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A new result on the exponential stability of solutions of non-linear neutral type periodic systems with variable delay

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Abstract

In this work, we consider a nonlinear time-varying delay system of neutral type differential equations with periodic coefficients and we obtain new sufficient conditions to ensure exponentially stability of zero solution of this system. The Lyapunov- Krasovskii functional approach is useful as a basic tool to prove the main general result of this paper.

1 Introduction

Recently [1-33], various kinds of stability, boundedness, asymptotic behavior of solutions as well as many other properties of solutions of numerous kind of differential equations and systems; i.e., various qualitative properties of solutions of ordinary, delay and neutral delay differential equations

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and system of differential equations without and with delay have received considerable interest. In particular, due to the applicability of neutral type differential equations and neutral type systems of differential equations with periodic coefficients to problems in the natural sciences, engineering, physics, medicine, population dynamics and so on, there is a continuing interest in obtaining new sufficient conditions for the stability, exponential stability of solutions, exponential decay of solutions at infinity and so on to that class of equations and systems, [2], [3], [7], [8], [13-15]. Meanwhile, it should be noted that in spite of the existence of numerous results on various qualitative behaviors of solutions of scalar neutral differential equations without periodic components and system of neutral differential equations without periodic components, there exists only a few results on the qualitative properties of system of neutral differential equations with periodic coefficients [2], [3], [13-15]. To the best of our knowledge, the reason for this is the difficulty of the topic for a system of neutral differential equations with periodic coefficients. Based on this, we would like to suggest investigating the qualitative properties of neutral differential systems with periodic coefficients.

In a recent paper, Matveeva [15] investigated exponentially stability of the zero solution of the neutral differential system with variable delay:

$$\begin{aligned}\frac{d}{dt}y(t) =& A(t)y(t) + B(t)y(t-\tau(t)) + C(t)\frac{d}{dt}y(t-\tau(t)) \\ &+ F(t, y(t), y(t-\tau(t)), \frac{d}{dt}y(t-\tau(t))).\end{aligned}$$

In [15], the author established sufficient conditions under which the zero solution of this system is exponential stable, and the solutions of this system satisfy the exponential decay at infinity. Here, we prove two results on the exponentially stability of the zero solution, where both results include sufficient conditions.

Motivated by the above system and the results in [15], we consider the following nonlinear differential systems of neutral type with periodic coefficients and time-varying delay:

$$\begin{aligned}\frac{d}{dt}y(t) =& A(t)G_1(y(t))y(t) + B(t)H_1(y(t-\tau(t))) + C(t)\frac{d}{dt}y(t-\tau(t)) \\ &+ F(t, y(t), y(t-\tau(t)), \frac{d}{dt}y(t-\tau(t))), \qquad (1)\end{aligned}$$

with

$$y(t) = \phi(t), t \in [-\tau_2, 0], \tau_2 > 0, \tau_2 \in \Re,$$
$$\phi(t) \in C^1[-\tau_2, 0], y(0^+) = \phi(0),$$

where $A(t), B(t), C(t) \in (\Re^+, \Re^{n\times n}$ with $A(t+T) \equiv A(t)$, $B(t+T) \equiv B(t)$, $C(t+T) \equiv C(t)$, $T > 0$, T is the period, $\tau(t) \in C^1(\Re^+, \Re^+)$, $\Re^+ = [0, \infty)$, $\tau(t)$ is the time-varying delay,

$$0 < \tau_1 \leq \tau(t) \leq \tau_2, \tau_3 \leq \tau'(t) \leq \tau_4 < 1, \tag{2}$$

$G_1, H_1 \in C(\Re^n, \Re^n)$ and $F \in C(\Re^+ \times \Re^{3n}, \Re^n), \Re^+ = [0, \infty)$.

The solution of (1) is defined as a continuous function on $[-\tau_2, \infty)$, continuously differentiable on $[-\tau_2, \infty)$ except for points $k\tau_1, k = 0, 1, 2, ...$, and satisfying (1) everywhere on $[0, \infty)$ except for points $k\tau_1, k = 0, 1, 2, ...$

We assume that the function $F(t, u, v, w)$ satisfies the Lipschitz condition with respect to u on every compact set $G \subset \Re^+ \times \Re^{3n}$,

$$\|F(t, u, v, w)\| \leq q_1\|u\| + q_2\|v\| + q_3\|w\|, t \geq 0, u, v, w \in \Re^n \tag{3}$$

for some constants $q_1, q_2, q_3 \geq 0$.

Based on the Lyapunov- Krasovskii functional approach, we give a new result on exponentially stability of the zero of a broad class of neutral equations, which includes sufficient conditions for the exponentially stability of the zero solution and without using any spectral information. Our result includes that of Matveeva [15, Theorem 2.3].

Throughout this paper, we use the following dot product and vector norm:

$$\langle x, z\rangle = \sum_{j=1}^{n} x_j \bar{z}_j, \|x\| = \sqrt{\langle x, x\rangle}.$$

Let us assume that

$$\|H_1(y(t-\tau(t))\| \leq K\|y(t-\tau(t))\|, K \geq 1.$$

2 Exponentially stability

Before presenting our exponentially stability result, we introduce some information. Let $H(t)$ and $K(s)$ be suitable $n \times n$ matrices, Define the following relations:

$$\begin{aligned}
\beta_1(t) &= 2\|H(t)\| + (2\|A(t)\|\|G_1(y(t)\| + q_1)\|L(0)\|,\\
\beta_2(t) &= (2K\|B(t)\| + q_2)\|L(0)\|,\\
\beta_3(t) &= (2\|C(t)\| + q_3)\|L(0)\|, \qquad (4)\\
\alpha_1(t) &= q_1\beta_1(t) + \frac{q_1\beta_2(t) + q_2\beta_1(t)}{2} + \frac{q_1\beta_3(t) + q_3\beta_1(t)}{2},\\
\alpha_2(t) &= q_2\beta_2(t) + \frac{q_2\beta_1(t) + q_1\beta_2(t)}{2} + \frac{q_2\beta_3(t) + q_3\beta_2(t)}{2},\\
\alpha_3(t) &= q_3\beta_3(t) + \frac{q_3\beta_1(t) + q_1\beta_3(t)}{2} + \frac{q_3\beta_2(t) + q_2\beta_3(t)}{2}, \qquad (5)
\end{aligned}$$

and the matrix

$$Q^\alpha(t) = Q(t) - \begin{bmatrix} \alpha_1(t)I & 0 & 0 \\ 0 & \alpha_2(t)I & 0 \\ 0 & 0 & \alpha_3(t)I \end{bmatrix}, \qquad (6)$$

where I is the identity matrix.

We use the following notation:

$$\begin{aligned}
P(t) =& Q_{11}(t) - \alpha_1(t)I - [Q_{12}(t) - Q_{13}(t)(Q_{33}(t) - \alpha_3(t)I)^{-1}Q_{23}^*(t)]\\
&\times [Q_{22}(t) - \alpha_2(t)I - Q_{23}(t)(Q_{33}(t) - \alpha_3(t)I)^{-1}Q_{23}^*(t)]^{-1}\\
&\times [Q_{12}(t) - Q_{13}(t)(Q_{33}(t) - \alpha_3(t)I)^{-1}Q_{23}^*(t)]^*\\
&- Q_{13}(t)(Q_{33}(t) - \alpha_3(t)I)^{-1}Q_{13}^*(t), \qquad (7)
\end{aligned}$$

where the matrices $Q_{ij}(t)$ are defined in the proof of the main result of [11]. Next, it is not difficult to show that the matrix $P(t)$ is positve definite. We denote the minimal eigenvalues of the matrices $P(t)$ and $H(t)$ by $p_{min}(t) >$ and $h_{min}(t) > 0$, respectively.

Let $y(t)$ be the solution to the initial value problem (1). We consider the Lyapunov-Krasovskii functional:

$$\begin{aligned}
V(t, y) =& \langle H(t)y(t), y(t)\rangle + \int_{t-\tau(t)}^{t} \langle K(t-s)y(s), y(s)\rangle ds\\
&+ \int_{t-\tau(t)}^{t} \langle L(t-s)y'(s), y'(s)\rangle ds. \qquad (8)
\end{aligned}$$

From (8), it is obvious that

$$\begin{aligned} V(0,\varphi) =& \langle H(0)\varphi(0), \varphi(0)\rangle + \int_{-\tau(0)}^{0} \langle K(-s)\varphi(s), \varphi(s)\rangle ds \\ &+ \int_{-\tau(0)}^{0} \langle L(-s)\frac{d}{ds}\varphi(s), \frac{d}{ds}\varphi(s)\rangle ds. \end{aligned}$$

We now give our main result on the exponential stability of solutions.

Theorem 1. Suppose that there are matrices $H(t) \in C^1[0,T], K(s)$ and $L(s)$ in $C^1[0,\tau_2]$ such that

$$\begin{aligned} H(t) =& H^*(t), t \in [0,T], H(0) = H(t) > 0, && (9) \\ K(s) =& K^*(s) > 0, K'(s) < 0, s \in [0,\tau_2], && (10) \\ L(s) =& L^*(s) > 0, L'(s) < 0, s \in [0,\tau_2]. \end{aligned}$$

Let $k, l > 0$ be maximal numbers such that

$$\begin{aligned} & K'(s) + kK(s) \leq 0, \\ & L'(s) + lL(s) \leq 0, s \in [0,\tau_2], \\ & \gamma(t) = min\{\frac{p_{min}(t)}{\|H(t)\|}, k, l\} > 0. \end{aligned}$$

The matrix

$$Q(t) = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) & Q_{13}(t) \\ Q_{12}^*(t) & Q_{22}(t) & Q_{23}(t) \\ Q_{13}^*(t) & Q_{23}^*(t) & Q_{33}(t) \end{bmatrix} \quad (11)$$

is positive definite for $t \in [0,T]$, where

$$\begin{aligned} Q_{11}(t) =& -\frac{d}{dt}H(t) - H(t)A(t)G_1(y(t)) - A^*(t)G_1^*(y(t))H(t) \\ & - K(0) - A^*(t)G_1^*(y(t))L(0)A(t)G_1(y(t)), \\ Q_{12}(t) =& - H(t)B(t) - A^*(t)G_1^*(y(t))L(0)B(t), \\ Q_{13}(t) =& - H(t)C(t) - A^*(t)G_1^*(y(t))L(0)C(t), \\ Q_{22}(t) =& (1-\tau_4)K(\tau_2) - B^*(t)L(0)B(t), \\ Q_{23}(t) =& - B^*(t)L(0)C(t), \\ Q_{33}(t) =& (1-\tau_3)^{-1}L(\tau_2) - C^*(t)L(0)C(t). \end{aligned}$$

Then the zero solution of the system (1) is exponentially stable.

Proof. Let $y(t)$ be the solution to the initial value problem (1). The proof of this theorem is based on the Lyapunov-Krasovskii functional (8). Differentiating (8) with respect to the independent variable t, we get:

$$\begin{aligned}\frac{d}{dt}V(t,y) =&\langle H'(t)y(t),y(t)\rangle + \langle H(t)y'(t),y(t)\rangle + \langle H(t)y(t),y'(t)\rangle \\ &+ \langle K(0)y(t),y(t)\rangle - (1-\tau'(t))\langle K(\tau(t))y(t-\tau(t)),y(t-\tau(t))\rangle \\ &+ \int_{t-\tau(t)}^{t}\langle \frac{d}{dt}K(t-s)y(s),y(s)\rangle ds + \langle L(0)y'(t),y'(t)\rangle \\ &- (1-\tau'(t))^{-1}\langle L(\tau(t))\frac{d}{dt}y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t))\rangle \\ &+ \int_{t-\tau(t)}^{t}\langle \frac{d}{dt}L(t-s)y'(s),y'(s)\rangle ds.\end{aligned}$$

If we use the differential system (1), then we get

$$\begin{aligned}\frac{d}{dt}V(t,y) =&\langle H'(t)y(t),y(t)\rangle \\ &+ \langle H(t)[A(t)G_1(y(t))y(t) + B(t)H_1(y(t-\tau(t))) + C(t)\frac{d}{dt}y(t-\tau(t)) \\ &+ F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t)))],y(t)\rangle \\ &+ \langle H(t)y(t),A(t)G_1(y(t))y(t) + B(t)H_1(y(t-\tau(t))) + C(t)\frac{d}{dt}y(t-\tau(t)) \\ &+ F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t)))\rangle \\ &+ \langle K(0)y(t),y(t)\rangle - (1-\frac{d}{dt}\tau(t))\langle K(\tau(t))y(t-\tau(t)),y(t-\tau(t))\rangle \\ &+ \int_{t-\tau(t)}^{t}\langle \frac{d}{dt}K(t-s)y(s),y(s)\rangle ds \\ &+ \langle L(0)[A(t)G_1(y(t))y(t) + B(t)H_1y(t-\tau(t)) + C(t)\frac{d}{dt}y(t-\tau(t)) \\ &+ F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t)))],A(t)G_1(y(t))y(t) + B(t)H_1y(t-\tau(t))) \\ &+ C(t)\frac{d}{dt}y(t-\tau(t)) + F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t))) \\ &- (1-\frac{d}{dt}\tau(t))^{-1}\langle L(\tau(t))\frac{d}{dt}y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t)))\rangle \\ &+ \int_{t-\tau(t)}^{t}\langle \frac{d}{dt}L(t-s)\frac{d}{ds}y(s),\frac{d}{ds}y(s)\rangle ds.\end{aligned}$$

In view of the estimates (2), (9), (10), we get

$$
\begin{aligned}
\frac{d}{dt}V(t,y) \leq & \langle \frac{d}{dt}H(t)y(t), y(t)\rangle \\
& + \langle H(t)[A(t)G_1(y(t))y(t) + B(t)H_1(y(t-\tau(t))) + C(t)\frac{d}{dt}y(t-\tau(t)) \\
& + F(t, y(t), y(t-\tau(t)), \frac{d}{dt}y(t-\tau(t)))], y(t)\rangle \\
& + \langle H(t)y(t), A(t)G_1(y(t))y(t) + B(t)H_1(y(t-\tau(t))) + C(t)\frac{d}{dt}y(t-\tau(t)) \\
& + F(t, y(t), y(t-\tau(t)), \frac{d}{dt}y(t-\tau(t)))\rangle \\
& + \langle K(0)y(t), y(t)\rangle - (1-\tau_4)\langle K(\tau_2)y(t-\tau(t)), y(t-\tau(t))\rangle \\
& + \int_{t-\tau(t)}^{t} \langle \frac{d}{dt}K(t-s)y(s), y(s)\rangle ds \\
& + \langle L(0)[A(t)G_1(y(t))y(t) + B(t)H_1 y(t-\tau(t)) + C(t)\frac{d}{dt}y(t-\tau(t)) \\
& + F(t, y(t), y(t-\tau(t)), \frac{d}{dt}y(t-\tau(t)))], A(t)G_1(y(t))y(t) + B(t)H_1 y(t-\tau(t)) \\
& + C(t)\frac{d}{dt}y(t-\tau(t)) + F(t, y(t), y(t-\tau(t)), \frac{d}{dt}y(t-\tau(t))) \\
& - (1-\tau_3)^{-1}\langle L(\tau_2)\frac{d}{dt}y(t-\tau(t)), \frac{d}{dt}y(t-\tau(t))\rangle\rangle \\
& + \int_{t-\tau(t)}^{t} \langle \frac{d}{dt}L(t-s)\frac{d}{ds}y(s), \frac{d}{ds}y(s)\rangle ds.
\end{aligned}
$$

As a consequence of this inequality, we deduce

$$
\begin{aligned}
\frac{d}{dt}V(t,y) \leq & - \langle Q(t)\begin{bmatrix} y(t) \\ y(t-\tau(t)) \\ \frac{d}{dt}y(t-\tau(t)) \end{bmatrix}, \begin{bmatrix} y(t) \\ y(t-\tau(t)) \\ \frac{d}{dt}y(t-\tau(t)) \end{bmatrix}\rangle + W(t) \\
& + \int_{t-\tau(t)}^{t} \langle \frac{d}{dt}K(t-s)y(s), y(s)\rangle ds + \int_{t-\tau(t)}^{t} \langle \frac{d}{dt}L(t-s)\frac{d}{ds}y(s), \frac{d}{ds}y(s)\rangle ds,
\end{aligned} \tag{12}
$$

where $Q(t)$ is the matrix defined by (11) and

$$
\begin{aligned}
W(t) =&\langle H(t)F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t))),y(t)\rangle \\
&+ \langle H(t)y(t),F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t)))\rangle \\
&+ \langle L(0)F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t))), \\
&A(t)G_1(y(t))y(t)+B(t)H_1y(t-\tau(t))+C(t)\frac{d}{dt}y(t-\tau(t))\rangle \\
&+ \langle L(0)[A(t)G_1(y(t))y(t)+B(t)H_1y(t-\tau(t))+C(t)\frac{d}{dt}y(t-\tau(t))], \\
&F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t)))\rangle \\
&+ \langle L(0)F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t))),F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t)))\rangle.
\end{aligned}
$$

We now consider the function W. Then, using the inequalities

$$\|H_1(y(t-\tau(t)))\| \le K\|y(t-\tau(t))\|, K \ge 1,$$

(3) and equalities (4), (5), we have

$$
\begin{aligned}
W(t) \le&\langle H(t)F(t,y(t),y(t-\tau(t)), \\
&\frac{d}{dt}y(t-\tau(t))),y(t)\rangle + \langle H(t)y(t),F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t)))\rangle \\
&+ \langle L(0)F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t))), \\
&A(t)G_1(y(t))y(t)+KB(t)y(t-\tau(t))+C(t)\frac{d}{dt}y(t-\tau(t))\rangle \\
&+ \langle L(0)[A(t)G_1(y(t))y(t)+KB(t)y(t-\tau(t))+C(t)\frac{d}{dt}y(t-\tau(t))], \\
&F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t)))\rangle \\
&+ \langle L(0)F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t))),F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t)))\rangle
\end{aligned}
$$

$$
\begin{aligned}
\leq & 2\langle H(t)y(t), F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t)))\rangle \\
& + 2\langle L(0)[A(t)G_1(y(t))y(t) + KB(t)y(t-\tau(t)) + C(t)\frac{d}{dt}y(t-\tau(t))], \\
& F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t)))\rangle \\
& + \langle L(0)F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t))), F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t)))\rangle \\
\leq & 2\|H(t)\|\|y(t)\|\|F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t)))\| \\
& + 2\|L(0)\|\|A(t)G_1(y(t))y(t) + KB(t)y(t-\tau(t)) + C(t)\frac{d}{dt}y(t-\tau(t))\| \\
& \times \|F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t)))\| \\
& + \|L(0)\|\|F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t)))\|\|F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t)))\| \\
\leq & [2\|H(t)\|\|y(t)\| \\
& + 2\|L(0)\|(\|A(t)\|\|G_1(y(t))\|\|y(t)\| + K\|B(t)\|\|y(t-\tau(t))\| + \|C(t)\|\|\frac{d}{dt}y(t-\tau(t))\|) \\
& + \|L(0)\|\|F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t)))\|]\|F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t)))\| \\
\leq & [2\|H(t)\|\|y(t)\| \\
& + 2\|L(0)\|(\|A(t)\|\|G_1(y(t))\|\|y(t)\| + K\|B(t)\|\|y(t-\tau(t))\| + \|C(t)\|\|\frac{d}{dt}y(t-\tau(t))\|) \\
& + \|L(0)\|(q_1\|y(t)\| + q_2\|y(t-\tau(t))\| + q_3\|\frac{d}{dt}y(t-\tau(t))\|)] \\
& \times (q_1\|y(t)\| + q_2\|y(t-\tau(t))\| + q_3\|\frac{d}{dt}y(t-\tau(t))\|) \\
= & [(2\|H(t)\| + 2(\|A(t)\|\|G_1(y(t))\| + q_1)\|L(0)\|)\|y(t)\| \\
& + (2K\|B(t)\| + q_2)\|L(0)\|\|y(t-\tau(t))\| + (2\|C(t)\| + q_3)\|L(0)\|\|\frac{d}{dt}y(t-\tau(t))\|] \\
& \times (q_1\|y(t)\| + q_2\|y(t-\tau(t))\| + q_3\|\frac{d}{dt}y(t-\tau(t))\|).
\end{aligned}
$$

$$
\begin{aligned}
=&(\beta_1(t)\|y(t)\| + \beta_2(t)\|y(t-\tau(t))\| + \beta_3(t)\|\frac{d}{dt}y(t-\tau(t))\|)\\
&\times (q_1\|y(t)\| + q_2\|y(t-\tau(t))\| + q_3\|\frac{d}{dt}y(t-\tau(t))\|)\\
=&\beta_1(t)q_1\|y(t)\|^2 + \beta_2(t)q_2\|y(t-\tau(t))\|^2 + \beta_3(t)q_3\|\frac{d}{dt}y(t-\tau(t))\|^2\\
&+ \beta_1(t)q_2\|y(t)\|\|y(t-\tau(t))\| + \beta_1(t)q_3\|y(t)\|\|\frac{d}{dt}y(t-\tau(t))\|\\
&+ \beta_2(t)q_1\|y(t-\tau(t))\|\|y(t)\| + \beta_2(t)q_3\|y(t-\tau(t))\|\|\frac{d}{dt}y(t-\tau(t))\|\\
&+ \beta_3(t)q_1\|\frac{d}{dt}y(t-\tau(t))\|\|y(t)\| + \beta_3(t)q_2\|\frac{d}{dt}y(t-\tau(t))\|\|y(t-\tau(t))\|\\
=&(\beta_1(t)q_1 + \frac{\beta_1(t)q_2 + \beta_1(t)q_3 + \beta_2(t)q_1 + \beta_3(t)q_1}{2})\|y(t)\|^2\\
&+ (\beta_2(t)q_2 + \frac{\beta_1(t)q_2 + \beta_2(t)q_1 + \beta_2(t)q_3 + \beta_3(t)q_2}{2})\|y(t-\tau(t)))\|^2\\
&+ (\beta_3(t)q_3 + \frac{\beta_1(t)q_3 + \beta_2(t)q_3 + \beta_3(t)q_1 + \beta_3(t)q_2}{2})\|\frac{d}{dt}y(t-\tau(t))\|^2\\
=&\alpha_1(t)\|y(t)\|^2 + \alpha_2(t)\|y(t-\tau(t))\|^2 + \alpha_3(t)\|\frac{d}{dt}y(t-\tau(t))\|^2. \qquad (13)
\end{aligned}
$$

Combining the inequalities (12) and (13), we obtain

$$
\begin{aligned}
\frac{d}{dt}V(t,y) \leq& -\langle Q^\alpha(t)\begin{bmatrix} y(t) \\ y(t-\tau(t)) \\ \frac{d}{dt}y(t-\tau(t)) \end{bmatrix}, \begin{bmatrix} y(t) \\ y(t-\tau(t)) \\ \frac{d}{dt}y(t-\tau(t)) \end{bmatrix}\rangle\\
&+ \int_{t-\tau(t)}^{t} \langle \frac{d}{dt}K(t-s)y(s), y(s)\rangle ds\\
&+ \int_{t-\tau(t)}^{t} \langle \frac{d}{dt}L(t-s)\frac{d}{ds}y(s), \frac{d}{ds}y(s)\rangle ds, \qquad (14)
\end{aligned}
$$

where $Q^\alpha(t)$ is the matrix given by (6). For the next step,

$$
\langle Q^\alpha(t)\begin{bmatrix} y(t) \\ y(t-\tau(t)) \\ \frac{d}{dt}y(t-\tau(t)) \end{bmatrix}, \begin{bmatrix} y(t) \\ y(t-\tau(t)) \\ \frac{d}{dt}y(t-\tau(t)) \end{bmatrix}\rangle \geq \langle P(t)y(t), y(t)\rangle,
$$

where $P(t)$ is the positive definite Hermitian matrix given in (7). Using the assumption $\langle P(t)y(t), y(t)\rangle \geq p_{min}(t)\|y(t)\|^2$ and (14), we obtain

$$\frac{d}{dt}V(t,y) \leq -\langle p_{min}(t)y(t), y(t)\rangle + \int_{t-\tau(t)}^{t} \langle \frac{d}{dt}K(t-s)y(s), y(s)\rangle ds + \int_{t-\tau(t)}^{t} \langle \frac{d}{dt}L(t-s)\frac{d}{ds}y(s), \frac{d}{ds}y(s)\rangle ds.$$

On the other hand, it is also clear that

$$h_{min}(t)\|y(t)\|^2 \leq \langle H(t)y(t), y(t)\rangle \geq \|H(t)\|\|y(t)\|^2. \tag{15}$$

Hence, using (15), the assumptions $K'(s) + kK(s) \leq 0$ and $L'(s) + lL(s) \leq 0, s \in [0, \tau_2]$, it follows that

$$\frac{d}{dt}V(t,y) \leq -\frac{p_{min}(t)}{\|H(t)\|}\langle H(t)y(t), y(t)\rangle - k\int_{t-\tau(t)}^{t} \langle K(t-s)y(s), y(s)\rangle ds - l\int_{t-\tau(t)}^{t} \langle L(t-s)\frac{d}{ds}y(s), \frac{d}{ds}y(s)\rangle ds.$$

By the definition of the functional in (9), we get $\frac{d}{dt}V(t,y) \leq -\gamma(t)V(t,y)$, where $\gamma(t) = min(\frac{p_{min}(t)}{\|H(t)\|}, k, l)$. Integrating, we get $V(t,y) \leq V(0,\varphi)exp(-\int_0^t \gamma(\xi)d\xi)$, where $V(0,\varphi)$ is defined above. Using (15) and taking into account the definition of the functional (9), we infer that

$$\|y(t)\|^2 \leq \frac{1}{h_{min}(t)}\langle H(t)y(t), y(t)\rangle \leq \frac{V(t,y)}{h_{min}(t)} \leq \frac{V(0,\varphi)}{h_{min}(t)}exp(-\int_0^t \gamma(\xi)d\xi)$$

so that

$$\|y(t)\| \leq \sqrt{\frac{V(0,\varphi)}{h_{min}(t)}}exp(-\frac{1}{2}\int_0^t \gamma(\xi)d\xi).$$

3 Conclusion

In this work, we took into account a nonlinear differential systems of neutral type with periodic coefficients and a time-varying retardation. Using Lyapunov- Krasovskii functional approach, we derived new sufficient conditions related to the exponentially stability of zero solution of the considered system. By this discussion, we extended some results on the subject.

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