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On outer (σ, τ)-n-derivations and commutativity in prime near-rings

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Abstract

In this paper, we introduce the notion of outer (σ, τ)-n-derivations in a near-ring and investigate some properties. Moreover, we obtain additive commutative of a prime near-rings satisfying certain algebraic identities involving outer (σ, τ)-n-derivation. Furthermore, we investigate some conditions involving outer (σ, τ)-n-derivations for a near-ring to be a commutative ring.

1 Introduction

A near-ring is a generalization of a ring where two axioms are omitted (addition is not necessarily abelian and only one distributive law holds). By a near ring, we shall mean an algebraic system $(N, +, \cdot)$ where N is a nonempty set, and "+", "·" are binary operations on N, called addition and multiplication, respectively, such that
(i) $(N, +)$ is a group (not necessarily abelian),
(ii) $(N, \cdot)$ is a semigroup and
(iii) multiplication is left distributive with respect to addition:
$x \cdot (y + x) = x \cdot y + x \cdot z$ for all $x, y, z \in N$

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In view of (*iii*), one speaks more precisely of a "left near ring". A similar definition with (*iii*) replaced by the right distributive law gives rise to a "right near ring". For further details about the concepts and other results in near rings, we refer the reader to the treatises [3], [4], [5], [8], [9], [21] and [22].

In 1987, Bell and Mason [7] initiated the study of derivations in near rings. A mapping $d : N \longrightarrow N$ is said to be a derivation on a near ring N if (i) $d(x+y) = d(x) + d(y)$ and (ii) $d(xy) = xd(y) + d(x)y$ holds for all $x, y \in N$. It was shown by Wang [23] that condition (ii) is equivalent to $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in N$, which facilitates the study of derivations in near rings.

Ashraf and et al. [10] defined a (σ, τ)-derivation in near rings as following.

Let $\sigma, \tau : N \longrightarrow N$ be two near-ring automorphisms of N. A mapping $d : N \longrightarrow N$ is called a (σ, τ)-derivation if (i) $d(x+y) = d(x) + d(y)$ and (ii) $d(xy) = \sigma(x)d(y) + d(x)\tau(y)$ for all $x, y \in N$.

In 2013, Ashraf and Siddeeque [14] introduced the notion of (σ, τ)-n-derivation in near-ring N, where n is a positive integer. Let $\sigma, \tau : N \longrightarrow N$ be two near-ring automorphisms of N. An n-additive (i.e. additive in each argument) mapping $D : N \times N \times \ldots \times N \longrightarrow N$ is called a (σ, τ)-n-derivation of N if

$$D(x_1y_1, x_2, \ldots, x_n) = D(x_1, x_2, \ldots, x_n)\sigma(y_1) + \tau(x_1)D(y_1, x_2, \ldots, x_n),$$

$$D(x_1, x_2y_2, \ldots, x_n) = D(x_1, x_2, \ldots, x_n)\sigma(y_2) + \tau(x_2)D(x_1, y_2, \ldots, x_n),$$

$$\vdots$$

$$D(x_1, x_2, \ldots, x_ny_n) = D(x_1, x_2, \ldots, x_n)\sigma(y_n) + \tau(x_n)D(x_1, x_2, \ldots, y_n),$$

hold for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N$.
Further, Ashraf and Siddeeque investigated some properties involving (σ, τ)-n-derivations of a prime near-ring N which force N to be a commutative ring. Our objective is to introduce a special type of derivation; namely, outer (σ, τ)-n-derivation in a near-ring, where n is a positive integer and study the commutativity behavior of prime near-rings which admit outer (σ, τ)-n-derivations satisfying certain properties.

2 Preliminaries

Throughout this paper, let N denote a (left) near-ring with center Z. Recall that a near-ring N is prime if $aNb = \{0\}$ with $a, b \in N$ implies $a = 0$ or $b = 0$ and semiprime in case $aNa = \{0\}$ with $a \in N$ implies $a = 0$. Let n be a positive integer and N^n denotes $N \times N \times \ldots \times N$($n$ terms). We begin with the following definition and lemma which are essential for developing the proofs of our main results.

Lemma 2.1. [7, *Lemma* 3]. *Let N be a prime near-ring. If $Z - \{0\}$ contains an element z for which $z + z \in Z$, then $(N, +)$ is abelian.*

Definition 2.2. *A map $D : N^n \longrightarrow N$ is called an n-additive mapping if*

$$D(x_1 + y_1, x_2, \ldots, x_n) = D(x_1, x_2, \ldots, x_n) + D(y_1, x_2, \ldots, x_n),$$
$$D(x_1, x_2 + y_2, \ldots, x_n) = D(x_1, x_2, \ldots, x_n) + D(x_1, y_2, \ldots, x_n),$$
$$\vdots$$
$$D(x_1, x_2, \ldots, x_n + y_n) = D(x_1, x_2, \ldots, x_n) + D(x_1, x_2, \ldots, y_n),$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N$.

Remark. *If D is an n-additive mapping, then it can be easily seen that*

$$\begin{aligned} D(0, x_2, \ldots, x_n) &= D(0 + 0, x_2, \ldots, x_n) \\ &= D(0, x_2, \ldots, x_n) + D(0, x_2, \ldots, x_n). \end{aligned}$$

Therefore, $D(0, x_2, \ldots, x_n) = 0$ for all $x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N$.

We also observe that

$$D(-x_1, x_2, \ldots, x_n) = -D(x_1, x_2, \ldots, x_n),$$

for all $x_1, x_2, \ldots, x_n \in N$.

In general, for $i = 1, 2, \ldots, n$ if $x_i = 0$, then

$$D(x_1, x_2, \ldots, x_i, \ldots, x_n) = 0$$

for all $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in N$ and

$$D(x_1, x_2, \ldots, -x_i, \ldots, x_n) = -D(x_1, x_2, \ldots, x_i, \ldots, x_n),$$

for all $x_1, x_2, \ldots, x_i, \in N$.

3 Outer (σ,τ)-n-derivation

In this section, we define an outer (σ,τ)-n-derivation in a near-ring and study its various properties.

Definition 3.1. *Let N be a near-ring and let $\sigma,\tau : N \longrightarrow N$ be automorphisms. A mapping $D : N^n \longrightarrow N$ is call an **outer** (σ,τ)**-n-derivation** on N if D is an $n-$additive mapping satisfying the relations*

$$D(x_1y_1,x_2,\dots,x_n)=\sigma(x_1)D(y_1,x_2,\dots,x_n)+D(x_1,x_2,\dots,x_n)\tau(y_1),$$
$$D(x_1,x_2y_2,\dots,x_n)=\sigma(x_2)D(x_1,y_2,\dots,x_n)+D(x_1,x_2,\dots,x_n)\tau(y_2),$$
$$\vdots$$
$$D(x_1,x_2,\dots,x_ny_n)=\sigma(x_n)D(x_1,x_2,\dots,y_n)+D(x_1,x_2,\dots,x_n)\tau(y_n),$$

for all $x_1,x_2,\dots,x_n,y_1,y_2,\dots,y_n\in N$.

Example. *Let* $N=\left\{\begin{bmatrix}0&0&0&0\\0&0&0&0\\w&x&0&0\\y&z&0&0\end{bmatrix}\;\middle|\; w,x,y,z\in\mathbb{R}\right\}$ *and define $D : N^n \longrightarrow N$ such that*

$$D\left(\begin{bmatrix}0&0&0&0\\0&0&0&0\\w_1&x_1&0&0\\y_1&z_1&0&0\end{bmatrix},\begin{bmatrix}0&0&0&0\\0&0&0&0\\w_2&x_2&0&0\\y_2&z_2&0&0\end{bmatrix},\dots,\begin{bmatrix}0&0&0&0\\0&0&0&0\\w_n&x_n&0&0\\y_n&z_n&0&0\end{bmatrix}\right)$$
$$=\begin{bmatrix}0&0&0&0\\0&0&0&0\\w_1w_2\dots w_n&x_1x_2\dots x_n&0&0\\y_1y_2\dots y_n&z_1z_2\dots z_n&0&0\end{bmatrix}.$$

Let $\sigma,\tau : N \longrightarrow N$ be such that

$$\sigma\left(\begin{bmatrix}0&0&0&0\\0&0&0&0\\w&x&0&0\\y&z&0&0\end{bmatrix}\right)=\begin{bmatrix}0&0&0&0\\0&0&0&0\\-w&-x&0&0\\-y&-z&0&0\end{bmatrix},\ \tau\left(\begin{bmatrix}0&0&0&0\\0&0&0&0\\w&x&0&0\\y&z&0&0\end{bmatrix}\right)=\begin{bmatrix}0&0&0&0\\0&0&0&0\\-y&-z&0&0\\-w&-x&0&0\end{bmatrix}.$$

It can be shown that N is a near-ring with respect to matrix addition and matrix multiplication, and D is a nonzero outer (σ,τ)-n-derivation, where σ and τ are automorphisms of N.

From now on, σ and τ will represent automorphisms of N.

Lemma 3.2. *Let $D : N^n \longrightarrow N$ be an $n-$additive mapping. Then D is an outer (σ,τ)-n-derivation on N if and only if*

$$D(x_1y_1, x_2, \ldots, x_n) = D(x_1, x_2, \ldots, x_n)\tau(y_1) + \sigma(x_1)D(y_1, x_2, \ldots, x_n),$$

$$D(x_1, x_2y_2, \ldots, x_n) = D(x_1, x_2, \ldots, x_n)\tau(y_2) + \sigma(x_2)D(x_1, y_2, \ldots, x_n),$$

$$\vdots$$

$$D(x_1, x_2, \ldots, x_ny_n) = D(x_1, x_2, \ldots, x_n)\tau(y_n) + \sigma(x_n)D(x_1, x_2, \ldots, y_n),$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N$.

Proof. Let D be an outer (σ,τ)-n-derivation on N.
Let $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N$. Then

$$\begin{aligned} D(x_1(y_1+y_1), x_2, \ldots, x_n) &= \sigma(x_1)D(y_1, x_2, \ldots, x_n) + \sigma(x_1)D(y_1, x_2, \ldots, x_n) \\ &\quad + D(x_1, x_2, \ldots, x_n)\tau(y_1) + D(x_1, x_2, \ldots, x_n)\tau(y_1) \end{aligned}$$

and

$$\begin{aligned} D(x_1y_1+x_1y_1, x_2, \ldots, x_n) &= \sigma(x_1)D(y_1, x_2, \ldots, x_n) + D(x_1, x_2, \ldots, x_n)\tau(y_1) \\ &\quad + \sigma(x_1)D(y_1, x_2, \ldots, x_n) + D(x_1, x_2, \ldots, x_n)\tau(y_1). \end{aligned}$$

Since $D(x_1(y_1+y_1), x_2, \ldots, x_n) = D(x_1y_1+x_1y_1, x_2, \ldots, x_n)$, we have

$$\begin{aligned} \sigma(x_1)D(y_1, x_2, \ldots, x_n) + D(x_1, x_2, \ldots, x_n)\tau(y_1) &= D(x_1, x_2, \ldots, x_n)\tau(y_1) \\ &\quad + \sigma(x_1)D(y_1, x_2, \ldots, x_n). \end{aligned}$$

Therefore,

$$D(x_1y_1, x_2, \ldots, x_n) = D(x_1, x_2, \ldots, x_n)\tau(y_1) + \sigma(x_1)D(y_1, x_2, \ldots, x_n).$$

Similarly,

$$D(x_1, x_2y_2, \ldots, x_n) = D(x_1, x_2, \ldots, x_n)\tau(y_2) + \sigma(x_2)D(x_1, y_2, \ldots, x_n)$$

$$\vdots$$

$$D(x_1, x_2, \ldots, x_ny_n) = D(x_1, x_2, \ldots, x_n)\tau(y_n) + \sigma(x_n)D(x_1, x_2, \ldots, y_n).$$

The converse can be shown in a similar manner. □

In a left near-ring N, the right distributive law does not hold in general. However, we can prove the following partial distributive properties in N.

Lemma 3.3. *Let D be an outer (σ, τ)-n-derivation on N. Then*

$$\begin{aligned}(D(x_1, x_2, \ldots, x_i y_i, \ldots, x_n))a &= \sigma(x_i)D(x_1, x_2, \ldots, y_i, \ldots, x_n)a \\ &\quad + D(x_1, x_2, \ldots, x_i, \ldots, x_n)\tau(y_i)a,\end{aligned}$$

for all $a \in N$, $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N$ and $i = 1, 2, \ldots, n$.

Proof. Let $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N$ and let $a \in N$.
Since $\tau : N \longrightarrow N$ is an automorphism, there exists $b \in N$ such that $\tau(b) = a$.
For each $i \in \{1, 2, \ldots, n\}$, consider

$$\begin{aligned}D(x_1, x_2, \ldots, x_i y_i b, \ldots, x_n) &= D(x_1, x_2, \ldots, (x_i y_i)b, \ldots, x_n) \\ &= \sigma(x_i)\sigma(y_i)D(x_1, x_2, \ldots, b, \ldots, x_n) \\ &\quad + D(x_1, x_2, \ldots, x_i y_i, \ldots, x_n)\tau(b),\end{aligned}$$

and

$$\begin{aligned}D(x_1, x_2, \ldots, x_i y_i b, \ldots, x_n) &= D(x_1, x_2, \ldots, x_i(y_i b), \ldots, x_n) \\ &= \sigma(x_i)D(x_1, x_2, \ldots, y_i b, \ldots, x_n) \\ &\quad + D(x_1, x_2, \ldots, x_i, \ldots, x_n)\tau(y_i)\tau(b) \\ &= \sigma(x_i)\sigma(y_i)D(x_1, x_2, \ldots, b, \ldots, x_n) \\ &\quad + \sigma(x_i)D(x_1, x_2, \ldots, y_i, \ldots, x_n)\tau(b) \\ &\quad + D(x_1, x_2, \ldots, x_i, \ldots, x_n)\tau(y_i)\tau(b).\end{aligned}$$

Combining the above two equalities and replacing $\tau(b)$ by a, we obtain

$$\begin{aligned}(D(x_1, x_2, \ldots, x_i y_i, \ldots, x_n))a &= \sigma(x_i)D(x_1, x_2, \ldots, y_i, \ldots, x_n)a \\ &\quad + D(x_1, x_2, \ldots, x_i, \ldots, x_n)\tau(y_i)a.\end{aligned}$$

The proof is complete. □

By Lemma 3.2, and Lemma 3.3, we obtain the following corollary.

Corollary 3.4. *Let D be an outer (σ, τ)-n-derivation on N. Then*

$$\begin{aligned}&(D(x_1, x_2, \ldots, x_i, \ldots, x_n)\tau(y_i) + \sigma(x_i)D(x_1, x_2, \ldots, y_i, \ldots, x_n))a \\ &\quad = D(x_1, x_2, \ldots, x_i, \ldots, x_n)\tau(y_i)a + \sigma(x_i)D(x_1, x_2, \ldots, y_i, \ldots, x_n))a,\end{aligned}$$

for all $a \in N$, $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N$ and $i = 1, 2, \ldots, n$.

Lemma 3.5. *Let N be a prime near-ring, D a nonzero outer (σ,τ)-n-derivation on N and $x \in N$.*
(i) *If $D(N,N,\ldots,N)x = \{0\}$ then $x = 0$.*
(ii) *If $xD(N,N,\ldots,N) = \{0\}$ then $x = 0$.*

Proof. (i) Suppose that $D(N,N,\ldots,N)x = \{0\}$. Since D is a nonzero mapping, there exist $x_1, x_2, \ldots, x_n \in N$ such that $D(x_1, x_2, \ldots, x_n) \neq 0$. For any $a \in N$. Since $\tau : N \longrightarrow N$ is an automorphism, there exists $b \in N$ such that $\tau(b) = a$. Then $D(x_1 b, x_2, \ldots, x_n)x = 0$.
By using hypothesis and Lemma 3.3, we get

$$D(x_1, x_2, \ldots, x_n)\tau(b)x = 0.$$

Since $\tau(b) = a$, $D(x_1 y_1, x_2, \ldots, x_n)ax = 0$.
Therefore, $D(x_1, x_2, \ldots, x_n)Nx = \{0\}$.
Since $D(x_1, x_2, \ldots, x_n) \neq 0$ and N is a prime near-ring, we conclude that $x = 0$.
(ii) This can be proved in a similar manner. □

Recall that a left near-ring N is called zero symmetric if $0x = 0$ for all $x \in N$. In the following lemma, we prove that if a near-ring N admits an outer (σ,τ)-n-derivation D, then N is a zero symmetric.

Lemma 3.6. *Let D be an outer (σ,τ)-n-derivation on N. Then N is a zero symmetric.*

Proof. Let $x \in N$. Since σ and τ are automorphisms, there exist $a, b \in N$ such that $\sigma(a) = x$ and $\tau(b) = x$. Then

$$\begin{aligned} D(a0b, x, \ldots, x) &= D(a(0b), x, \ldots, x) \\ &= \sigma(a)\sigma(0)D(b, x, \ldots, x) + \sigma(a)D(0, x, \ldots, x)\tau(b) \\ &\quad + D(a, x, \ldots, x)\tau(0)\tau(b) \\ &= 0D(b, x, \ldots, x) + 0\tau(b) + 0\tau(b), \end{aligned}$$

and

$$\begin{aligned} D(a0b, x, \ldots, x) &= D((a0)b, x, \ldots, x) \\ &= D(0b, x, \ldots, x) \\ &= \sigma(0)D(b, x, \ldots, x) + 0\tau(b). \end{aligned}$$

Combining the above two equalities and replacing $\tau(b)$ by x, we have $0x = 0$. Hence N is a zero symmetric. □

4 Commutativities of near-ring

Theorem 4.1. *Let N ba a prime near-ring and let D be a nonzero outer (σ,τ)-n-derivation on N. If $D(N,N,\ldots,N)\subseteq Z$, then N is a commutative ring.*

Proof. Suppose that $D(N,N,\ldots,N)\subseteq Z$. Since D is a nonzero outer (σ,τ)-n-derivation on N, there exist nonzero elements $x_1,x_2,\ldots,x_n\in N$, such that $D(x_1,x_2,\ldots,x_n)\in Z-\{0\}$. Then

$$D(x_1,x_2,\ldots,x_n)+D(x_1,x_2,\ldots,x_n)=D(x_1+x_1,x_2,\ldots,x_n)\in Z.$$

By Lemma 2.1, we have $(N,+)$ is abelian.
Next, we will show that N is commutative. Let $x,y\in N$. Since σ and τ are automorphisms, there exist $a,b\in N$ such that $\sigma(a)=x$ and $\tau(b)=y$.
Since $D(N,N,\ldots,N)\subseteq Z$, $D(ab,x_2,\ldots,x_n)y=yD(ab,x_2,\ldots,x_n)$.
By Lemma 3.3 and using $D(N,N,\ldots,N)\subseteq Z$, we have

$$D(b,x_2,\ldots,x_n)(xy-yx)=0.$$

Then $cD(b,x_2,\ldots,x_n)(xy-yx)=0$ for all $c\in N$.
By using $D(N,N,\ldots,N)\subseteq Z$,

$$D(b,x_2,\ldots,x_n)N(xy-yx)=\{0\}.$$

Since N is a prime near-ring, $D(b,x_2,\ldots,x_n)=0$ or $(xy-yx)=0$.
If $xy-yx=0$ then N is commutative.
Assume that $D(b,x_2,\ldots,x_n)=0$. Consider

$$xD(x_1b,x_2,\ldots,x_n)=D(x_1b,x_2,\ldots,x_n)x.$$

By Lemma 3.3 and using $D(N,N,\ldots,N)\subseteq Z$, we have

$$D(x_1,x_2,\ldots,x_n)(xy-yx)=0.$$

Since $D(N,N,\ldots,N)\subseteq Z$, $D(x_1,x_2,\ldots,x_n)N(xy-yx)=\{0\}$.
Since $D(x_1,x_2,\ldots,x_n)\neq 0$, $xy-yx=0$. This implies that N is commutative.
Therefore N is a commutative ring. □

Theorem 4.2. *Let N be a prime near-ring and let D_1 and D_2 be nonzero outer (σ,τ)-n-derivation on N. If*

$$D_1(x_1,x_2,\ldots,x_n)D_2(y_1,y_2,\ldots,y_n)=D_2(y_1,y_2,\ldots,y_n)D_1(x_1,x_2,\ldots,x_n)$$

for all $x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n\in N$, then $(N,+)$ is abelian.

Proof. Suppose that

$$D_1(x_1,x_2,\dots,x_n)D_2(y_1,y_2,\dots,y_n)=D_2(y_1,y_2,\dots,y_n)D_1(x_1,x_2,\dots,x_n)$$

for all $x_1,x_2,\dots,x_n,y_1,y_2,\dots,y_n\in N$.
Let $x,y,x_1,x_2,\dots,x_n,y_2,\dots,y_n\in N$. Then

$$\begin{aligned}&D_1(x_1+x_1,x_2,\dots,x_n)D_2(x+y,y_2,\dots,y_n)\\&\quad=D_2(x+y,y_2,\dots,y_n)D_1(x_1+x_1,x_2,\dots,x_n)\end{aligned}\tag{4.2.1}$$

By using the hypothesis, we get

$$\begin{aligned}&D_1(x_1+x_1,x_2,\dots,x_n)D_2(x+y,y_2,\dots,y_n)\\&\quad=D_1(x_1,x_2,\dots,x_n)D_2(x,y_2,\dots,y_n)\\&\quad\quad+D_1(x_1,x_2,\dots,x_n)D_2(x,y_2,\dots,y_n)\\&\quad\quad+D_1(x_1,x_2,\dots,x_n)D_2(y,y_2,\dots,y_n)\\&\quad\quad+D_1(x_1,x_2,\dots,x_n)D_2(y,y_2,\dots,y_n),\end{aligned}$$

and

$$\begin{aligned}&D_2(x+y,y_2,\dots,y_n)D_1(x_1+x_1,x_2,\dots,x_n)\\&\quad=D_1(x_1,x_2,\dots,x_n)D_2(x,y_2,\dots,y_n)\\&\quad\quad+D_1(x_1,x_2,\dots,x_n)D_2(y,y_2,\dots,y_n)\\&\quad\quad+D_1(x_1,x_2,\dots,x_n)D_2(x,y_2,\dots,y_n)\\&\quad\quad+D_1(x_1,x_2,\dots,x_n)D_2(y,y_2,\dots,y_n).\end{aligned}$$

Combining the above two equalities, we have

$$\begin{aligned}&D_1(x_1,x_2,\dots,x_n)D_2(x,y_2,\dots,y_n)+D_1(x_1,x_2,\dots,x_n)D_2(y,y_2,\dots,y_n)\\&\quad=D_1(x_1,x_2,\dots,x_n)D_2(y,y_2,\dots,y_n)+D_1(x_1,x_2,\dots,x_n)D_2(x,y_2,\dots,y_n).\end{aligned}$$

Hence, $D_1(x_1,x_2,\dots,x_n)D_2(x+y-x-y,y_2,\dots,y_n)=0$.
Therefore, $D_1(N,N,\dots,N)D_2(x+y-x-y,y_2,\dots,y_n)=\{0\}$,
for all $x,y,y_2,\dots,y_n\in N$.
By Lemma 3.5, we obtain

$$D_2(x+y-x-y,y_2,\dots,y_n)=0,\tag{4.2.2}$$

for all $x,y,y_2,\dots,y_n\in N$.
For $x,y,y_1,y_2,\dots,y_n\in N$,

$$\begin{aligned}&\sigma(y_1)D_2(x+y-x-y,y_2,\dots,y_n)+D_2(y_1,y_2,\dots,y_n)\tau(x+y-x-y)\\&\quad=D_2(y_1(x+y-x-y),y_2,\dots,y_n)\\&\quad=D_2(y_1x+y_1y-y_1x-y_1y,y_2,\dots,y_n)=0.\end{aligned}$$

Hence, $D_2(y_1, y_2, \ldots, y_n)\tau(x + y - x - y) = 0$.
Now, by using Lemma 3.5, we conclude that $\tau(x + y - x - y) = 0$, for all $x, y \in N$.
Since τ is an injective function, $x + y = y + x$ for all $x, y \in N$. Therefore $(N, +)$ is abelian. This completes the proof. □

Theorem 4.3. *Let N be a prime near-ring and let D_1 and D_2 be nonzero outer (σ, τ)-n-derivations on N. If*

$$D_1(x_1, x_2, \ldots, x_n)D_2(y_1, y_2, \ldots, y_n) = -D_2(x_1, x_2, \ldots, x_n)D_1(y_1, y_2, \ldots, y_n),$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N$, then $(N, +)$ is abelian.

Proof. The proof is similar to that of Theorem 4.2. □

Theorem 4.4. *Let N be a prime near-ring and let D_1 and D_2 be nonzero outer (σ, τ)-n-derivation. If*

$$\sigma(D_1(x_1, x_2, \ldots, x_n))D_2(y_1, y_2, \ldots, y_n) + D_2(x_1, x_2, \ldots, x_n)\tau(D_1(y_1, y_2, \ldots, y_n)) = 0,$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N$, then $(N, +)$ is abelian.

Proof. Assume that

$$\sigma(D_1(x_1, x_2, \ldots, x_n))D_2(y_1, y_2, \ldots, y_n) + D_2(x_1, x_2, \ldots, x_n)\tau(D_1(y_1, y_2, \ldots, y_n)) = 0$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N$.
Let $x, y \in N$. Replacing y_1 by $x + y$, we obtain

$$\sigma(D_1(x_1, x_2, \ldots, x_n))D_2(x + y - x - y, y_2, \ldots, y_n) = 0.$$

By hypothesis, $D_2(x_1, x_2, \ldots, x_n)\tau(D_1(x + y - x - y, y_2, \ldots, y_n)) = 0$,
for all $x, y, x_1, x_2, \ldots, x_n, y_2, \ldots, y_n \in N$.
By Lemma 3.3, we get
$\tau(D_1(x + y - x - y, y_2, \ldots, y_n)) = 0$ for all $x, y, y_2, \ldots, y_n \in N$.
Since τ is an injective function,
$D_1(x + y - x - y, y_2, \ldots, y_n) = 0$ for all $x, y, y_2, \ldots, y_n \in N$.
Arguing in a similar manner as in the proof in Theorem 4.2, we have $(N, +)$ is abelian. □

Theorem 4.5. *Let N be a prime near-ring and let D_1 and D_2 be nonzero outer (σ, τ)-n-derivation and $n \geqslant 2$. If*

$$\sigma(D_1(x_1, x_2, \ldots, x_n))D_2(y_1, y_2, \ldots, y_n) - D_2(x_1, x_2, \ldots, x_n)\tau(D_1(y_1, y_2, \ldots, y_n)) = 0$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N$, then $(N, +)$ is abelian.

Proof. Assume that

$$\sigma(D_1(x_1, x_2, \ldots, x_n))D_2(y_1, y_2, \ldots, y_n) - D_2(x_1, x_2, \ldots, x_n)\tau(D_1(y_1, y_2, \ldots, y_n)) = 0,$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N$.
Let $x, y \in N$. Replacing y_1 by $x + y$, we have

$$\begin{aligned} &\sigma(D_1(x_1, x_2, \ldots, x_n))D_2(x + y, y_2, \ldots, y_n) \\ &-D_2(x_1, x_2, \ldots, x_n)\tau(D_1(x + y, y_2, \ldots, y_n)) = 0 \end{aligned} \tag{4.5.1}$$

for all $x_1, x_2, \ldots, x_n, y_2, \ldots, y_n \in N$.
Consider

$$\begin{aligned} &-D_2(x_1, x_2, \ldots, x_n)\tau(D_1(x + y, y_2, \ldots, y_n)) \\ &\quad = D_2(x_1, x_2, \ldots, x_n)\tau(-D_1(x + y, y_2, \ldots, y_n)) \\ &\quad = D_2(x_1, x_2, \ldots, x_n)\tau(D_1(x + y, -y_2, \ldots, y_n)) \\ &\quad = D_2(x_1, x_2, \ldots, x_n)\tau(D_1(x, -y_2, \ldots, y_n)) \\ &\qquad + D_2(x_1, x_2, \ldots, x_n)\tau(D_1(y, -y_2, \ldots, y_n)) \\ &\quad = D_2(x_1, x_2, \ldots, x_n)\tau(D_1(-x, y_2, \ldots, y_n)) \\ &\qquad + D_2(x_1, x_2, \ldots, x_n)\tau(D_1(-y, y_2, \ldots, y_n)) \\ &\quad = D_2(x_1, x_2, \ldots, x_n)\tau(D_1(-x - y, y_2, \ldots, y_n)). \end{aligned}$$

By using the hypothesis, we get

$$\begin{aligned} &D_2(x_1, x_2, \ldots, x_n)\tau(D_1(-x - y, y_2, \ldots, y_n)) \\ &\quad = \sigma(D_1(x_1, x_2, \ldots, x_n))D_2(-x - y, y_2, \ldots, y_n). \end{aligned}$$

Then

$$\begin{aligned} &-D_2(x_1, x_2, \ldots, x_n)\tau(D_1(x + y, y_2, \ldots, y_n)) \\ &\quad = \sigma(D_1(x_1, x_2, \ldots, x_n))D_2(-x - y, y_2, \ldots, y_n). \end{aligned}$$

From (4.5.1), it follows that

$$\sigma(D_1(x_1, x_2, \ldots, x_n))D_2(x + y - x - y, y_2, \ldots, y_n) = 0.$$

By using hypothesis again, we have

$$D_2(x_1, x_2, \ldots, x_n)\tau(D_1(x + y - x - y, y_2, \ldots, y_n)) = 0,$$

for all $x, y, x_1, x_2, \ldots, x_n, y_2, \ldots, y_n \in N$.
Arguing in a similar manner as in the proof in Theorem 4.4, we have $(N, +)$ is abelian. □

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