mathcal{H}$.

Theorem 3.11. *Let* (X, μ) *be a generalized topological* $\mu_\omega T_1-$*space. If* (X, μ) *is a* $\mu_\omega M-$ *normal space, then* (X, μ) *is a collectionwise* $\mu_\omega-$*normal space.*

Proof. Let E, F be disjoint $\mu_\omega-$closed subsets of X. Then , by Lemma 3.2, there exists $\mu_\omega-$open sets $U(E, F)$ and $U(F, E)$ such that $U(E, F) \cap U(F, E) = \phi$. Let $\mathcal{H}$ be a discrete family of $\mu_\omega-$closed subsets of (X, μ). For each $H \in \mathcal{H}$, define:

$$U'(H) = U\big(H, \bigcup\{H^* \in \mathcal{H} : H^* \neq H\}\big).$$

Then $U(H)$ in a $\mu_\omega-$open set. Moreover, for $H_1 \neq H_2$,

$$U'(H_1) \subseteq U(H_1, H_2) \subseteq X - H_2,$$

$$U'(H_2) \subseteq U(H_2, H_1) \subseteq X - H_1,$$

and

$$U'(H_1) \cap U'(H_2) \subseteq U(H_1, H_2) \cap U(H_2, H_1) = \phi,$$

and hence $\mathcal{U}' = \{U'(H_1) : H_1 \in \mathcal{H}\}$ is the required collection. Consequently, (X, μ) is a collectionwise $\mu_\omega-$normal space.

□

4 $\mu_\omega-$stratifiable and $\mu_\omega-$semistratifiable in Generalized Topological Spaces

Definition 4.1. *A generalized topological* $\mu_\omega T_1-$ *space* (X, μ) *is called* $\mu_\omega-$*semistratifiable if there exists* $\mu_\omega-$*open sets* $\{S(H, n)\}_{n=1}^{\infty}$ *where* H *is a* $\mu_\omega-$*closed set and* $n \in \mathbb{N}$ *such that*

(a) If $H \subseteq K$ *are* $\mu_\omega-$*closed subsets of* X, *then* $S(H, n) \subseteq S(K, n)$ *for all* $n \in \mathbb{N}$,

(b) $H = \bigcap \{S(H,n) : n \in \mathbb{N}\}$ *for each* $\mu_\omega-$*closed set* $H \subseteq X$,
in addition, if $\{S(H,n)\}_{n=1}^{\infty}$ *satisfies:*

(c) $H = \bigcap \{\overline{S(H,n)}^{\mu_\omega} : n \in \mathbb{N}\}$, *then* X *is called* $\mu_\omega-$*stratifiable space.*

Theorem 4.2. *A generalized topological* $\mu_\omega T_1-$ *space* (X,μ) *is* $\mu_\omega-$*stratifiable if and only if it is* $\mu_\omega M-$ *normal space and* $\mu_\omega-$*semi-stratifiable space.*

Proof. ($\Leftarrow$) Let $K \subseteq X$. Since (X,μ) is a $\mu_\omega-$semi-stratifiable space, there exists $\mu_\omega-$ open subsets $\{T(K,n)\}_{n=1}^{\infty}$ of (X,μ) satisfying Definition 4.1. In addition, since (X,μ) is a $\mu_\omega M-$ normal space, there exists $\mu_\omega-$ open subsets $\{U(K, X - T(K,n))\}_{n=1}^{\infty}$ of (X,μ) satisfying Definition 3.1. Now, the set

$$S(K,n) = U(K, X - T(K,n))$$

is the required $\mu_\omega-$ open subset.

($\Rightarrow$) Let E, F be disjoint $\mu_\omega-$closed subsets of $(X,\mu.)$ Since (X,μ) is a $\mu_\omega-$stratifiable space, there exists a sequence $\{S(E,n)\}_{n=1}^{\infty}$ of $\mu_\omega-$open subsets of (X,μ). Let

$$U(E,F) = \bigcup_{n=1}^{\infty} \left(S(E,n) - \overline{S(F,n)}^{\mu_\omega}\right).$$

Then $U(E,F)$ is an $\mu_\omega-$open set and $E \subseteq U(E,F)$ since $E \cap F = \phi$. Let $x \notin X - F$. Then $x \in \overline{S(F,n)}^{\mu_\omega}$ for all $n \in \mathbb{N}$ and so

$$S(F,n) \cap (X - E) \neq \phi$$

and

$$\big(S(F,n) \cap (X - E)\big) \bigcap U(E,F) = \phi.$$

Therefore, $x \notin \overline{U(E,F)}^{\mu_\omega}$, for a pair (E', F') of $\mu_\omega-$closed subsets of (X,μ) with $E \subseteq E'$ and $F' \subseteq F$. We have $U(E,F) \subseteq U(E',F')$. Consequently, (X,μ) is a $\mu_\omega M-$ normal space.

□

Before focusing on a product Theorem considering $\mu_\omega M-$ normal, we need the following definition.

Definition 4.3. *Let* (X,μ) *be a generalized topological space and let* $A \subseteq X$. *Then a point* $x \in X$ *is called* $\mu_\omega-$ *limit point of* A *if every* $\mu_\omega-$*open set containing* x *contains at least one point of* A *different from* x.

Lemma 4.4. *Let (X, μ) be a generalized topological space and let $A \subseteq X$. Then A is $\mu_\omega-$ closed if and only if A contains all $\mu_\omega-$ limit point of A.*

Theorem 4.5. *Let (X, μ) be a generalized topological space. If $X \times Y$ is a $\mu_\omega M-$ normal space, then either no subset of X has a $\mu_\omega-$limit point or Y is $\mu_\omega-$stratifiable space.*

Proof. Suppose $\mathcal{A} = \{A_\alpha : \alpha \in \Delta, |\Delta| < \omega_0\}$ is a subset of a generalized topological space (X, μ) having a $\mu_\omega-$limit point a. Assume $a \notin \mathcal{A}$. Let $A^* = A \cup \{a\}$. Since $X \times Y$ is a $\mu_\omega M-$ normal, $A^* \times Y$ is also a $\mu_\omega M-$ normal. Let $F \subseteq Y$ be a $\mu_\omega-$ closed set. Consider the sets

$$H(F) = \{(x, y) \in A^* \times Y : y \in F \quad \text{and} \quad x \notin A\},$$

$$M(F) = \{(x, y) \in A^* \times Y : y \in A^*\}.$$

Clearly, $H(F)$ and $M(F)$ are $\mu_\omega-$ separated subsets of $A^* \times Y$ and hence, by Lemma 3.2, there exists a $\mu_\omega-$ open set $U(H(F), M(F))$. Finally, for $n \in \mathbb{N}$, let $S(F, n) = \{y \in Y : (A_\alpha, y) \in U(H(F), M(F))\}$. Then $\{S(F, n)\}_{n=1}^{\infty}$ is a $\mu_\omega-$open set satisfying the conditions of $\mu_\omega-$stratifiable space. □

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