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Existence investigation of a fourth order semi-linear weighted problem

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Abstract

In this paper, we investigate the existence or the nonexistence of positive solutions for some semi-linear biharmonic elliptic problems with Dirichlet boundary conditions. First, we consider that the nonlinearity is asymptotically linear and we prove that the bifurcation phenomena occurs by using a variational method and ”mountain pass” approach. In our proof, we will neither suppose the Ambrosetti-Rabionowitz condition nor any replacement condition on the nonlinearity. Secondly, we treat the same problem with super-linearity at infinity in the subcritical case.

1 Introduction and Main Results

Since the pioneering paper of Mironescu and Rădulescu [21], partial differential equations with asymptotically linear nonlinearities attracted a lot of attentions. In [20, 21, 24, 14], the authors considered the harmonic problem with non linearity of the form $f(x,u)=\lambda g(u)$ where $g(u)$ is a positive, increasing and convex smooth function and $\lim\limits_{u\to\infty}\dfrac{g(u)}{u}=\ell<\infty$. In [1], Dammak et al. studied the effect of small perturbation for the existence of

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the bifurcation phenomena. With the same type of nonlinearities and fourth order elliptic differential equations, we refer the reader to [2, 10, 11, 28]. In [34], different conditions are assumed for the asymptotically nonlinearities and different type of results are proved. Later, in [33], Yue and Zhengping considered two boundary biharmonic problems and supposed that:
F1. $f(x,t)$ is continuous on $\overline{\Omega} \times \mathbb{R}$, non-negative and $f(x,t) \equiv 0$ for $t \leq 0$ and $x \in \overline{\Omega}$.
F2. $\lim_{t\to 0} \frac{f(x,t)}{t} = p(x)$ and $\lim_{t\to\infty} \frac{f(x,t)}{t} = \ell < \infty$ uniformly in x such that $0 \leq p(x) \in L^\infty(\Omega)$, $\|p(x)\|_\infty < \lambda_1$, where $\lambda_1 > 0$ denotes the first eigenvalue of the given boundary condition operator.
F3. $\frac{f(x,t)}{t}$ is a nondecreasing function on $t > 0$. The authors proved the existence of a number λ^* such that there exists a positive solution if $\ell > \lambda^*$ and there is no positive solution if $\ell < \lambda^*$.
When f is super-linear, we refer the reader to [4, 2, 7, 18, 28, 32] and the references therein. In the recent years, the study of partial differential equations with a weight function and bi-laplace operator is considered an interesting topic since they arise from thin film theory, micro-electro-mechanical systems and surface diffusion on solids. Also, in flow in Hele-Shaw cells and interface dynamics, and other fields of science related to traveling waves in suspension bridges and radar imaging , see for example [3, 12, 13, 17, 22].

In what follows, we consider the following nonlinear Dirichlet elliptic weighted problem

$$\begin{cases} \Delta^2 u - \operatorname{div}(a(x)\nabla u) = f(x,u) & \text{in} \quad \Omega, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega = B(0,1)$ the unitary ball in $\mathbb{R}^N$, $N \geq 2$ and $a(x)$ is a continuous function defined on $\overline{\Omega}$. This type of equations was first introduced by Turing in the modeling morphogenesis phenomena in Biology and in population dynamics [31].

Definition 1.1. *Let $u \in H_0^2(\Omega)$. u is called a solution of* (1.1) *if*

$$\int_\Omega \Delta u \Delta \varphi dx + \int_\Omega a(x)\nabla u \nabla \varphi dx = \int_\Omega f(x,u)\varphi dx, \quad \text{for all} \ \ \varphi \in H_0^2(\Omega). \tag{1.2}$$

In order to find a solution to problem (1.1), consider the energy I defined on $H_0^2(\Omega)$ by

$$I(u) = \frac{1}{2}\int_\Omega [|\Delta u|^2 + a(x)|\nabla u|^2]dx - \int_\Omega F(x,u)dx, \tag{1.3}$$

where

$$F(x,s) = \int_0^s f(x,t)dt.$$

A non zero critical point of I gives a solution of the (1.1) and by the condition $(F1)$ and the maximum principle, we obtain a positive solution. As variational method, we use the mountain pass approach. When Ambrosetti and Rabionovitz in [5, 23] introduce the Mountain Pass Theorem, they also supposed a technical condition in order to apply the Theorem and prove the compactness condition of type Palais-Smaile: Suppose that there exist $\theta > 2$ and $M > 0$ satisfying

$$0 < \theta F(x,t) \leq f(x,t)t, \quad \text{for all } \ |t| \geq M \ \text{ and } \ x \in \Omega. \tag{AR}$$

When we have asymptotically linear nonlinearities, we cannot suppose such condition since by using the condition (AR), we have $\lim\limits_{t\to\infty} \dfrac{F(x,t)}{t^2} = \infty$. So, by the same condition (AR), we get $\lim\limits_{t\to\infty} \dfrac{f(x,t)}{t} = \infty$ and this contradicts the fact that $f(x,t)$ is asymptotically linear at ∞. So, in our case, we cannot suppose the condition (AR).
In some papers, the authors introduce other condition in order to replace the condition (AR) as in [8, 9, 16, 26, 29, 30, 34] and the references therein. In this paper, we will use the "Mountain Pass Theorem" without any replacement condition.

Before stating our results, let $\|.\|_p$ denote the standard $L^p(\Omega)$-norm for $1 \leq p < \infty$ and suppose that the function $a(x)$ satisfies

(A) $\quad 0 < a^- \leq a(x)$, where $a^- = \inf\limits_{x\in\Omega} a(x)$.

Considering the scalar product on $H_0^2(\Omega)$ given by

$$< u, v >= \int_\Omega [\Delta u \Delta v + a(x)\nabla u.\nabla v] \ dx,$$

and set the associated norm

$$\|u\| := \Big(\int_\Omega [|\Delta u|^2 + a(x)|\nabla u|^2] \ dx\Big)^{\frac{1}{2}}.$$

Let φ_1 be a normalized, positive eigenfunction corresponding to the first eigenvalue λ_1; that is,

$$\begin{cases} \Delta^2\varphi_1 - \operatorname{div}(a(x)\nabla\varphi_1) &= \lambda_1\varphi_1 \quad \text{in} \quad \Omega \\ \varphi_1 = \dfrac{\partial\varphi_1}{\partial n} &= 0 \qquad \text{on} \quad \partial\Omega, \\ \|\varphi_1\|_2 &= 1, \end{cases} \tag{1.4}$$

and r^* the critical Sobolev exponent:

$$r^* = \begin{cases} \frac{2N}{N-4} & \text{if} \quad N > 4 \\ \infty & \text{else.} \end{cases}$$

We state our results as follows:

Theorem 1.2. *Suppose that the conditions $(F1)$ and $(F2)$ hold and $\ell \in (0,\infty)$. Then, we have*

(i) The equation (1.1) does not have positive solutions if $\ell < \lambda_1$ and $(F3)$ holds.

(ii) The equation (1.1) has a positive solution if $\ell > \lambda_1$.

(iii) If $\ell = \lambda_1$ and $(F3)$ holds, then equation (1.1) has a positive solution $u \in H_0^2(\Omega)$ if and only if there exists a constant $c > 0$ such that $u = c\varphi_1$ and $f(x,u) = \lambda_1 u$ a.e. in Ω.

When the nonlinearity is super-linear, we suppose that it is subcritical and we prove the following result:

Theorem 1.3. *Let $\ell = \infty$ and $(F1) - (F3)$ hold.*
Suppose that $\lim_{t\to\infty} \dfrac{f(x,t)}{t^{r-1}} = 0$, uniformly in x, for some $r \in (2, r^*)$. Then the problem (1.1) has a positive solution.

This paper is organized as follows:
Section 2 contains some useful definitions and some preliminary Lemmas. In section 3, we prove the main results.
Throughout the paper, we use the notation$A$ or C to represent a constant that may not be the same from one line to another.

2 Preliminary results

Here, we recall some basic definitions and properties and we prove some elementary Lemmas. First, recall the following definition:

Definition 2.1. *Let X be a Banach space and I be a given functional in $C^1(X, \mathbb{R})$.*

(i) A sequence $\{u_n\} \subset X$ is called a Palais-Smale (PS) sequence of I if $I(u_n)$ is bounded in $\mathbb{R}$ and $I'(u_n) \to 0$ as $n \to \infty$.

(ii) We say that I satisfies the Palais-Smale (PS) condition if every (PS) sequence is relatively compact in X; this means, $\{u_n\}$ has a convergent subsequence in the space X.

In the proof of our main results, we use the following classical Mountain Pass Theorem.

Proposition 2.2. *[5] Let X be a Banach space and $I : X \to \mathbb{R}$ a functional of class C^1 satisfying $I(0) = 0$ and*

(i) There exist $\rho, \quad \beta > 0$ with $I(u) \geq \beta$, $\forall u \in \partial B(0, \rho)$.

(ii) There exists $x_1 \in X$ satisfying $\|x_1\| > \rho$ and $I(x_1) < 0$.

(iii) I satisfies The Palais-Smale condition.

Then I has a critical point u and $I(u) \geq \beta > 0$.

In the sequel, we need the following elementary result:

Lemma 2.3. *$(H_0^2(\Omega), \|.\|)$ is a Hilbert space and the norm $\|.\|$ is equivalent to the standard norm $\|.\|_{H^2(\Omega)}$ in $H_0^2(\Omega)$.*

Proof. The equivalence between the two norms follows from the condition (A) and the Poincaré inequality. Since $(H_0^2(\Omega), \|.\|_{H^2(\Omega)})$ is a Banach space, $(H_0^2(\Omega), \|.\|)$ is a Hilbert space. □

In the following two Lemmas, we prove that the functional I satisfies the two geometric properties of the Proposition 2.2.

Lemma 2.4. *Suppose that $(F1) - (F2)$ hold and $\ell \in (0, \infty)$ is finite. Then we have* *(i) There exist $\rho, \beta > 0$ such that $I(u) \geq \beta$ for all $u \in \partial B(0,\rho)$ in $H_0^2(\Omega)$.*

(ii) If $\lambda_1 < \ell < \infty$, then $I(t\varphi_1) \to -\infty$ as $t \to \infty$.

Proof. (i) For all $\varepsilon > 0$ and all $q > 1$, from the condition $(F2)$, we can find
$A = A(\varepsilon) \geq 0$ such that

$$F(x,t) \leq (\|p(x)\|_\infty + \varepsilon)t + A|t|^q \tag{2.5}$$

for all $x \in \Omega$ and $t \in \mathbb{R}$. Then

$$F(x,t) \leq \frac{1}{2}(\|p(x)\|_\infty + \varepsilon)t^2 + A|t|^{q+1}. \tag{2.6}$$

But

$$I(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u)\, dx, \tag{2.7}$$

and so

$$I(u) \geq \frac{1}{2}\|u\|^2 - \frac{1}{2}(\|p(x)\|_\infty + \varepsilon)\|u\|_2^2 - A\|u\|_{q+1}^{q+1}.$$

When $2 < q+1 < r^*$, using the Sobolev embedding theorem, we have

$$\|u\|_{q+1}^{q+1} \leq C\|u\|^{q+1}, \tag{2.8}$$

for some constant C. Then

$$I(u) \geq \frac{1}{2}\|u\|^2 - \frac{1}{2}(\|p(x)\|_\infty + \varepsilon)\|u\|_2^2 - AC\|u\|^{q+1}.$$

Because of the characterization of λ_1, we have

$$I(u) \geq \frac{1}{2}(1 - \frac{\|p(x)\|_\infty + \varepsilon}{\lambda_1})\|u\|^2 - AC\|u\|^{q+1}.$$

From $(F2)$, there exists $\varepsilon > 0$ such that $\|p(x)\|_\infty + \varepsilon < \lambda_1$. Then, we take $\|u\| = \rho$ small enough in order to get $I(u) \geq \beta$ for some $\beta > 0$ sufficiently small.

(ii) Let $t > 0$. We have

$$I(t\varphi_1) = \frac{t^2}{2}\int_\Omega [|\Delta u|^2 + a(x)|\nabla\varphi_1|^2]dx - \int_\Omega F(x,t\varphi_1)\, dx$$

and so

$$I(t\varphi_1) = \frac{t^2}{2}\lambda_1 - \int_\Omega F(x, t\varphi_1)\, dx. \tag{2.9}$$

On the other hand, from the condition $(F2)$, we get

$$\lim_{t\to\infty} \frac{F(x,t)}{t^2} = \frac{\ell}{2}\cdot$$

So, by (2.9) and Fatou's Lemma, we obtain

$$\lim_{t\to\infty} \frac{I(t\varphi_1)}{t^2} \leq \frac{1}{2}\lambda_1 - \int_\Omega \lim_{t\to\infty} \frac{F(x,t\varphi_1)}{(t\varphi_1)^2}\varphi_1^2\, dx$$

and then

$$\lim_{t\to\infty} \frac{I(t\varphi_1)}{t^2} \leq \frac{1}{2}(\lambda_1 - \ell).$$

When $\lambda_1 < \ell$, we get $\lim_{t\to\infty} I(t\varphi_1) = -\infty$. □

Lemma 2.5. *Assume that* $(F1) - (F3)$ *hold,* $\ell = \infty$ *and* $\lim_{t\to\infty} \frac{f(x,t)}{t^{r-1}} = 0$ *uniformly in* x*, for some* $r \in (2, r^*)$*. Then, the following results hold.*

(i) There exist positive constants ρ *and* β *satisfying* $I(u) \geq \beta$ *for all* $u \in H_0^2(\Omega)$ *with* $\|u\| = \rho$.

(ii) $\lim_{t\to\infty} I(t\varphi_1) = -\infty$.

Proof. (i) In this subcritical case, the condition

$$\lim_{t\to\infty} \frac{f(x,t)}{t^{r-1}} = 0, \quad \text{for some} \;\; r \in (2, r^*)$$

and the first part of the condition $(F2)$ give that for any $\varepsilon > 0$, there exist $A = A(\varepsilon) \geq 0$ satisfying for all $t \in \mathbb{R}$ and $x \in \Omega$

$$F(x,t) \leq \frac{1}{2}(\|p(x)\|_\infty + \varepsilon)t^2 + A|t|^r. \tag{2.10}$$

From (2.7) and (2.10), we get

$$I(u) \geq \frac{1}{2}\|u\|^2 - \frac{1}{2}(\|p(x)\|_\infty + \varepsilon)\|u\|_2^2 - A\|u\|_r^r.$$

Since $2 < r < r^*$, by the Sobolev embedding theorem, we have $\|u\|_r^r \leq C\|u\|^r$ and then

$$I(u) \geq \frac{1}{2}\|u\|^2 - \frac{1}{2}(\|p(x)\|_\infty + \varepsilon)\|u\|_2^2 - AC\|u\|^r.$$

So

$$I(u) \geq \frac{1}{2}(1 - \frac{\|p(x)\|_\infty + \varepsilon}{\lambda_1})\|u\|^2 - AC\|u\|^r.$$

Now, we can choose $\varepsilon > 0$ such that $\|p(x)\|_\infty + \varepsilon < \lambda_1$ and $\|u\| = \rho$ small enough in order to have $I(u) \geq \beta$ for some $\beta > 0$ sufficiently small.

(ii) Since the positive function φ_1 is in $C(\Omega)$. Let $\Omega_0 \subset \mathbb{R}^N$ such that $\Omega_0 \subset \overline{\Omega_0} \subset \Omega$ and let $\alpha > 0$ such that $\varphi_1(x) \geq \alpha > 0$ for all $x \in \Omega_0$. From $(F3)$, we obtain

$$0 \leq 2F(x,t) \leq tf(x,t). \tag{2.11}$$

Then, $\frac{F(x,t)}{t^2}$ is a nondecreasing function with respect to $t > 0$ for a.e. $x \in \Omega_0$, so, for all $x \in \Omega_0$ and $t > 0$ we have

$$\frac{F(x, t\varphi_1(x))}{t^2\varphi_1^2(x)} \geq \frac{F(x, t\alpha)}{t^2\alpha^2}. \tag{2.12}$$

Since $\ell = \infty$,

$$\lim_{t\to\infty} \frac{F(x,t)}{t^2} = \infty.$$

Therefore, for all $A > 0$, there exist t_0 such that for $t \geq t_0$ and $x \in \Omega_0$, we have $\frac{F(x,t\varphi_1(x))}{t^2\varphi_1^2(x)} \geq A$. Therefore

$$\frac{I(t\varphi_1)}{t^2} = \frac{1}{2}\int_\Omega |\Delta\varphi_1|^2 + a(x)|\varphi_1|^2 dx - \int_\Omega \frac{F(x,t\varphi_1)}{(t\varphi_1)^2}\varphi_1^2\, dx.$$

We get

$$\frac{I(t\varphi_1)}{t^2} \leq \frac{1}{2}\lambda_1 \int_\Omega \varphi_1^2 dx - A\int_{\Omega_0} \varphi_1^2\, dx,$$

and so

$$\frac{I(t\varphi_1)}{t^2} \leq \frac{1}{2}\lambda_1 - A\alpha|\Omega_0|.$$

For a large $A > 0$, we obtain

$$\frac{I(t\varphi_1)}{t^2} \leq C < 0$$

and then $\lim_{t\to\infty} I(t\varphi_1) = -\infty$. □

We end this section by the following Lemma and we omit the proof which is similar to these in [30, 34].

Lemma 2.6. *Suppose that* $(F3)$ *holds and there exists a sequence* (u_n) *in* $H_0^2(\Omega)$ *such that*

$$< I'(u_n), u_n > \to 0$$

where I *the functional given by (1.3). Then, up to a subsequence,*

$$I(su_n) \leq \frac{1+s^2}{2n} + I(u_n), \quad \textit{for all} \;\; s > 0. \tag{2.13}$$

3 Proof of the Main Results

Proof of the Theorem 1.2 (i) Consider $0 < \ell < \lambda_1$ and let $u \in H_0^2(\Omega)$ be a positive solution of the equation (1.1). From the equation (1.2) and $(F1) - (F3)$, we have

$$\int_\Omega |\Delta u|^2 + a(x)|\nabla u|^2 dx = \int_\Omega f(x,u)u dx \leq \int_\Omega \ell u^2 dx. \tag{3.14}$$

So, $\lambda_1 \leq \ell$ which contradicts our hypothesis. Then, the equation (1.1) does not have a positive solution.

(ii) Let $< \lambda_1 < \ell < \infty$. From Lemma 2.4, the geometric properties are satisfied. By Proposition 2.2, we only have to prove the (PS) condition for the energy I induced by (1.3). Let $\{u_n\}$ be a (PS) sequence of the functional I, where

$$I(u_n) = \frac{1}{2}\|u_n\| - \int_\Omega F(x,u_n)dx. \tag{3.15}$$

Up to a subsequence, we have

$$I(u_n) \to \mu \quad \text{as} \;\; n \to \infty, \tag{3.16}$$

for some $\mu \in \mathbb{R}$ and

$$\|I'(u_n)\|_* \to 0 \quad \text{as} \;\; n \to \infty. \tag{3.17}$$

Step 1. Here, we prove that if $\{u_n\}$ is bounded in $H_0^2(\Omega)$, then $\{u_n\}$ has a convergent subsequence.

Since $\{u_n\}$ is bounded in the reflexive space $(H_0^2(\Omega), \|.\|)$ and by using compact Sobolev embedding theorem, up to a subsequence, we get

$$u_n \rightharpoonup u, \quad \text{weakly in} \quad H_0^2(\Omega),$$
$$u_n \to u, \quad \text{in} \quad L^2(\Omega)$$
and
$$u_n(x) \to u(x), \quad \text{a.e in} \quad \Omega.$$

By using (3.17), we get

$$\int_\Omega [\Delta u_n \Delta\varphi + a(x)\ \nabla u_n.\nabla\varphi]dx - \int_\Omega f(x,u_n)\varphi dx \to 0, \quad \forall\varphi \in H_0^2(\Omega). \tag{3.18}$$

Then, we obtain

$$\Delta^2 u_n - \operatorname{div}(a(x)\nabla u_n) - f(x,u_n) \to 0 \quad \text{in} \quad H_0^{-2}(\Omega), \tag{3.19}$$

where $H_0^{-2}(\Omega)$ is the dual space of $H_0^2(\Omega)$.
From $(F2)$, we have that $f(x,u_n) \to f(x,u)$ in $L^2(\Omega)$ and we get

$$\Delta^2 u_n - \operatorname{div}(a(x)\nabla u_n) \to f(x,u) \quad \text{in} \quad H_0^{-2}(\Omega). \tag{3.20}$$

As in [19], we can prove easily that the operator $L = -\Delta^2 - \operatorname{div}(a(x)\nabla)$ is an isomorphism from $H_0^2(\Omega)$ into $H_0^{-2}(\Omega)$. As consequence

$$u_n \to L^{-1}(f(x,u)) \quad \text{in} \quad H_0^2(\Omega). \tag{3.21}$$

Step 2 In this second step, we show that $\{u_n\}$ is bounded in $H_0^2(\Omega)$.
By contradiction, suppose that the sequence $\{u_n\}$ is not bounded in $(H_0^2(\Omega), \|.\|)$.
Then,

$$\|u_n\| \to \infty \quad \text{as} \quad n \to \infty$$

up to a subsequence. Now, let

$$w_n = \frac{u_n}{\|u_n\|} \quad \text{and} \quad t_n = \|u_n\|.$$

Since the sequence w_n is bounded in $H_0^2(\Omega)$, there exists $w \in H_0^2(\Omega)$ such that $w_n \rightharpoonup w$ weakly in $H_0^2(\Omega)$, $w_n \to w$ in $L^2(\Omega)$ and $w_n(x) \to w(x)$ a.e in Ω. By using the condition $(F2)$, we can find a constant $M > 0$ satisfying

$$\frac{f(x,t)}{t} \le M, \quad \text{for all} \quad t > 0 \quad \text{and} \quad x \in \Omega; \tag{3.22}$$

that is,

$$\int_\Omega \frac{f(x,u_n)}{\|u_n\|^2} u_n dx = \int_\Omega \frac{f(x,u_n)}{u_n} w_n^2 dx \le M \int_\Omega w_n^2 dx. \tag{3.23}$$

From (3.18), for $\varphi = u_n$ and after multiplication by $\dfrac{1}{\|u_n\|^2}$, we get

$$1 - \int_\Omega \frac{f(x,u_n)}{u_n} w_n^2 dx \to 0.$$

therefore $w \not\equiv 0$, from (3.23). We consider again (3.18), and we write that for all $\varphi \in H_0^2(\Omega)$,

$$\int_\Omega [\Delta w_n \Delta\varphi + a(x)\nabla w_n.\nabla\varphi]dx - \int_\Omega \frac{f(x,u_n)}{u_n} w_n \varphi dx \to 0. \tag{3.24}$$

Now, for the first term, we know that (as in the first step),

$$\int_\Omega [\Delta w_n \Delta\varphi + a(x)\nabla w_n.\nabla\varphi]dx \to \int_\Omega [\Delta w \Delta\varphi + a(x)\nabla w.\nabla\varphi]dx. \tag{3.25}$$

For the second term in (3.24), we consider

$$\Omega_+ := \{x \in \Omega; \quad w(x) > 0\}$$

and as $u_n(x) = \|u_n\| w_n(x)$, we have that

$$\lim_{n\to\infty} u_n(x) = \infty, \quad \text{for all} \;\; x \in \Omega_+.$$

Set

$$g_n(x) := \frac{f(x,u_n(x))}{u_n(x)} \;\text{ if }\; u_n(x) > 0 \;\text{ and }\; g_n(x) = 0 \;\text{ if }\; u_n(x) \le 0.$$

The sequence $\{g_n\}$ is bounded on Ω and so it is weakly star convergent in $L^\infty(\Omega)$, up to a subsequence, to a function g.
By $(F2)$, the function $g(x) = \ell$, for all $x \in \Omega_+$. We have $w_n \to w$ in $L^2(\Omega)$. Then $w_n^+ \to w^+$ in $L^2(\Omega)$ where $w_n^+ = \frac{w_n + |w_n|}{2}$.
Then

$$\int_\Omega \frac{f(x,u_n)}{u_n} w_n \varphi dx = \int_\Omega g_n(x) w_n \varphi dx = \int_\Omega g_n(x) w_n^+ \varphi dx;$$

that is,

$$\int_\Omega \frac{f(x,u_n)}{u_n} w_n \varphi dx \to \int_\Omega g(x) w^+ \varphi dx.$$

which means

$$\int_\Omega \frac{f(x,u_n)}{u_n} w_n \varphi dx \to \ell \int_\Omega w^+ \varphi dx. \tag{3.26}$$

From (3.25) and (3.26), it follows that

$$\int_\Omega [\Delta w \ \Delta\varphi + a(x)\nabla w.\nabla\varphi] \ dx \to \int_\Omega \ell w^+ \varphi dx, \forall \varphi \in H_0^2(\Omega). \tag{3.27}$$

We get

$$\begin{cases} \Delta^2 w - \operatorname{div}(a(x)\nabla w) &= \ell w^+ \quad \text{in} \quad \Omega \\ \dfrac{\partial w}{\partial n} = w &= 0 \qquad \text{on} \quad \partial\Omega. \end{cases} \tag{3.28}$$

By the maximum principle, w is a positive solution and the equation (3.28) insures that $w = \varphi_1$ and $\ell = \lambda_1$. This is contradicts the hypothesis and so the sequence $\{u_n\}$ is bounded in $H_0^2(\Omega)$.
By Proposition 2.2, the problem (1.1) has a nontrivial solution. From the condition $(F1)$ and the maximum principle, the solution is positive.

(iii) Let $\ell = \lambda_1$ and let u be a positive solution for the problem (1.1). By taking $\varphi = \varphi_1$ in (1.2), we get

$$\int_\Omega [\Delta u \Delta\varphi_1 + a(x)\nabla u.\nabla\varphi_1] \ dx = \int_\Omega f(x,u)\varphi_1 dx. \tag{3.29}$$

If we multiply the first equation in (1.4) by u and integrate, we obtain

$$\int_\Omega [\Delta u \Delta\varphi_1 + a(x)\nabla u.\nabla\varphi_1] \ dx = \ell \int_\Omega u\varphi_1 dx. \tag{3.30}$$

So,

$$\int_\Omega (f(x,u) - \ell u)\varphi_1 dx = 0.$$

Since $\varphi_1 > 0$ and by using the conditions $(F2)$ and $(F3)$, we get $f(x,u) = \ell u$ a.e. in Ω. But $\ell = \lambda_1$ and then u is an eigenfunction associated to the simple eigenvalue λ_1, so $u = c\varphi_1$ for some constant $c > 0$ and we have $f(x,u) = \lambda_1 u$.
Conversely, let $\ell = \lambda_1$, $u = c\varphi_1$ for some constant $c > 0$ and $f(x,u) = \lambda_1 u$. Then u is a solution of the equation (1.1). □

Proof of the Theorem 1.3 We study the equation (1.1) when $\ell = \infty$. In view of the Lemma 2.5 and the Proposition 2.2, we have to prove that I satisfies the PS condition and then (1.1) has a positive solution.

Let $\{u_n\}$ a (PS) sequence. Up to a subsequence, $\{u_n\}$ satisfies (3.16) and (3.17). The step 1 of the proof of Theorem 1.2 (*ii*) is still valid here, so we have to prove that the sequence $\{u_n\}$ is bounded in $H_0^2(\Omega)$.
We argue by contradiction. Suppose that $\{u_n\}$ is not bounded in $H_0^2(\Omega)$. For a positive real number d, set

$$w_n = \frac{u_n}{d\|u_n\|}, \quad t_n = \frac{1}{d\|u_n\|}.$$

So, there exists $w \in H_0^2(\Omega)$ such that, up to a subsequence, $w_n \rightharpoonup w$ in $H_0^2(\Omega)$, $w_n \to w$ in $L^2(\Omega)$ and $w_n \to w$ a.e in Ω.
As a consequence,

$$w_n^+ \to w^+ \quad \text{in} \quad L^2(\Omega),$$

and

$$w_n^+ \to w^+ \quad \text{a.e. in} \quad \Omega.$$

$$I(w_n) = \frac{1}{2}\|w_n\|^2 - \int_\Omega F(x, w_n)dx.$$

From the condition $(F1)$, we get

$$I(w_n) = \frac{1}{2}\|w_n\|^2 - \int_\Omega F(x, w_n^+)dx. \tag{3.31}$$

In order to use Lemma 2.6 and find a contradiction, we will prove that $w^+ \equiv 0$.
Let $\Omega_+ = \{x \in \Omega; w^+(x) > 0\}$.
For $x \in \Omega_+$

$$u_n^+(x) = dw_n^+(x)\|u_n\| \to \infty$$

and so, for any $K > 0$, there exists $n_0 = n_0(x) > 0$ such that for all $n \geq n_0$, we have

$$\frac{f(x, u_n^+(x))}{u_n^+(x)}(w_n^+(x))^2 \geq K. \tag{3.32}$$

Also, $w_n^+(x) \to w^+(x.)$ T hen there exists $n_1 = n_1(x) > 0$ such that for all $n \geq n_1$, we have

$$w_n^+(x) \geq \frac{w^+(x)}{2}. \tag{3.33}$$

From (3.32) and (3.33), we get

$$\frac{f(x, u_n^+(x))}{u_n^+(x)}(w_n^+(x))^2 \geq K\frac{(w^+(x))^2}{4}$$

for all $n \geq \max(n_0, n_1)$. So, for all $x \in \Omega_+$,

$$\lim_{n\to\infty} \frac{f(x, u_n^+(x))}{u_n^+(x)}(w_n^+(x))^2 \geq K\frac{(w^+(x))^2}{4}. \tag{3.34}$$

Now, by using (3.18), we have

$$\|u_n\|^2 - \int_\Omega f(x, u_n)u_n\ dx \to 0,$$

and so,

$$\frac{1}{d^2} - \int_\Omega \frac{f(x, u_n)}{u_n}(w_n)^2\ dx \to 0.$$

Then, from $(F1)$, we get

$$\int_\Omega \frac{f(x, u_n^+)}{u_n^+}(w_n^+)^2\ dx \to \frac{1}{d^2}.$$

$$\begin{aligned}
\frac{1}{d^2} &= \lim_{n\to\infty} \int_\Omega \frac{f(x, u_n^+(x))}{u_n^+(x)}(w_n^+(x))^2 dx \\
&\geq \lim_{n\to\infty} \int_{\Omega_+} \frac{f(x, u_n^+(x))}{u_n^+(x)}(w_n^+(x))^2 dx \\
&\geq \int_{\Omega_+} \lim_{n\to\infty} \frac{f(x, u_n^+(x))}{u_n^+(x)}(w_n^+(x))^2 dx \\
&\geq \frac{K}{4}\int_{\Omega_+} (w^+(x))^2 dx,
\end{aligned}$$

$\forall K > 0$, so $|\Omega_+| = 0$ and then $w^+ \equiv 0$.
Return to the equality (3.31), we obtain

$$\lim_{n\to\infty} I(w_n) = \frac{1}{2d^2}. \tag{3.35}$$

If we apply Lemma 2.5, we have up to a subsequence,

$$I(w_n) = I(t_n u_n) \leq \frac{1}{2n}(1 + t_n^2) + I(u_n). \tag{3.36}$$

$$\frac{1}{2d^2} \leq d \tag{3.37}$$

for all $d > 0$. So Theorem 1.3 is proved. □

4 Conclusion

We studied Weighted Biharmonic Problems with asymptotically linear nonlinearities. Under some suitable conditions, we proved the existence of positive solutions. Moreover, we proved the existence of bifurcation phenomena. Furthermore, we proved that the same technique worked when the nonlinearity is superlinear and subcritical at infinity. In the proof of the existence, we use variational methods without using the Ambrosetti-Rabionowitz condition (AR) or any replacement condition.

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