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A Mixed Ramsey Problem Revisited

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Abstract

It is shown that for each integer $n > 1$, $\{k\colon \exists$ an edge coloring of K_n with exactly k colors appearing in which exactly 2 colors appear on the edges of each K_3 in $K_n\}= \{g\left(n\right), \ldots, n-1\}$, a block of consecutive integers in which $g\left(n\right) \in \{\lceil 2\log_5 n\rceil, \lceil 2\log_5 n\rceil + 1\}$. This is a sharpening of a result of Chung and Graham from 1983 [1].

1 Introduction

In 1983 Chung and Graham [1] obtained a wonderful result that would now be regarded as a "mixed Ramsey" theorem.

Theorem 1. *For each positive integer k let $f\left(k\right)$ be the largest integer n such that the edges of K_n can be colored with no more than k colors appearing so that each K_3 subgraph has exactly 2 colors appearing on its edges.* (That is, no $K_3 \subseteq K_n$ is either monochromatic or rainbow.) *Then,*

$$f\left(k\right) = \begin{cases} 5^{\frac{k}{2}} & \text{if } k \text{ is even} \\ 2\cdot 5^{\frac{k-1}{2}} & \text{if } k \text{ is odd} \end{cases}$$

During the same era, roughly 1976 - 1990, Vitaly Voloshin [3] was developing his ideas about mixed hypergraphs and their proper colorings. A mixed

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hypergraph is a triple $\mathscr{H} = (V; C, D)$ in which V, the set of vertices of $\mathscr{H}$, is a non-empty set and $C, D \subseteq 2^V$ are sets of subsets of V. These subsets are *hyperedges*, or *edges*. A proper coloring of $\mathscr{H}$ is a coloring of V such that no $c \in C$ is rainbow (that is, 2 different elements of c bear the same color), and no $d \in D$ is monochromatic (that is 2 different elements of d bear different colors). The Voloshin spectrum of $\mathscr{H}$ is $\{k\colon$ there is a proper coloring of $\mathscr{H}$ with exactly k colors appearing$\}=$ VSPEC($\mathscr{H}$).

In an important family of mixed hypergraphs, the vertices are edges of an ordinary graph and the hyperedges are edge sets of particular subgraphs. For simple graphs G, X, Y, we set $V = E(G)$, the edge set of G, $C = \{E(X') : X'$ is a subgraph of G isomorphic to $X\}$, and $D = \{E(Y') : Y'$ is a subgraph of G isomorphic to $Y\}$. Then a proper coloring of $\mathscr{H} = (V; C, D)$ is a coloring of G's edges such that no copy of X in G is rainbow and no copy of Y in G is monochromatic. We will denote the Voloshin spectrum of $\mathscr{H}$ by $\mathrm{VSPEC}'(G; X, Y)$.

Suppose that X and Y are graphs with $|E(Y)| > 1$ and k is a positive integer. Ramsey's theorem implies that for all n sufficiently large, depending on Y and k, for every edge coloring of K_n with less than or equal to k colors appearing, there must be a monochromatic copy of Y somewhere in K_n. Therefore, if X and Y are given, it is natural for Ramsey theorists to ask: "for each positive integer k, what is the largest $n = f_{X,Y}(k)$ such that $\{1, \ldots, k\} \cap \mathrm{VSPEC}'(K_n; X, Y) \neq \emptyset$?"

Of course, the question would not be posed in this way! But this is the first question answered by Chung and Graham in Theorem 1, in the case $X = Y = K_3$. Our question is: "for each integer $n > 1$, what is the Voloshin spectrum $\mathrm{VSPEC}'(K_n; K_3, K_3)$?" We will begin by showing the smallest and largest element of $\mathrm{VSPEC}'(K_n; K_3, K_3)$.

2 Smallest Element of $\mathrm{VSPEC}'(K_n; K_3, K_3)$

Lemma 2. *The smallest element of VSPEC*$'(K_n; K_3, K_3)$ *is either* $\lceil 2\log_5 n\rceil$ *or* $\lceil 2\log_5 n\rceil + 1$.

Proof. Let f be as in Theorem 1, let the smallest element of $\mathrm{VSPEC}'(K_n; K_3, K_3)$ be denoted $g(n)$, and let $\mathscr{H}_n$ denote the mixed hypergraph of which $\mathrm{VSPEC}'(K_n; K_3, K_3)$ is the Voloshin spectrum.

Claim. $g(n)$ is the value of k satisfying $f(k-1) < n \leq f(k)$.

Proof. By Theorem 1, if $f(k-1) < n \leq f(k)$, then there is a proper coloring of $\mathscr{H}_n$ with no more than k colors appearing, and there is no proper coloring of $\mathscr{H}_n$ with $k-1$ or fewer colors appearing, so there must be a proper coloring of $\mathscr{H}_n$ with exactly k colors appearing. Therefore $k \in \mathrm{VSPEC}'(K_n; K_3, K_3)$, so $g(n) \leq k$.

On the other hand, if there is a proper coloring of $\mathscr{H}$ with exactly $r \leq k-1$ colors appearing, then n would be $\leq f(k-1)$. Since $f(k-1) < n \leq f(k)$, it follows that there is no such r; therefore k is the smallest element of $\mathrm{VSPEC}'(K_n; K_3, K_3)$. □

Suppose that $k = g(n)$ is even. By Theorem 1 and our previous Claim,

$$2 \cdot 5^{\frac{k-2}{2}} = f(k-1) < n \leq f(k) = 5^{\frac{k}{2}}$$

"Solving" for k, we obtain

$$2\log_5 n \leq k < 2\log_5 n + 2(1 - \log_5 2)$$

Because k is an integer and $2(1-\log_5 2) < 2$, it follows that $k \in \{\lceil 2\log_5 n \rceil, \lceil 2\log_5 n \rceil + 1\}$.

When $k = g(n)$ is odd we obtain

$$1 - 2\log_5 2 + 2\log_5 n \leq k < 2\log_5 n + 1$$

whence $k \in \{\lceil 2\log_5 n \rceil, \lceil 2\log_5 n \rceil + 1\}$. □

Given n, how does one decide whether $k = g(n)$ is $\lceil 2\log_5 n \rceil$ or $\lceil 2\log_5 n \rceil + 1$? It is the value of k such that $f(k-1) < n \leq f(k)$.

Example 3. For instance, if $n = 19$, $\lceil 2\log_5 19 \rceil = 4$, and we see that $10 = 2 \cdot 5^{\frac{3-1}{2}} < 19 \leq 5^{\frac{4}{2}} = 25$, so $g(19) = 4$. Now consider $n = 51$. Then $\lceil 2\log_5 n \rceil = 5$. Clearly $51 \not\leq f(5) = 50$, so $g(51) = 6$.

Now we will find the largest element of $\mathrm{VSPEC}'(K_n; K_3, K_3)$. The following is well known (see [2]), but we supply a proof for the reader's convenience.

3 Largest Element of $\mathrm{VSPEC}'(K_n; K_3, K_3)$

Lemma 4. *Suppose G is a simple connected graph on n vertices and $E(G)$ is colored with n or more colors appearing. Then there is a rainbow cycle in G with respect to this coloring.*

Proof. Choose n edges of G with different colors. Let H be the subgraph of G induced by these edges. Then H is a subgraph with n edges on no more than n vertices. So H contains a cycle and that cycle is rainbow. □

Corollary 5. *The greatest number of colors with which the edges of a simple connected graph on n vertices can be colored so that there is no rainbow cycle is less than or equal to $n-1$.*

The following theorem is proved in [2]; we will supply a short proof here.

Theorem 6. *If G is a connected simple graph on $n \geq 1$ vertices, then there is a rainbow-cycle-forbidding edge coloring of G with exactly $n-1$ colors appaearing.*

Proof. The proof will be by induction on n. Clearly the conclusion holds when $n = 1$.

Suppose that $n > 1$. Let T be a spanning tree in G. Take any $e \in E(T)$; $T-e$ is the disjoint union of two trees, T_1 and T_2. Let $R = V(T_1)$, $S = V(T_2)$. Then R and S partition $V(G)$ and the induced subgraphs $G[R]$, $G[S]$ are connected, since each has a spanning connected subgraph.

By the induction hypothesis, if $X \in \{G[R], G[S]\}$ then $E(X)$ can be colored with $|V(X)|-1$ colors appearing so that there are no rainbow cycles in X. We arrange for the sets of colors on the edges of $G[R]$, $G[S]$ to be disjoint. We complete the coloring of $E(G)$ by coloring the edges of the edge cut $[R,S] = \{f \in E(G) : \text{one end of } f \text{ is in } R, \text{the other in } S\}$ with a color not appearing in $G[R] \cup G[S]$.

Note that $e \in [R,S]$, so $[R,S]$ is non-empty. Therefore, the number of colors appearing on G is $|R|-1+|S|-1+1 = |R|+|S|-1 = n-1$. There are no rainbow cycles in $G[R]$, nor in $G[S]$. If a cycle in G has a vertex in R and a vertex in S, then the cycle must have at least two edges in $[R,S]$, and so must have a color repeated on its edges. Thus the coloring of G is rainbow-cycle-forbidding. □

This leads us to conclude the following.

Corollary 7. *For all $n \geq 1$, the largest element of VSPEC$'(K_n; K_3, K_3)$ is $n-1$.*

4 VSPEC$'(K_n; K_3, K_3)$

Theorem 8. *For all $n \geq 1$, VSPEC$'(K_n; K_3, K_3) = \{k \colon g(n) \leq k \leq n-1\}$ where $g(n) \in \{\lceil 2\log_5 n \rceil, \lceil 2\log_5 n \rceil + 1\}$.*

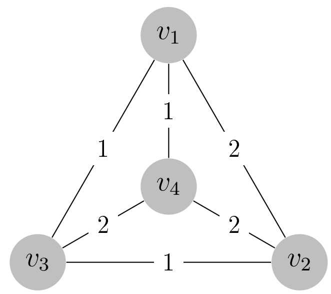

Figure 1: K_4 with a 2-edge-coloring such that every K_3 has exactly two colors.

Proof. This proof is by induction on n. For $n = 1$, $\text{VSPEC}'(K_1; K_3, K_3) = \{0\}$. For $n = 2, \text{VSPEC}'(K_2; K_3, K_3) = \{1\}$. For $n = 3, \text{VSPEC}'(K_3; K_3, K_3) = \{2\}$. For $n = 4$, consider Figure 1 as it shows that the $\min(\text{VSPEC}'(K_4; K_3, K_3)) = 2$. This is consistent with Lemma 2, since $\lceil 2 \log_5 4 \rceil = 2$. Also, by Corollary 7, $n - 1 = 4 - 1 = 3$. So $\text{VSPEC}'(K_4; K_3, K_3) = \{2, 3\}$.

We will show that K_n is exactly $k-$edge-colorable (so that exactly 2 colors appear on each K_3 in K_n), when $g(n) < k < n - 1$ for $n > 4$. Note that $g(n-1) \leq g(n) \leq k - 1$. Let $v \in V(K_n)$. By the induction hypothesis $K_n - v$ is exactly $(k-1)-$edge-colorable. Consider the join of $K_n - v$ and v, that is, K_n. Let all edges incident to v be colored with a kth color. Clearly the resulting coloring of the edges of K_n with exactly k colors appearing admits neither monochromatic nor rainbow K_3's. So $\text{VSPEC}'(K_n; K_3, K_3) = \{k \colon g(n) \leq k \leq n - 1\}$ for $n \geq 1$. ☐

References

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