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On Trace of Symmetric Bi-derivations on Rings

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Abstract

The purpose of this paper is to prove some results concerning symmetric bi-derivations on prime and semiprime rings. Moreover, we characterize symmetric bi-derivations when their traces satisfies certain conditions on prime and semiprime rings. Furthermore, the commutativity of a prime ring satisfying certain identity involving symmetric bi-derivations of a prime ring is discussed.

1 Introduction

Throughout, R will represent a ring. A ring R is prime if $aRb = 0$ with $a, b \in R$ implies $a = 0$ or $b = 0$ and semiprime in case $aRa = 0$ with $a \in R$ implies $a = 0$. For an integer $n > 1$, an element $x \in R$ is called $n-$torsion free if $nx = 0$ implies $x = 0$. A ring R is called an $n-$torsion free ring if every element in R is $n-$torsion free. Moreover, a ring R is called an $n!-$torsion free if it is $d-$torsion free for any divisor d of $n!$. For any $x, y \in R$, the symbol $[x, y]$ will represent the commutator $xy - yx$ and the symbol $< x, y >$ stands for the skew-commutator $xy + yx$.

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A mapping $F : R \times R \longrightarrow R$ is said to be symmetric if $F(x, y) = F(y, x)$ for all $x, y \in R$. A mapping $f : R \longrightarrow R$ defined by $f(x) = F(x, x)$, where $F : R \times R \longrightarrow R$ is symmetric mapping, is called a trace of F. An additive mapping $d : R \longrightarrow R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Gy. Maksa [1] introduced the concept of a symmetric bi-derivation on a ring R (see also [2],where an example can be seen). A symmetric bi-additive mapping $F : R \times R \longrightarrow R$ is called a symmetric bi-derivation if $F(xy, z) = F(x, z)y + xF(y, z)$ for all $x, y, z \in R$. Obviously, in this case also the relation $F(x, yz) = F(x, y)z + yF(x, z)$ for all $x, y, z \in R$ holds. It was shown in [2] that symmetric bi-derivations are related to general solution of some functional equations. J. Vukman proved some results concerning symmetric bi-derivation on prime and semiprime rings [5, 6].

The aim of this work is to investigate some results on the notion of symmetric bi-derivation on rings which partially extend some results of Vukman [5]. Moreover we also investigate some conditions involving traces of symmetric bi-derivations, derivations and endomorphisms in a prime ring which turn the prime ring into commutative ring.

2 Preliminaries

This section gives some basic definitions that have been used in the sequel.

Definition 2.1. *A mapping $F : R \times R \longrightarrow R$ is called symmetric if $F(x, y) = F(y, x)$ holds for all $x, y \in R$. A mapping $f : R \longrightarrow R$ defined by $f(x) = F(x, x)$ for all $x \in R$, where $F : R \times R \longrightarrow R$ is a symmetric mapping, is called a trace of F. A mapping $F : R \times R \longrightarrow R$ is called bi-additive if for all $x, y, z \in R$, the following conditions hold:*
(i) $F(x + y, z) = F(x, z) + F(y, z)$,
(ii) $F(x, y + z) = F(x, y) + F(x, z)$.
The following are some basic properties of a symmetric bi-additive mapping $F : R \times R \longrightarrow R$ with the trace f of F.
The proofs of these properties are straightforward and hence omitted.
(1) $f(x + y) = f(x) + f(y) + 2F(x, y)$ for all $x, y \in R$,
(2) $F(x, 0) = F(0, x) = 0$ for all $x \in R$,
(3) $F(-x, y) = -F(x, y) = F(x, -y)$ for all $x, y \in R$,
(4) $f(-x) = f(x)$ for all $x \in R$,
(5) $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ for all $x, y \in R$.

Definition 2.2. *Let $F : R \times R \longrightarrow R$ be a symmetric bi-additive mapping. We call F symmetric bi-derivation on R if it satisfies the following condition $F(xy, z) = F(x, z)y + xF(y, z)$ for all $x, y, z \in R$.*
Obviously, a symmetric bi-derivation F on R satisfies the relation $F(x, yz) = F(x, y)z + yF(x, z)$ for all $x, y, z \in R$.
The following lemmas will be used in our proofs.

Lemma 2.3. ([3]) *Let R be a prime ring and $d : R \longrightarrow R$ be a derivation. If $a \in R$ such that $ad(x) = 0$ (resp. $ad(x) = 0$) for all $x \in R$, then $a = 0$ or $d(x) = 0$ for all $x \in R$.*

Lemma 2.4. ([4]) *Let R be a $2-$torsion prime ring. If $a, b \in R$ such that $axb + bxa = 0$ for all $x \in R$, then $a = 0$ or $b = 0$.*

3 Main results

Theorem 3.1. *Let R be a prime ring. Let $F : R \times R \longrightarrow R$ be a symmetric bi-derivation with trace f. Let $d : R \longrightarrow R$ be a derivation. If $[d(x), x] = f(x)$ for all $x \in R$ then R is commutative or $d(x) = 0$ for all $x \in R$.*

Proof. Suppose that

$$[d(x), x] = f(x) \text{ for all } x \in R. \tag{3.1.1}$$

The linearization of (3.1.1) gives

$$[d(y), x] + [d(x), y] = 2F(x, y), \text{ for all } x, y \in R. \tag{3.1.2}$$

Replacing x by yx in (3.1.2), we obtain

$$y([d(y), x] + [d(x), y]) + d(y)[x, y] = 2yF(x, y) \text{ for all } x, y \in R. \tag{3.1.3}$$

And using (3.1.2), it follows that

$$d(y)[x, y] = 0, \text{ for all } x, y \in R. \tag{3.1.4}$$

By lemma 2.3, implies $d(y) = 0$ for all $y \in R$ or $[x, y]$ for all $x, y \in R$. The proof is complete.

□

Theorem 3.2. *Let R be a $2-$torsion free prime ring. Let $F : R \times R \longrightarrow R$ be a symmetric bi-derivation with trace f and $d : R \longrightarrow R$ be a derivation. If $g : R \longrightarrow R$ be endomorphism such that $F(d(x), x) = g(x)$ for all $x \in R$ then $F(x, y) = 0$ for all $x, y \in R$ or $d(x) = 0$ for all $x \in R$.*

Proof. Suppose that g is an endomorphism satisfying the relation

$$F(d(x),x)=g(x) \text{ for all } x\in R. \quad (3.2.1)$$

The linearization of (3.2.1) gives

$$F(d(y),x)+F(d(x),y)=0 \text{ for all } x,y\in R. \quad (3.2.2)$$

Replacing y by yx in (3.2.2), we obtain

$$(F(d(y),x)+F(d(x),y))x+d(y)f(x)+F(y,x)d(x)+2yg(x)=0. \quad (3.2.3)$$

From (3.2.2) and (3.2.3), it follows that

$$d(y)f(x)+F(y,x)d(x)+2yg(x)=0 \text{ for all } x,y\in R. \quad (3.2.4)$$

Replacing y by xy in (3.2.4) and using (3.2.4), we have

$$d(x)yf(x)+f(x)yd(x)=0 \text{ for all } x,y\in R.$$

By Lemma 2.4, It follows that $d(x)=0$ for all $x\in R$ or $f(x)=0$ for all $x\in R$. Since f is the trace of F, $F(x,x)=0$ for all $x\in R$.
By linearization, we have $F(x,y)+F(y,x)=0$ for all $x,y\in R$.
Since F is symmetric and R is a $2-$torsion free ring, $F(x,y)=0$ for all $x,y\in R$. The proof of the theorem is complete.

□

Theorem 3.3. *Let R be a $2-$torsion free prime ring. Let $F:R\times R\longrightarrow R$ be a symmetric bi-derivation with trace f. If $d:R\longrightarrow R$ be a derivation such that $F(d(x),x)=f(x)$ for all $x\in R$ then*
(i) $F(x,y)=0$ for all $x,y\in R$ or $d(x)=0$ for all $x\in R$,
(ii) $<d(x),f(x)>=0$ for all $x\in R$.

Proof. Suppose that $d:R\longrightarrow R$ is a derivation satisfying the relation

$$F(d(x),x)=f(x) \text{ for all } x\in R. \quad (3.3.1)$$

The linearization of (3.3.1) gives

$$F(d(y),x)+F(d(x),y)=2F(x,y) \text{ for all } x,y\in R. \quad (3.3.2)$$

Replacing y by yx in (3.3.2), we get

$$(F(d(y),x)+F(d(x),y))x+d(y)f(x)+F(y,x)d(x)=2F(x,y)x.$$

And using (3.3.2), it follows that

$$d(y)f(x) + F(y,x)d(x) = 0 \text{ for all } x, y \in R. \tag{3.3.3}$$

Replacing y by xy in (3.3.3) and using (3.3.3), we obtain

$$d(x)yf(x) + f(x)yd(x) = 0 \text{ for all } x, y \in R. \tag{3.3.4}$$

By lemma 2.4 implies $d(x) = 0$ for all $x \in R$ or $f(x) = 0$ for all $x \in R$.
If $f(x) = 0$ for all $x \in R$ then $F(x, y) = 0$ for all $x, y \in R$.
Replacing y by x in (3.3.3), we obtain $d(x)f(x)+f(x)d(x) = 0$ for all $x \in R$, which means that $< d(x), f(x) >= 0$ for all $x \in R$. The proof of the theorem is complete.

□

Theorem 3.4. *Let R be a $2-$torsion free prime ring. Suppose there exist symmetric bi-derivations $F_1 : R \times R \longrightarrow R$ and $F_2 : R \times R \longrightarrow R$ such that $F_1(f_2(x), x) = f_1(x)$ for all $x \in R$, where f_1 and f_2 are the traces of F_1 and F_2, respectively. Then*
(i) $F_1(x, y) = 0$ for all $x, y \in R$ or $F_2(x, y) = 0$ for all $x, y \in R$,
(ii) $< f_1(x), f_2(x) >= 0$ for all $x \in R$.

Proof. Suppose that $F_1, F_2 : R \times R \longrightarrow R$ are symmetric bi-derivations satisfying the relation

$$F_1(f_2(x), x) = f_1(x) \text{ for all } x \in R. \tag{3.4.1}$$

The linearization of (3.4.1), gives

$$F_1(f_2(x), x) + F_1(f_2(y), x) + 2F_1(F_2(x, y), x) + F_1(f_2(x), y)$$
$$+F_1(f_2(y), y) + 2F_1(F_2(x, y), y) - f_1(x) - f_1(y) - 2F_1(x, y) = 0.$$

Using (3.4.1), we get

$$F_1(f_2(y), x) + 2F_1(F_2(x, y), x) + F_1(f_2(x), y) + 2F_1(F_2(x, y), y) = 2F_1(x, y)$$

for all $x, y \in R$

(3.4.2)

Replacing x by $-x$ in (3.4.2) and comparing this new equation with (3.4.2), we obtain $4F_1(F_2(x, y), x) + 2F_1(f_2(x), y) = 0$. Since R is a $2-$ torsion free prime ring, we have

$$2F_1(F_2(x, y), x) + F_1(f_2(x), y) = 0 \text{ for all } x, y \in R. \tag{3.4.3}$$

Replacing y by xy in (3.4.3), we get

$$2f_1(x)y + 2f_2(x)F_1(y,x) + 2f_1(x)F_2(x,y)$$
$$+f_1(x)y + x(2F_1(F_2(x,y),x) + F_1(f_2(x),y)) = 0.$$

Hence, by (3.4.3) we get

$$3f_1(x)y + 2f_2(x)F_1(x,y) + 2f_1(x)F_2(x,y) = 0 \text{ for all } x,y \in R. \tag{3.4.4}$$

Replacing x by $2x$ in (3.4.4) and using the fact that R is a $2-$torsion free prime ring, we have

$$3f_1(x)y + 4f_2(x)F_1(x,y) + 4f_1(x)F_2(x,y) = 0 \text{ for all } x,y \in R. \tag{3.4.5}$$

By comparing (3.4.4) and (3.4.5), we obtain

$$2f_2(x)F_1(x,y) + 2f_1(x)F_2(x,y) = 0 \text{ for all } x,y \in R.$$

Since is a 2-torsion free prime ring, we get

$$f_2(x)F_1(x,y) + f_1(x)F_2(x,y) = 0 \text{ for all } x,y \in R. \tag{3.4.6}$$

Replacing y by yx in (3.4.6) and using (3.4.6), we obtain

$$f_2(x)yf_1(x) + f_1(x)yf_2(x) = 0 \text{ for all } x,y \in R. \tag{3.4.7}$$

By Lemma 2.4, implies $f_1(x) = 0$ for all $x \in R$ or $f_2(x) = 0$ for all $x \in R$, which means that $F_1(x,y) = 0$ for all $x,y \in R$ or $F_2(x,y) = 0$ for all $x,y \in R$. Replacing y by x in(3.4.6), we obtain $f_2(x)f_1(x) + f_1(x)f_2(x) = 0$ for all $x \in R$. That is $< f_1(x), f_2(x) >= 0$ for all $x \in R$. The proof of the theorem is complete. □

Theorem 3.5. *Let R be a $2-$torsion free semiprime ring. Suppose there exist symmetric bi-derivations $F_1 : R \times R \longrightarrow R$ and $F_2 : R \times R \longrightarrow R$ such that $F_1(f_1(x),x) = f_2(x)$ for all $x \in R$, where f_1 and f_2 are the traces of F_1 and F_2, respectively. Then $F_1(x,y) = 0$ for all $x,y \in R$.*

Proof. Suppose that $F_1, F_2 : R \times R \longrightarrow R$ are symmetric bi-derivations satisfying the relation

$$F_1(f_1(x),x) = f_2(x) \text{ for all } x \in R. \tag{3.5.1}$$

The linearization of (3.5.1) gives

$$F_1(f_1(y),x) + 2F_1(F_1(x,y),x) + F_1(f_1(x),y) + 2F_1(F_1(x,y),y) = 2F_2(x,y)$$

for all $x, y \in R$

(3.5.2)

Replacing x by $-x$ in (3.5.2) and comparing this new equation with (3.5.2), we obtain $4F_1(F_1(x,y),x) + 2F_1(f_1(x),y) = 0$ for all $x, y \in R$. Since R is a 2−torsion free semiprime ring, we have

$$2F_1(F_1(x,y),x) + F_1(f_1(x),y) = 0 \text{ for all } x, y \in R. \tag{3.5.3}$$

Replacing y by xy in (3.5.3) and using (3.5.3), we obtain

$$3F_1(f_1(x),x)y + 4f_1(x)F_1(x,y) = 0 \text{ for all } x, y \in R. \tag{3.5.4}$$

Replacing y by yx in (3.5.4) and using (3.5.4), we obtain $4f_1(x)yf_1(x) = 0$ for all $x, y \in R$. Since R is a 2−torsion free semiprime ring, we get

$$f_1(x)yf_1(x) = 0 \text{ for all } x, y \in R. \tag{3.5.5}$$

Since R is semiprime ring, $f_1(x) = 0$ for all $x \in R$. Hence $F_1(x,y) = 0$ for all $x, y \in R$. The proof of the theorem is complete. □

In case $F_1 = F_2$ in Theorem 3.5, we get the following corollary.

Corollary 3.6. *Let R be a 2−torsion free semiprime ring.Let $F : R \times R \longrightarrow R$ be a symmetric bi-derivation with trace f. If $F(f(x),x) = f(x)$ for all $x \in R$, Then $F(x,y) = 0$ for all $x, y \in R$.*

Theorem 3.7. *Let R be a 2−torsion free semiprime ring.*
Let $F : R \times R \longrightarrow R$ be a symmetric bi-derivation with the trace f.
If $g : R \longrightarrow R$ is an endomorphism such that $F(f(x),x) = g(x)$ for all $x \in R$ then $F(x,y) = 0$ for all $x, y \in R$.

Proof. Suppose that $g : R \longrightarrow R$ is an endomorphism satisfying the relation

$$F(f(x),x) = g(x) \text{ for all } x \in R. \tag{3.7.1}$$

The linearization of (3.7.1), gives

$$F(f(y),x)+2F(F(x,y),x)+F(f(x),y)+2F(F(x,y),y) = 0 \text{ for all } x, y \in R. \tag{3.7.2}$$

Replacing x by $-x$ in (3.7.2) and comparing this new equation with (3.7.2) that $4F(F(x,y),x) + 2F(f(x),y) = 0$ for all $x, y \in R$.
Since R is a 2−torsion free semiprime ring, we get

$$2F(F(x,y),x) + F(f(x),y) = 0 \text{ for all } x, y \in R. \tag{3.7.3}$$

Replacing y by yx in (3.7.3) and using (3.7.3), we obtain

$$4F(x,y)f(x) + 3yg(x) = 0 \text{ for all } x, y \in R. \tag{3.7.4}$$

Replacing y by xy in (3.7.4) and using (3.7.4), we obtain $4f(x)yf(x) = 0$ for all $x, y \in R$. Since R is a $2-$torsion free semiprime ring,

$$f(x)yf(x) = 0 \text{ for all } x, y \in R. \tag{3.7.5}$$

Since R is semiprime, $f(x) = 0$ for all $x \in R$. Hence $F(x,y) = 0$ for all $x, y \in R$. The proof of the theorem is complete.

□

Theorem 3.8. *Let R be a $3!-$torsion free prime ring. Let $F : R \times R \longrightarrow R$ be a symmetric bi-derivation with trace f. Let $d : R \longrightarrow R$ be a derivation. If $d([f(x), x]) = 0$ for all $x \in R$, then $< [f(x), x], d(x) >= 0$ for all $x \in R$.*

Proof. Suppose that

$$d([f(x), x]) = 0 \text{ for all } x \in R. \tag{3.8.1}$$

The linearization of (3.8.1), gives

$$d([f(y), x]) + 2d([F(x,y), x]) + d([f(x), y]) + 2d([F(x,y), y]) = 0 \text{ for all } x, y \in R. \tag{3.8.2}$$

Replacing x by $-x$ in (3.8.2) we obtain by comparing this new equation with (3.8.2) that

$$2d([f(y), x]) + 4d([F(x,y), y]) = 0 \text{ for all } x, y \in R.$$

Since R is a $3!-$torsion free prime ring, we have

$$d([f(y), x]) + 2d([F(x,y), y]) = 0 \text{ for all } x, y \in R. \tag{3.8.3}$$

Replacing x by yx in (3.8.3) and using (3.8.3), we obtain

$$\begin{aligned} 3[f(y), y]d(x) + d(y)[f(y), x] + 2d(f(y))[x, y] \\ +2f(y)d([x, y]) + 2d(y)[F(x,y), y] = 0, \text{ for all } x, y \in R. \end{aligned} \tag{3.8.4}$$

Replacing y by x in (3.8.4) and using R is a $3!-$ torsion prime ring, we get

$$[f(x), x]d(x) + d(x)[f(x), x] = 0 \text{ for all } x \in R.$$

Therefore $< [f(x), x], d(x) >= 0$ for all $x \in R$. The proof of the theorem is complete. □

Theorem 3.9. *Let R be a $3!-$torsion free prime ring. Let $F : R \times R \longrightarrow R$ be a symmetric bi-derivation with trace f. Let $d : R \longrightarrow R$ be a derivation. If $g : R \longrightarrow R$ is an endomorphism such that $d([f(x), x]) = g(x)$ for all $x \in R$, then $< [f(x), x], d(x) >= 0$ for all $x \in R$.*

Proof. Suppose that

$$d([f(x), x]) = g(x) \text{ for all } x \in R. \tag{3.9.1}$$

The linearization of (3.9.1) gives

$$d([f(y), x]) + 2d([F(x, y), x]) + d([f(x), y]) + 2d([F(x, y), y]) = 0 \text{ for all } x, y \in R. \tag{3.9.2}$$

Replacing x by $-x$ in (3.9.2) we obtain by comparing this new equation with (3.9.2) that $2d([f(y), x]) + 4d([F(x, y), y]) = 0$ for all $x, y \in R$. And using the fact that R is a $3!-$torsion free prime ring, we get

$$d([f(y), x]) + 2d([F(x, y), y]) = 0 \text{ for all } x, y \in R. \tag{3.9.3}$$

Replacing y by x in (3.9.3) and using R is a $3!-$torsion prime ring, we get

$$d([f(x), x]) = 0, \text{ for all } x, y \in R. \tag{3.9.4}$$

By Theorem 3.8, we get

$$< [f(x), x], d(x) >= 0 \text{ for all } x \in R.$$

The proof of the theorem is complete. □

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